MICROLOCAL ANALYSIS

1. Quantization

1.1. **Cotangent bundle.** Let M be a smooth dimensional manifold. The cotangent bundle T^*M is a vector bundle over M whose fiber at $x \in M$ is given by the set of linear forms $\xi : T_xM \to \mathbb{R}$. If $\kappa : M \to N$ is a smooth diffeomorphism between two smooth manifolds M and N, the natural forward action of κ on vectors is by the differential map

$$d\kappa(x) : T_xM \to T_{\kappa(x)}N, \qquad v \mapsto d\kappa(x)v.$$

The transpose map $d\kappa(x) : T^*_{\kappa(x)}N \to T^*_xM$ is defined by the identity

$$(d\kappa(x)^\top \xi, v) := (\xi, d\kappa(x)v), \qquad \forall v \in T_xM.$$

The natural forward action of κ is by the inverse transpose of the differential:

$$d\kappa(x)^{-\top} : T^*_xM \to T^*_{\kappa(x)}N, \qquad \xi \mapsto d\kappa(x)^{-\top}\xi.$$

This can be packaged into a single map $\widetilde{\kappa} : T^*M \to T^*N$, called the symplectic lift of κ, defined as

$$\widetilde{\kappa}(x,\xi) := (\kappa(x), d\kappa(x)^{-\top}\xi). \tag{1.1}$$

1.2. **Quantization in $\mathbb{R}^n$.** The cotangent bundle $T^*\mathbb{R}^n$ of $\mathbb{R}^n$ can be naturally identified with $T^*\mathbb{R}^n \simeq \mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. The space $S^m(\mathbb{R}^n)$ of *symbols* of order m is defined as the set of smooth functions $a \in C^\infty(T^*\mathbb{R}^n)$ satisfying the following bounds: for all $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, there exists $C := C(U,\alpha,\beta) > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \qquad |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C\langle \xi \rangle^{m-|\alpha|}. \tag{1.2}$$

Note that $a \times b \in S^{m+m'}(\mathbb{R}^n)$ if $a \in S^m(\mathbb{R}^n)$, $b \in S^{m'}(\mathbb{R}^n)$.

Denote by $\mathrm{Op}_{\mathbb{R}^n}$ the following quantization procedure: for $f \in C^\infty(\mathbb{R}^n)$ and $a \in S^m(\mathbb{R}^n)$, let

$$\mathrm{Op}_{\mathbb{R}^n}(a)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_y} e^{i\xi\cdot(x-y)}a(x,\xi)f(y)\mathrm{d}y\mathrm{d}\xi. \tag{1.3}$$

Such an operator is called a *pseudodifferential operator*. In general, this integral does not converge absolutely; it is to be understood as an *oscillatory integral*, that is it converges after sufficiently many integrations by parts. If $a(x,\xi) = \sum_{|\alpha|\leq m} c_\alpha \xi^\alpha$, where $c_\alpha \in \mathbb{C}$, then

$$\mathrm{Op}_{\mathbb{R}^n}(a) = \sum_{|\alpha| \ \leq m} c_\alpha D_x^\alpha,$$

where D_x^α denotes the operator $i^{|\alpha|}\partial_x^\alpha = i^{|\alpha|}\partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n}$.

1.3. **Quantization on closed manifolds.** Let M be a smooth closed manifold. The space $S^m(M)$ of symbols of order $m \in \mathbb{R}$ is defined as the set of smooth functions $a \in C^\infty(T^*M)$ such that for any local chart $\kappa : U \to V \subset \mathbb{R}^n$, where $U \subset M$ is an open subset, one has $(\widetilde{\kappa}^{-1})^* a \in S^m(\mathbb{R}^n)$. Here, $\widetilde{\kappa}$ denotes the lift to the cotangent bundle, see (1.1). This symbol class is well-defined on M (independently of the choice of diffeomorphism κ) because $S^m(\mathbb{R}^n)$ is invariant by the action of diffeomorphisms.

Similarly to (1.3), it is possible to define a quantization procedure $\mathrm{Op} : S^m(M) \to \Psi^m(M)$ on M by considering a cover of M by small open charts, and using (1.3) in each coordinate patch. The set of *pseudodifferential operators* of order $m \in \mathbb{R}$ is then defined as

$$\Psi^k(\mathcal{M}) := \left\{ \mathrm{Op}(a) + R \mid a \in S^m(M), R \in \Psi^{-\infty}(M) \right\}.$$

Here, $\Psi^{-\infty}(M)$ denotes the set of operators mapping boundedly distributions to smooth functions, that is $R : \mathcal{D}'(M) \to C^\infty(M)$. Equivalently, $R : H^{-N}(M) \to H^N(M)$ is bounded as a map between these Sobolev spaces for any $N \geq 0$. It can be checked that $\Psi^m(\mathcal{M})$ is intrinsically defined on M (it only depends on the smooth structure of M), and independent of the choice of quantization Op.

2. Properties of pseudodifferential operators

We now summarize the main properties of pseudodifferential operators.

2.1. **Principal symbol.** There exists a well-defined *principal symbol map*

$$\sigma^{(m)} : \Psi^m(\mathcal{M}) \to S^m(T^*\mathcal{M})/S^{m-1}(T^*\mathcal{M})$$

with the property that for any $A \in \Psi^m(M)$, for any representative a in the equivalence class $[\sigma_A^{(m)}]$, one has $A - \mathrm{Op}(a) \in \Psi^{m-1}(M)$. In particular, the following holds: if $A \in \Psi^m(M)$

$$\sigma_A^{(m)} = 0 \Longleftrightarrow A \in \Psi^{m-1}(M).$$

We often omit the index m in the notation. Note that some (pseudo)differential operators admit a preferred choice of representative in the class σ_A which is m-homogeneous as a function of ξ. For instance, if $A = \sum_{|\alpha| \ \leq m} c_\alpha(x) D_x^\alpha$ is a differential operator in $\mathbb{R}^n$, a natural choice of representative for σ_A is $a(x,\xi) := \sum_{|\alpha| \ = m} c_\alpha(x)\xi^\alpha$. Very often, we forget that σ_A is only an equivalence class, and identify the principal symbol with an actual function on T^*M.

Example 2.1. If g is a metric on M, and Δ_g denotes the corresponding Laplace-Beltrami operator, then $\sigma_\Delta(x,\xi) = |\xi|_g^2$. If X is a vector field on M (which acts by derivation on functions), then $\sigma_X(x,\xi) = i\langle \xi, X(x)\rangle$. Both are homogeneous with respect to ξ.

2.2. **Composition.** Let $A \in \Psi^m(M)$, $B \in \Psi^{m'}(M)$. Then $A \circ B \in \Psi^{m+m'}(M)$ and

$$\sigma_{A \circ B} = \sigma_A \times \sigma_B.$$

In particular, if $a \in S^m(M)$ and $b \in S^{m'}(M)$, $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(ab) + R$, where $R \in \Psi^{m+m'-1}(M)$.

2.3. **Adjoint.** Let μ be a smooth nowhere vanishing measure on M. If $A : C^\infty(M) \to C^\infty(M)$ is a bounded linear operator, its *formal adjoint* is the unique operator $A^* : C^\infty(M) \to C^\infty(M)$ such that for all $f, f' \in C^\infty(M)$:

$$\langle Af, f' \rangle_{L^2(M,\mu)} = \langle f, A^* f' \rangle_{L^2(M,\mu)}.$$

If $A \in \Psi^m(M)$, then $A^* \in \Psi^m(M)$ and $\sigma_{A^*} = \overline{\sigma}_A$.

Example 2.2. Let $A := X$. Then $A^* = -X - \operatorname{div}_\mu(X)$ since for all $f, f' \in C^\infty(M)$:

$$\begin{aligned} \langle Xf, f' \rangle_{L^2(M,\mu)} &= \int_M Xf(x) f(x) \mathrm{d}\mu(x) \\ &= -\int_M f(x)(Xf'(x) + \operatorname{div}_\mu(X) f'(x)) \mathrm{d}\mu(x) \\ &= \langle f, (-X - \operatorname{div}_\mu(X)) f' \rangle_{L^2(M,\mu)}. \end{aligned}$$

(Verify this as an exercise.) Observe that $\sigma_A(x,\xi) = i(\xi, X(x))$ and $\sigma_{A^*}(x,\xi) = \sigma_{-X}(x,\xi) = -i(\xi, X(x)) = \overline{\sigma}_A(x,\xi)$.

2.4. **Action of diffeomorphisms.** Let $\kappa : M \to M$ be a smooth diffeomorphism, $\widetilde{\kappa} : T^*M \to T^*M$ denotes its symplectic lift as defined in (1.1). Let $A \in \Psi^m(M)$. Then

$$B := \kappa^* \circ A \circ (\kappa^{-1})^* \in \Psi^m(M),$$

and $\sigma_B = \widetilde{\kappa}^* \sigma_A$, that is $\sigma_B(x,\xi) = \sigma_A(\kappa(x), d\kappa(x)^{-\top}\xi)$.

Example 2.3. Let $a \in C^\infty(M)$, and define $Af(x) := a(x)f(x)$. Then $A \in \Psi^0(M)$ with principal symbol $\sigma_A(x,\xi) = a(x)$. (Here, a is identified with a function on T^*M by pullback.) In addition, $Bf = (\kappa^* a) f$ so $\sigma_B = \kappa^* a$.

2.5. **Ellipticity.** Let g be an arbitrary metric on TM; it induces a natural metric on T^*M that is still denoted by g. We recall that the Japanese bracket of a covector ξ is defined as $\langle \xi \rangle := (1 + |\xi|_g^2)^{1/2}$. An operator $A \in \Psi^m(M)$ is *elliptic* if there exist constants $C, R > 0$ such that the following holds:

$$|\sigma_A(x,\xi)| \geq C \langle \xi \rangle^m, \qquad \forall \xi \in T_x^*M, |\xi| \geq R. \tag{2.1}$$

Observe that (2.1) is independent of the choice of metric g and representative for the principal symbol $\sigma_A \in S^m/S^{m-1}$. Typically, $A = \Delta_g \in \Psi^2(M)$ is elliptic since its principal symbol is $\sigma_{\Delta_g}(x,\xi) = |\xi|_g^2$. On the other hand, $A = X \in \Psi^1(M)$ (vector field) is *not* elliptic.

The important property of elliptic operators is that they are invertible modulo smoothing remainders: if $A \in \Psi^m(M)$ is elliptic, then there exists an elliptic operator $B \in \Psi^{-m}(M)$, and $R_L, R_R \in \Psi^{-\infty}(M)$ such that

$$BA = \mathbb{1} + R_L, \qquad AB = \mathbb{1} + R_R. \tag{2.2}$$

Observe that $\sigma_B = 1/\sigma_A \in S^{-m}(M)$. The operator B is called a *parametrix* for A, or a *quasi-inverse*. In particular, if A is *invertible* (that is $A : C^\infty(M) \to C^\infty(M)$ is an isomorphism), then the true inverse A^{-1} is a pseudodifferential operator $A^{-1} \in \Psi^{-m}(M)$ and for any parametrix B of A, $B - A^{-1} \in \Psi^{-\infty}(M)$.

In addition, (2.2) implies that distributional solutions to $Au = 0, u \in \mathcal{D}'(M)$ are smooth, that is $u \in C^\infty(M)$. Indeed, if $Au = 0$, then $BAu = 0 = u + R_L u$, that is $u = -R_L u \in C^\infty(M)$ since R_L is smoothing.

3. Boundedness on functional spaces

3.1. **The Calderon-Vaillancourt theorem.** If $A \in \Psi^0(M)$, then $A : L^2(M,\mu) \to L^2(M,\mu)$ is bounded. In addition, letting

$$\mathbf{M} := \limsup_{x \in M, |\xi| \to +\infty} |\sigma_A(x,\xi)|,$$

one has that for all $\varepsilon > 0$, there exists $A_\varepsilon \in \Psi^0(M)$ and $K_\varepsilon \in \Psi^{-\infty}(M)$ such that $\|A_\varepsilon\|_{L^2 \to L^2} \leq \mathbf{M} + \varepsilon$ and $A = A_\varepsilon + K_\varepsilon$. This result is called the Calderon-Vaillancourt theorem.

In particular, this implies that operators of negative order are compact on $L^2(M)$. Indeed, let $A \in \Psi^{-m}(M)$ with $m > 0$. Then $\mathbf{M} = 0$. Hence $K_\varepsilon \to_{\varepsilon \to 0} A$ in $\mathcal{L}(L^2(M,\mu))$. However, smoothing operators are compact on $L^2(M)$ (verify this as an exercise). Since compact operators form a closed subset for the operator norm topology, this shows that A is also compact on $L^2(M,\mu)$.

3.2. **Sobolev spaces.** For $s \in \mathbb{R}$, one can define the Sobolev space $H^s(M)$ as follows. There exists an invertible elliptic operator $\Lambda_s \in \Psi^s(M)$ with principal symbol $\sigma_{\Lambda_s}(x,\xi) = \langle \xi \rangle^s$ such that $H^s(M)$ is the completion of $C^\infty(M)$ with respect to the norm

$$\|f\|_{H^s(M)} := \|\Lambda_s f\|_{L^2(M)}.$$

For $s = 0$, $H^0(M) = L^2(M)$. The Sobolev spaces are intrinsic, that is they only depend on the differentiable structure of the closed manifold M. Note that, by construction, $\Lambda_s : H^s(M) \to L^2(M)$ is an isometry. This implies that for all $s \in \mathbb{R}$, an operator $A \in \Psi^m(M)$ is bounded as a map

$$A : H^{s+m}(M) \to H^s(M).$$

Indeed, this is equivalent to showing that $B := \Lambda_s \circ A \circ \Lambda_{s+m}^{-1}$ is bounded on $L^2(M)$. However, observe that $B \in \Psi^0(M)$ and the latter is bounded on $L^2(M)$ by the Calderon-Vaillancourt theorem.

Finally, for $s > s'$, the space $H^s(M)$ embeds compactly into $H^{s'}(M)$ (this is known as the Kato-Rellich theorem). Indeed, this is equivalent to showing that $\Lambda_{s'}\Lambda_s^{-1}$ is compact on $L^2(M)$, which is an immediate consequence of the fact that $\Lambda_{s'}\Lambda_s^{-1} \in \Psi^{s'-s}(M)$ has strictly negative order as $s' - s < 0$.

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