EXERCISE CLASSES
INTRODUCTION TO GEOMETRIC ANALYSIS

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In all the exercises, unless stated explicitly, (M, g) is a closed (i.e. compact without boundar) Riemannian manifold of dimension n and $\Delta_g \geq 0$ is the nonnegative Laplacian acting on functions.

1. Laplace eigenvalues and eigenfunctions

1.1. Examples.

Exercise 1.1 (Laplace spectrum on the sphere)**.** The purpose of this exercise is to compute the Laplace eigenvalues on the sphere $(\mathbb{S}^n, g_{\mathrm{can}})$ equipped with the round metric. Let $\mathbf{P}_m(\mathbb{R}^{n+1})$ be the space of homogeneous polynomials of degree $m \geq 0$ on $\mathbb{R}^{n+1}$, and

$$\mathbf{H}_m(\mathbb{R}^{n+1}) := \{u \in \mathbf{P}_m(\mathbb{R}^{n+1}), \mid \Delta u = 0\},$$

the space of harmonic homogeneous polynomials.

We introduce the operator

$$\partial : \mathbf{P}_m(\mathbb{R}^{n+1}) \to \mathrm{Diff}^m, \qquad \partial(\sum_{|\alpha|=m} c_\alpha x^\alpha) = \sum_{|\alpha|=m} c_\alpha \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}.$$

and define the scalar product

$$\langle P, Q\rangle := \partial(P)\overline{Q}, \qquad \forall P, Q \in \mathbf{P}_m(\mathbb{R}^{n+1}). \tag{1.1}$$

(1) Compute $\mathbf{H}_0(\mathbb{R}^{n+1}), \mathbf{H}_1(\mathbb{R}^{n+1})$.
(2) Verify that (1.1) defines indeed a Hermitian scalar product (or Euclidean scalar product in restriction to real-valued polynomials).
(3) Compute the dimension of $\mathbf{P}_m(\mathbb{R}^{n+1})$.
(4) Let $R \in \mathbf{P}_k(\mathbb{R}^{n+1})$ and define

$$m_R : \mathbf{P}_m(\mathbb{R}^{n+1}) \to \mathbf{P}_{m+k}(\mathbb{R}^{n+1}), \qquad m_R(P) := PR.$$

Show that $m_R^* : \mathbf{P}_{m+k}(\mathbb{R}^{n+1}) \to \mathbf{P}_m(\mathbb{R}^{n+1})$ is given by $m_R^*(P) = \partial(\overline{R})P$. What do you get for $R := |x|^2$?
(5) Show that for $m \geq 2$,

$$\mathbf{P}_m(\mathbb{R}^{n+1}) = \mathbf{H}_m(\mathbb{R}^{n+1}) \oplus |x|^2\mathbf{P}_{m-2}(\mathbb{R}^{n+1}).$$

Deduce that $\mathbf{P}_m(\mathbb{R}^{n+1}) = \oplus_{k\geq 0}|x|^{2k}\mathbf{H}_{m-2k}(\mathbb{R}^{n+1})$ (with the convention that $\mathbf{H}_{m-2k} = \{0\}$ for $m - 2k < 0$).
(6) Let $f \in C^\infty(\mathbb{R}^{n+1})$. Show that

$$\Delta_{\mathbb{S}^n}(f|_{\mathbb{S}^n}) = (\Delta f + n\partial_r f + \partial_r^2 f)|_{\mathbb{S}^n}.$$

(7) Show that if $f \in \mathbf{H}_m(\mathbb{R}^{n+1})$, then $P := f|_{\mathbb{S}^{n+1}}$ satisfies $\Delta_{\mathbb{S}^n} P = m(m+n-1)P$.
(8) Deduce the Laplace spectrum of $(\mathbb{S}^n, g_{\mathrm{can}})$.

Exercise 1.2 (Zonal harmonics and irreducible representations)**.** The group $\mathrm{SO}(n+1)$ acts on $\mathbf{P}_m(\mathbb{R}^{n+1})$ by pullback, that is if $f \in \mathbf{P}_m(\mathbb{R}^{n+1})$, $A \in \mathrm{SO}(n+1)$, then

$$A^* f(x) := f(Ax).$$

The purpose of this exercise is to prove that this action preserves $\mathbf{H}_m(\mathbb{R}^{n+1})$ and that this representation of $\mathrm{SO}(n+1)$ is *irreducible*, that is it does not preserve any vector subspace of $\mathbf{H}_m(\mathbb{R}^{n+1})$ except $\{0\}$ and $\mathbf{H}_m(\mathbb{R}^{n+1})$ itself.

(1) Show that $\mathrm{SO}(n+1)$ preserves $\mathbf{H}_m(\mathbb{R}^{n+1})$.
(2) Show that $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ (where $\mathrm{SO}(n)$ is seen as a subgroup of $\mathrm{SO}(n+1)$ by the diagonal embedding).

The group $\mathrm{SO}(n+1)$ acts on $C^\infty(S^n)$ by pullback, that is $A^* f(x) := f(Ax)$ for all $f \in C^\infty(S^n), x \in S^n$. A function $f \in C^\infty(S^n)$ is said to be *zonal* if it is invariant by $\mathrm{SO}(n)$. If $V \subset C^\infty(S^n)$ is a subspace, we shall denote by $Z(V)$ its subspace of zonal functions. We write $x = (x_0, ..., x_n) \in \mathbb{R}^{n+1}$ for the coordinates on $\mathbb{R}^{n+1}$.

(1) Show that $Z(V)$ is a vector space.
(2) Show that the restriction to S^n of the polynomials

$$x_0^m, x_0^{m-2}|x|^2, ..., x_0^{m-2[m/2]}|x|^{2[m/2]} \in \mathbf{P}_m(\mathbb{R}^{n+1})$$

are zonal.
(3) Show that $f \in C^\infty(S^n)$ is zonal if and only if it only depends on the x_0-variable.
(4) Deduce that $Z(\mathbf{P}_m(\mathbb{R}^{n+1}))$ is given precisely by the span of the above list.
(5) Show that if $V \subset C^\infty(S^n)$ is a non-zero finite-dimensional subspace invariant by $\mathrm{SO}(n+1)$, then $\dim Z(V) \geq 1$. *Hint: Fix $x_0 \in S^n$ and consider $\phi : V \to \mathbb{C}, f \mapsto f(x_0)$.*
(6) Show that if $\dim Z(V) = 1$, then V is irreducible.
(7) Deduce that $\mathbf{H}_m(\mathbb{R}^{n+1})$ is an irreducible representation of $\mathrm{SO}(n+1)$.

Exercise 1.3 (Laplace spectrum on the torus)**.** Let Λ be a lattice in $\mathbb{R}^n$ and set $\mathbb{T}^n := \mathbb{R}^n/\Lambda$. Define Λ^* the dual lattice, as the set of vectors $\lambda^* \in \mathbb{R}^n$ such that $\langle \lambda^*, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$.

(1) Show that $f_{\lambda^*}(x) := e^{2i\pi\langle \lambda^*, x\rangle}$ is a well-defined function on $\mathbb{T}^n$ and a Laplace eigenfunction for the eigenvalue $4\pi^2|\lambda^*|^2$.
(2) Show that it forms a basis of $L^2(\mathbb{T}^n)$.

On a closed Riemannian manifold (M, g), Weyl's law describes the asymptotic growth of Laplace eigenvalues. It was proved by Weyl [Wey11] in 1911. It shows that

$$\#\{\mu \leq T, \mu \text{ eigenvalue of } \Delta\} \sim (2\pi)^{-n} T^{n/2} \omega_n \operatorname{vol}_g(M),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ (and the eigenvalues are counted with multiplicities).

(3) Prove Weyl's law on the torus.

Exercise 1.4 (Laplace spectrum on $\mathrm{U}(n)$)**.** The purpose of this exercise is to characterize Laplace eigenfunctions on the group $\mathrm{U}(n)$ with constant modulus.

(1) Show that the map $\psi : \mathrm{SU}(n) \times \mathrm{U}(1) \to \mathrm{U}(n)$ given by $(w, z) \mapsto z^{-1}w$ is a $\mathbb{Z}_n$ bundle map. What is the $\mathbb{Z}_n$ action here?

Let $g_{\mathrm{SU}(n)}$ be a bi-invariant metric on $\mathrm{SU}(n)$, and let $g_{\mathrm{U}(1)} = \mathrm{d}\theta^2$ be the standard metric on $\mathrm{U}(1) = \mathbb{R}/2\pi\mathbb{Z}$.

(2) Show that the product metric $g_{\mathrm{SU}(n)} \oplus g_{\mathrm{U}(1)}$ on the product $\mathrm{SU}(n) \times \mathrm{U}(1)$ descends to a bi-invariant metric $g_{\mathrm{U}(n)}$ on $\mathrm{U}(n)$, and that ψ is a local isometry.

(3) Let $f \in C^\infty(\mathrm{U}(n))$ such that $\Delta_{\mathrm{U}(n)} f = \mu f$ for some $\mu > 0$. Show that

$$\Delta_{\mathrm{SU}(n)\times\mathrm{U}(1)} \psi^* f = \mu \psi^* f.$$

(4) We now further assume that $|f| = 1$ on $\mathrm{U}(n)$. Show that $\psi^* f(w, \theta) = a(w) e^{ik\theta}$, where $a \in C^\infty(\mathrm{SU}(n))$ is an eigenfunction of $\Delta_{\mathrm{SU}(n)}$ associated to the eigenvalue $\lambda \geq 0$, $k \in \mathbb{Z}$ and $k^2 + \lambda = \mu$.

(5) Show that $|\nabla a|$ is constant. *Hint: Compute* $\Delta_{\mathrm{SU}(n)} |a|^2$.

(6) Show that $\nabla a = 0$. Deduce that a is constant and $\lambda = 0$. *Hint: Use that* $\mathrm{SU}(n)$ *is simply connected.*

(7) Show that $\det : \mathrm{U}(n) \to \mathrm{U}(1)$ is a submersion with fibers diffeomorphic to $\mathrm{SU}(n)$.

(8) Deduce from the previous questions that the function f is of the form $f = \det^* u$ for some function $u \in C^\infty(\mathrm{U}(1))$. Show that u is an eigenfunction of the Laplacian on the circle.

1.2. **General results.** Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 1$. We recall that any smooth function $f \in C^\infty(M)$ can be uniquely decomposed as

$$f = \sum_{j=0}^{+\infty} f_j \varphi_j, \qquad f_j = \langle f, \varphi_j \rangle_{L^2(M)} \in \mathbb{C}, \tag{1.2}$$

where $\varphi_j \in C^\infty(M)$ is a Laplace eigenfunction associated to λ_j such that $\|\varphi_j\|_{L^2(M)} = 1$. Furthermore, the eigenvalues $\lambda_0 = 0 < \lambda_1 \leq ...$ are counted with multiplicities and sorted in increasing order.

Exercise 1.5. Let $\sigma \in \mathbb{R}$. Express the spectrum of $\Delta_{e^{2\sigma}g}$ in terms of that of Δ_g.

Exercise 1.6 (Products of Riemannian manifolds)**.** Let (M, g) and (N, h) be two closed Riemannian manifolds. Compute the eigenfunctions and eigenvalues of the Laplacian on $(M \times N, g \oplus h)$.

Exercise 1.7 (Rayleigh Theorem. Max-min Theorem)**.** (1) Show that for $f \in H^1(M), f \neq 0$ such that $\langle f, \varphi_j \rangle_{L^2(M)} = 0$ for all $0 \leq j \leq k-1$, one has

$$\lambda_k \leq \frac{\int_M |\nabla f|^2 \,\mathrm{vol}_g}{\int_M |f|^2 \,\mathrm{vol}_g},$$

with equality if and only if f is an eigenfunction of λ_k.

(2) Let $v_0, ..., v_{k-1} \in L^2(M)$ and set $H := \{f \in H^1(M) \mid \langle f, v_i\rangle_{L^2(M)} = 0\}$. Show that
$$\inf_{f\in H, f\neq 0} \frac{\int_M |\nabla f|^2 \operatorname{vol}_g}{\int_M |f|^2 \ \operatorname{vol}_g} \leq \lambda_k.$$

Exercise 1.8 (Manifolds with boundary)**.** Let (M, g) be a smooth connected manifold with boundary. Let ν be the outward pointing unit vector field on the boundary ∂M. Recall that if X is a vector field on M, its divergence $\operatorname{div}(X)$ is defined such that $\mathcal{L}_X \operatorname{vol}_g = \operatorname{div}(X) \operatorname{vol}_g$.

(1) Show that for all vector fields $X \in C^\infty(M, TM)$,
$$\int_M \operatorname{div}(X) \operatorname{vol}_g = \int_{\partial M} X \cdot \nu \operatorname{vol}_{g|\partial M}.$$
(2) More generally, show that for all $u \in C^\infty(M), X \in C^\infty(M, TM)$,
$$\int_M u \ \operatorname{div}(X) \operatorname{vol}_g = -\int_M \nabla u \cdot X \operatorname{vol}_g + \int_{\partial M} u \ X \cdot \nu \operatorname{vol}_{g|\partial M}$$
(3) Deduce that for all $u, v \in C^\infty(M)$,
$$\int_M u \ \Delta v \operatorname{vol}_g = \int_M \nabla u \cdot \nabla v \operatorname{vol}_g - \int_{\partial M} u \ \nabla v \cdot \nu \operatorname{vol}_{g|\partial M}.$$
(4) The space $H^1_0(M)$ is defined as the completion of $C^\infty_{\text{comp}}(M)$ with respect to the H^1-norm. Show that for all $u \in H^1_0(M), X \in C^\infty(M, TM)$,
$$\int_M u \ \operatorname{div}(X) \operatorname{vol}_g = \int_M \nabla u \cdot X \operatorname{vol}_g.$$

The *double* of M is the manifold M^{double} obtained by gluing two copies of M along a cylinder $\partial M \times [-1, 1]$. More precisely,
$$M^{\text{double}} := M \times \{-1\} \sqcup \partial M \times [-1, 1] \sqcup M \times \{1\}/\sim,$$
where $(x, \pm 1)_{M\times\{\pm 1\}} \sim (x, \pm 1)_{\partial M\times[-1,1]}$ for all $x \in \partial M$. We will admit that M^{double} has the structure of a nice smooth closed manifold. Observe that there is a natural embedding $M \hookrightarrow M^{\text{double}}$. We extend the metric g to a smooth metric g^{double} such that $g^{\text{double}} = g$ on $M \times \{\pm 1\}$ and g^{double} is arbitrary on $\partial M \times [-1, 1]$.

(5) Draw a picture of M^{double}.
(6) In this doubling process, it is key to guarantee that $g^{\text{double}} = g$ on $M \times \{\pm 1\}$. Why do we need to glue a cylinder then?
(7) Given $f \in H^1_0(M)$, we define Ef as the function on M^{double} such that $Ef = f$ on $M \times \{-1\}$ and $Ef = 0$ on $\partial M \times [-1, 1]$ and $Ef = -f$ on $M \times \{1\}$. Compute ∇Ef and show that $E : H^1_0(M) \to H^1(M)$ is continuous.
(8) Using the Poincaré inequality on M^{double}, deduce that there exists $C > 0$ such that for all $f \in H^1_0(M)$, $\|f\|_{L^2(M)} \leq C\|\nabla f\|_{H^1(M)}$.
(9) Show that the inclusion $H^1_0(M) \hookrightarrow L^2(M)$ is compact.
(10) Given $\rho \in C^\infty(M)$, solve $\Delta f = \rho$ with $f \in H^1_0(M)$ in the weak sense (that is, when applied to $\varphi \in C^\infty_{\text{comp}}(M^\circ)$).
(11) Show that the solution is unique.

(12) Show that $\rho \mapsto \Delta^{-1}\rho$ is defined on $L^2(M)$ and compact.

Exercise 1.9. For $s \in \mathbb{R}$, we introduce the Sobolev space $H^s(M)$ as the completion of $C^\infty(M)$ for the norm

$$\|f\|^2_{H^s(M)} := \sum_{j=1} \langle \lambda_j \rangle^s |f_j|^2.$$

(1) Show that for all $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that:

$$\|\varphi_j\|_{L^\infty} \leq C_\varepsilon \lambda_j^{n/4+\varepsilon}, \qquad \forall j \geq 0.$$

(2) Let $f \in C^\infty(M)$. Show that for all $N > 0$, there exists $C_N > 0$ such that:

$$|f_j| \ \leq C_N \lambda_j^{-N}, \qquad \forall j > 0.$$

(3) Let $u \in \mathcal{D}'(M)$ be a distribution. Show that (1.2) still holds for a distribution. Bound the coefficients $(f_j)_{j\geq 0}$.
(4) Show that the heat equation

$$(\partial_t + \Delta_g)u(t) = 0, \qquad u(t) = u_0 \in \mathcal{D}'(M)$$

admits a unique solution $u \in C^\infty((0,\infty), C^\infty(M))$ and that $R(t) : u_0 \mapsto u(t)$ is a smoothing operator for $t > 0$.

2. Elliptic operators

2.1. Examples. General results.

Exercise 2.1 (Total symbol vs. principal symbol)**.** Let $\Delta := -(\partial_x^2 + \partial_y^2)$ be the Laplacian in the standard coordinates (x, y) of $\mathbb{R}^2$. Let

$$[0,\infty) \times (\mathbb{R}/2\pi\mathbb{Z}) \ni (r,\theta) \mapsto re^{i\theta} \in \mathbb{R}^2,$$

be the polar coordinates.

(1) Show that, in polar coordinates, the expression of Δ is given by

$$\Delta = -(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\ \partial_\theta^2).$$

(2) What is the full symbol of Δ in the standard coordinates?
(3) What is the full symbol of Δ in the polar coordinates?
(4) Is the full symbol invariant by this change of coordinates?
(5) Show that the principal symbol of Δ is invariant by this change of coordinates.

Exercise 2.2 (Laplacian and isometries)**.** Let (M, g) be a complete Riemannian manifold, and $\phi : M \to M$ a smooth diffeomorphism. Show that Δ_g commutes with ϕ if and only if ϕ is an isometry.

Exercise 2.3 (Examples of (non-)elliptic operators)**.** Compute the principal symbol of the following operators and explain if they are elliptic or not.

(1) $P_1 := X : C^\infty(M) \to C^\infty(M)$, where $X \in C^\infty(M, TM)$ is a vector field on M (seen as a differential operator of order 1);

(2) $P_2 := a : C^\infty(M,E) \to C^\infty(M,E)$, where a is a fiberwise endomorphism of E (i.e. $a(x) \in \mathrm{End}(E_x)$ for all $x \in M$);
(3) $P_3 := d, d^*$ and $d + d^*$ acting on forms;
(4) $P_4 := \nabla : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$, where $E \to M$ is a complex vector bundle and ∇ a connection on E;
(5) $P_5 := P \circ Q : C^\infty(M,E) \to C^\infty(M,G)$, where $P : C^\infty(M,E) \to C^\infty(M,F)$ and $Q : C^\infty(M,F) \to C^\infty(M,G)$.

Exercise 2.4 (Basic properties of elliptic operators)**.** Let $P : C^\infty(M,E) \to C^\infty(M,F)$ be an elliptic differential operator of order m.

(1) Show that for all $s \in \mathbb{R}$, there exists $C > 0$ such that:
$$\|f\|_{H^{s+m}} \leq C\|Pf\|_{H^s}, \qquad \forall f \in H^{s+m}(M,E) \cap (\ker P)^\perp,$$
where $(\ker P)^\perp$ is defined with respect to the L^2 inner product on sections of E.

We now aim to show that for all $f \in C^\infty(M,E)$, there exists a unique pair (u,v) such that $u \in C^\infty(M,E)$ and $u \perp \ker P$, $v \in C^\infty(M,F)$ and $v \in \ker P^*$, and
$$f = Pu + v. \tag{2.1}$$

(3) Show that $P^*P : C^\infty(M,E) \to C^\infty(M,E)$ is elliptic.
(4) Show that for all $s \in \mathbb{R}$,
$$P^*P : H^{s+m}(M,E) \cap (\ker P)^\perp \to H^s(M,E) \cap (\ker P)^\perp$$
is an isomorphism. Prove that it also holds for C^∞ sections.
(5) Deduce (2.1).

2.2. **Isometries. Killing vector fields.**

Exercise 2.5 (Killing vector fields)**.** Let (M,g) be a closed Riemannian manifold. We introduce the operator
$$D_g : C^\infty(M, T^*M) \to C^\infty(M, S^2T^*M), \qquad D_g\alpha := \mathcal{S}(\nabla_g \alpha),$$
where ∇_g is the Levi-Civita connection and $\mathcal{S} : T^*M^{\otimes 2} \to S^2T^*M$ is the orthogonal projector onto symmetric 2-tensors given by
$$\mathcal{S}(\alpha_1 \otimes \alpha_2) := (\alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1)/2.$$

(1) Let $X \in C^\infty(M,TM)$ be a vector field. Show that $\mathcal{L}_X g = 2D_g X^\sharp$, where $\sharp : TM \to T^*M$ is the musical isomorphism.
(2) Show that D_g is elliptic.
(3) Let $\phi : (M,g) \to (M,g)$ be an isometry. Show that it is determined by the pair $(\phi(p), d\phi_p)$ where $p \in M$ is arbitrary (i.e. there exists at most one isometry ϕ such that $\phi(p)$ and $d\phi_p$ are given).
(4) Let $\mathcal{A} := \mathbf{1} - \mathcal{S}$. For $p \in M$, show that the map
$$\ker D_g \to T_pM \oplus \Lambda^2 T_pM, \qquad X \mapsto (X(p), \mathcal{A}\nabla X(p))$$
is injective.

(5) Deduce that $\dim \ker D_g \leq n(n+1)/2$.
(6) Give an example that saturates this bound.

By working more, one could show that the group of isometries is a compact Lie group G of dimension $\leq n(n+1)/2$. This was proved by Myers and Steenrod [MS39] in 1939. A geometric structure (such as a metric) with finite-dimensional automorphism group is called a *rigid* structure; this was introduced and studied by Gromov [Gro88] in the 1980s. The theory of rigid structure is very rich and has deep interactions with dynamical systems, see the classification of contact Anosov flows with smooth subbundles by Benoist, Foulon and Labourie for instance [BFL90].

Exercise 2.6 (Isometries and geodesic flow dynamics)**.** Let (M, g) be a closed Riemannian manifold, $SM := \{v \in TM, \mid |v|_g = 1\}$ the unit tangent bundle. For $v \in SM$, denote by $t \mapsto \gamma_v(t)$ the unit-speed geodesic generated by v. The geodesic flow is defined on SM by

$$\varphi_t(v) := (\gamma_v(t), \dot{\gamma}_v(t)). \tag{2.2}$$

Let $X \in C^\infty(SM, T(SM))$ be the generator of $(\varphi_t)_{t\in\mathbb{R}}$.

(1) Verify that (2.2) defines indeed a flow.
(2) Compute X for $(M, g) = (\mathbb{R}^n, g_{\mathrm{euc}})$.
(3) For $m \geq 0$, define $\pi_m^* : C^\infty(M, S^m T^*M) \to C^\infty(SM)$ by $\pi_m^* f(v) := f_{\pi(v)}(v, ..., v)$. Show that $\pi_{m+1}^* D_g = X\pi_m^*$. *Hint: use normal coordinates.*
(4) Deduce that, if the geodesic flow is transitive on SM, then the space of Killing vector fields is trivial.

Exercise 2.7. Let (M, g) be a closed Riemannian manifold. For $f \in C^\infty(M, S^2T^*M)$, show that there exists a unique pair (p, h) such that $h \in C^\infty(M, S^2T^*M)$, $D_g^* h = 0$ and $p \in C^\infty(M, T^*M)$ and $p \perp \ker D_g$ such that

$$f = D_g p + h.$$

This statement is an infinitesimal version of Ebin's slice theorem which will be proved in Exercise 4.1.

2.3. Operator theory.

Exercise 2.8 (Compact operators)**.** Let E be a Banach space. Denote by $\mathcal{L}(E)$ the space of bounded linear operators on E and $\mathcal{K}(E)$ the set of compact operators, that is $A \in \mathcal{K}(E)$ iff for all bounded sequences $(x_n)_{n\,\geq 0}$ in E, the sequence $(A(x_n))_{n\geq 0}$ admits a converging subsequence.

(1) Show that $\mathcal{K}(E)$ is closed (for the operator norm on $\mathcal{L}(E)$).
(2) Show that $\mathcal{K}(E)$ is a two-sided ideal of $\mathcal{L}(E)$.
(3) Given $K \in \mathcal{K}(E)$, $\lambda \in \mathbb{C} \setminus \{0\}$, show that $\ker(K + \lambda)$ finite-dimensional and $\mathrm{ran}(K + \lambda)$ is closed with finite codimension.
(4) For $K \in \mathcal{K}(E)$, show that the spectrum $\sigma(K) \subset \mathbb{C}$ of K is discrete on $\mathbb{C}\setminus\{0\}$. Prove that each non-zero eigenvalue is associated to a finite-dimensional eigenspace.
(5) Prove that $K \in \mathcal{K}(E)$ is compact iff $K^* \in \mathcal{K}(E^*)$ is compact (Schauder's theorem).

(6) Assume that E is a Hilbert space now. Show that all compact operators are limits of finite rank operators.

Exercise 2.9 (Fredholm operators. Index of elliptic operators)**.** The purpose of this exercise is to study Fredholm operators. Let E be a Banach space. Let $\mathcal{F}(E)$ be the set of Fredholm operators i.e. bounded linear operators with finite dimensional kernel, and closed image of finite codimension (finite cokernel), and define the index as

$$\mathrm{ind} : \mathcal{F}(E) \to \mathbb{Z}, \qquad \mathrm{ind}(A) := \dim\ker A - \dim \mathrm{coker} A.$$

(1) Let $A \in \mathcal{L}(E)$. Show that $\dim \mathrm{coker}\, A = \dim \ker A^*$.
(2) Let $K \in \mathcal{K}(E)$. Show that $\mathbb{1} + K \in \mathcal{F}(E)$ and $\mathrm{ind}(\mathbb{1} + K) = 0$.
(3) Show that for $A \in \mathcal{F}(E)$, there exists $B \in \mathcal{L}(E)$ such that

$$BA = \mathbb{1} - \Pi_{\ker A}, \qquad AB = \Pi_{\mathrm{ran}\, A},$$

where $\Pi_{\ker A}$ is a finite rank projection onto $\ker A$, $\Pi_{\mathrm{ran}\, A}$ a projection on $\mathrm{ran}\, A$ with finite cokernel.
(4) Show that $A \in \mathcal{F}(E)$ if and only if there exists $B \in \mathcal{L}(E)$ and $K_1, K_2 \in \mathcal{K}(E)$ such that

$$AB = \mathbb{1} + K_1, \qquad BA = \mathbb{1} + K_2.$$

(5) Let $A, B \in \mathcal{F}(E)$. Show that $AB \in \mathcal{F}(E)$ and

$$\mathrm{ind}(AB) = \mathrm{ind}(A) + \mathrm{ind}(B).$$

(6) Let $A \in \mathcal{F}(E)$. Show that there exists $\varepsilon > 0$ such that for all $P \in \mathcal{L}(E)$ with $\|P\| < \varepsilon$, $A + P \in \mathcal{F}(E)$ with $\mathrm{ind}(A + P) = \mathrm{ind}(A)$.
(7) Let $A \in \mathcal{F}(E)$. Show that for all $K \in \mathcal{K}(E)$, $A + K \in \mathcal{F}(E)$ and $\mathrm{ind}(A) = \mathrm{ind}(A + K)$.

3. Estimates on λ_1

Exercise 3.1 (Coarea formula)**.** Let (M, g) be a closed manifold. For $f \in C^\infty(M)$ show that:

(1) If f is positive, then:

$$\int_M f\,\mathrm{vol}_g = \int_0^{+\infty} \mathrm{vol}(f^{-1}(t, \infty))\mathrm{d}t.$$

(2) If f is a Morse function, then:

$$\int_M |\nabla f| \ \mathrm{vol}_g = \int_0^{+\infty} \mathrm{vol}_{n-1}(f^{-1}(t))\mathrm{d}t. \tag{3.1}$$

Formula (3.1) is called the coarea formula. It is actually valid for all $f \in C^\infty(M)$ (admitted).

Exercise 3.2 (Isoperimetric inequality and sharp Sobolev inequalities)**.** Let (M, g) be a complete Riemannian manifold. We say that $C > 0$ is an isoperimetric constant in (M, g) if for all relatively compact domain $\Omega \subset M$ with smooth boundary,

$$C \leq \frac{\mathrm{vol}_{n-1}(\partial\Omega)^n}{\mathrm{vol}_n(\Omega)^{n-1}}$$

We say that the Sobolev embedding $W^{1,1}(M) \hookrightarrow L^{n/(n-1)}(M)$ is satisfied with constant $C > 0$ if for all $u \in C^\infty_{\mathrm{comp}}(M)$,

$$\|u\|_{L^{n/(n-1)}(M)} \leq C\|\nabla u\|_{L^1(M)}.$$

The purpose of this exercise is to investigate the relationship between these two constants.

(1) Let (N, g) be a smooth Riemannian manifold with boundary, and ν be the inward pointing unit vector field on ∂N. Show that

$$\phi : [0, \varepsilon) \times \partial N \to N, \qquad (t, x) \mapsto \exp_x(t\nu(x))$$

is a diffeomorphism to a local neighborhood of ∂N for $\varepsilon > 0$ small enough.

(2) Show that $g = dt^2 + h_t$, where h_t is a smooth metric on ∂N.

(3) Show that the Sobolev inequality with constant C implies the isoperimetric inequality with constant $C^{-(n-1)}$.

Conversely, we want to show that the isoperimetric inequality with constant C implies the Soblev inequality with constant $C^{-1/n}$.

(4) Show that it suffices to show the Sobolev inequality for $f \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with $f \geq 0$.

(5) Assume $f \geq 0$, and $f \in C^\infty_{\mathrm{comp}}(M)$. Show that

$$\int_M |f|^{\frac{n}{n-1}}\mathrm{d}x = \frac{n}{n-1}\int_0^{+\infty} \mathrm{vol}_n(f \geq t)t^{\frac{1}{n-1}}\mathrm{d}t,$$
$$\int_M |\nabla f|\mathrm{d}x \geq C\int_0^{+\infty} \mathrm{vol}_n(f \geq t)^{\frac{n-1}{n}}\mathrm{d}t.$$

(6) Deduce the Sobolev inequality.

It is a classical theorem that the isoperimetric inequality is saturated in $\mathbb{R}^n$ for Ω being the unit ball, see Exercise 3.3. However, proving similar isoperimetric inequalities in general Riemannian manifolds is still an open question.

Exercise 3.3 (Isoperimetric inequality in $\mathbb{R}^n$)**.** The purpose of this exercise is to prove the isoperimetric inequality in $\mathbb{R}^n$, namely for all relatively compact domain Ω with smooth boundary $\partial\Omega$,

$$\frac{\mathrm{vol}_{n-1}(\partial B_1)^n}{\mathrm{vol}_n(B_1)^{n-1}} \leq \frac{\mathrm{vol}_{n-1}(\partial\Omega)^n}{\mathrm{vol}_n(\Omega)^{n-1}}, \tag{3.2}$$

with equality if and only if $\Omega = B_1$ is the open unit ball. We will follow closely the proof by Cabre [Cab17].

(1) Let $\psi : \Omega \to B_1$ be a smooth surjective map preserving orientation. Show that

$$\mathrm{vol}(B_1) \leq \int_\Omega \det \mathrm{d}\psi(x)\ \mathrm{d}x.$$

(2) Let A be a real symmetric nonnegative matrix. Show that $\det(A)^{1/n} \leq \mathrm{Tr}(A)/n$.

Consider the following Neumann problem in Ω:

$$\Delta u = \mathrm{vol}_{n-1}(\partial\Omega)/\,\mathrm{vol}_n(\Omega), \qquad \partial_\nu u = 1, \tag{3.3}$$

where ∂_ν is the exterior normal derivative on $\partial\Omega$. Define

$$\Gamma := \{x \in \Omega,\ u(y) \geq u(x) + \nabla u(x) \cdot (y - x), \forall y \in \overline{\Omega}\}.$$

We aim to show that the map $\nabla u : \Omega \to \mathbb{R}^n$ is surjective onto the unit ball. More precisely, $B_1 \subset \nabla u(\Gamma)$. Let $v \in B_1$ and consider $x \in \overline{\Omega}$, a minimum of the function

$$\overline{\Omega} \ \ni y \mapsto u(y) - v \cdot y \in \mathbb{R}.$$

(This is the Legendre transform of u.)

(3) First, show that (3.3) admits a unique solution.
(4) Explain geometrically what Γ is for u.
(5) Show that x cannot lie on the boundary $\partial\Omega$.
(6) Deduce that $v = \nabla u(x)$. Conclude on the surjectivity of $\nabla u : \Gamma \to B_1$.
(7) Show that $\det \mathrm{d}\nabla u(x) \geq 0$ for all $x \in \Gamma$.
(8) Show that

$$\mathrm{vol}_n(B_1) \leq \int_\Gamma \det \mathrm{d}\nabla u(x)\ \mathrm{d}x.$$

(9) Show that

$$\det \mathrm{d}\nabla u(x) \leq (\Delta u(x)/n)^n, \quad \forall x \in \Gamma.$$

(10) Deduce that

$$\mathrm{vol}_n(B_1) \leq \left(\frac{\mathrm{vol}_{n-1}(\partial\Omega)}{n\,\mathrm{vol}_n(\Omega)}\right)^n \mathrm{vol}_n(\Omega).$$

(11) Finally, prove the isoperimetric inequality (3.2).
(12) Show that equality (3.2) holds if and only if Ω is a ball.

Exercise 3.4 (Cheeger's inequality)**.** The purpose of this exercise is to establish Cheeger's inequality, proved by Cheeger [Che70] in 1970. Set

$$h := \inf_\Sigma \frac{\mathrm{vol}_{n-1}(\Sigma)}{\min(\mathrm{vol}_n(M_1), \mathrm{vol}_n(M_2))},$$

where Σ runs over all codimension 1 submanifolds disconnecting M, that is $M \setminus \Sigma = M_1 \sqcup M_2$. Then:

$$\lambda_1(M, g) \geq h^2/4. \tag{3.4}$$

Let f be a real-valued eigenfunction for the eigenvalue λ_1, and further assume that 0 is a regular value of f. Set $M_\pm := \{\pm f \geq 0\}, M_0 = \{f = 0\}$. Up to switching the role of f and $-f$, we further assume $\mathrm{vol}(M_+) \leq \mathrm{vol}(M_-)$.

(1) First, give a heuristic argument to explain why it is reasonable that h is a good measure of λ_1.

(2) Show that

$$\int_{M_+} |\nabla f|^2 \operatorname{vol}_g = \lambda_1 \int_{M_+} f^2 \operatorname{vol}_g .$$

(3) Deduce that

$$\lambda_1 \geq \frac{1}{4} \left(\frac{\int_{M_+} |\nabla(f^2)| \operatorname{vol}_g}{\int_{M_+} f^2 \operatorname{vol}_g} \right)^2 .$$

(4) Using the coarea formula, show that

$$\int_{M_+} |\nabla(f^2)| \operatorname{vol}_g \geq h \int_{M_+} f^2 \operatorname{vol}_g,$$

and conclude.

(5) Treat the general case where 0 is not necessarily a regular value.

Cheeger's inequality is known to be sharp in some cases. An upper bound for λ_1 involving h was also proved by Buser [Bus82] in 1982. The Cheeger constant also plays an important role in graph theory.

Exercise 3.5 (Hersch's theorem)**.** The purpose of this exercise is to prove Hersch's theorem [Her70]. In the following, $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and g_{can} denotes the restriction of the Euclidean metric to S^2.

Theorem (Hersch, 1970)**.** *Let g be a metric on S^2 such that $\operatorname{vol}_g(S^2) = \operatorname{vol}_{g_{\text{can}}}(S^2)$. Then:*

$$\lambda_1(g) \leq \lambda_1(g_{\text{can}}).$$

(1) What is the value $\lambda_1(g_{\text{can}})$?

(2) Let g be a metric on S^2. Show that there exists a conformal map $\phi : (S^2, g) \to (S^2, g_{\text{can}})$ (i.e. such that $\phi_* g = e^{2\sigma} g_{\text{can}}$ for some $\sigma \in C^\infty(S^2)$.)

(3) Let $f \in C^\infty(S^2)$. Show that:

$$|\nabla^g f|_g^2 \operatorname{vol}_g = |\nabla^{e^{2\sigma} g} f|_{e^{2\sigma} g}^2 \operatorname{vol}_{e^{2\sigma} g} .$$

The previous equality shows that $|\nabla^g \bullet|_g^2 \operatorname{vol}_g$ is a conformal invariant in dimension 2. Is it still true in higher dimensions?

We now further assume that $\operatorname{vol}_g(S^2) = \operatorname{vol}_{g_{\text{can}}}(S^2)$. Let c_1, c_2, c_3 be the coordinate functions on S^2, that is $c_i(x) := g_{\mathbb{R}^3}(x, \mathbf{e}_i)$, where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

(4) Show that there exists $i \in \{1, 2, 3\}$ such that

$$\frac{\int_{S^2} |\nabla^g \phi^* c_i|_g^2 \operatorname{vol}_g}{\int_{S^2} |\phi^* c_i|^2 \operatorname{vol}_g} \leq 2.$$

Have we proved the theorem?

It remains to show that ϕ can be chosen such that $\int c_i \,\mathrm{vol}_{\phi_* g} = 0$. Given a (smooth) measure μ on S^2, define its barycenter $b(\mu) \in B_1$, the unit ball in $\mathbb{R}^3$, as:

$$b(\mu) = \left(\int_{S^2} c_1 \ \mu, \int_{S^2} c_2 \ \mu, \int_{S^2} c_3 \ \mu \right) \in B_1.$$

We also introduce the stereographic projection

$$p(x,y,z) = \frac{x+iy}{1-z} \in \mathbb{C}, \qquad p^{-1}(z) = \left(\frac{2\Re(z)}{1+|z|^2}, \frac{2\Im(z)}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2} \right),$$

and the conformal transformation $\phi_\lambda : z \mapsto \lambda z$ on $\mathbb{C}P^1 = \mathbb{C} \sqcup \{\infty\}$ for $\lambda > 0$.

(5) What is the group of conformal transformations on (S^2, g_{can})?

(6) Show that for all $A \in \mathrm{SO}(3)$,

$$A(b(\mu)) = b(A_* \mu).$$

(7) Compute $b(\phi_{\lambda *}\mu)$. Deduce $\lim_{\lambda \to 0} b(\phi_{\lambda *}\mu)$ and $\lim_{\lambda \to +\infty} b(\phi_{\lambda *}\mu)$.

(8) Given an arbitrary $v \in S^2$, choose $A \in \mathrm{SO}(3)$ such that $Av = \partial_z$, and define

$$\phi_{v,\lambda} := A^{-1} \circ \phi_\lambda \circ A.$$

Show that $\phi_{v,\lambda}$ is independent of the choice of matrix $A \in \mathrm{SO}(3)$ satisfying $Av = \partial_z$.

(9) What are $\lim_{\lambda \to 0} b(\phi_{v,\lambda_*}\mu)$ and $\lim_{\lambda \to +\infty} b(\phi_{v,\lambda_*}\mu)$?

(10) For $\lambda \geq 0$ fixed, define

$$S_\lambda := \{ b(\phi_{v,\lambda_*}\mu) \mid v \in S^2 \}.$$

Compute S_0, S_1 and S_∞. Conclude.

(11) Finally, prove Hersch's theorem.

Hersch's theorem can be refined and similar results can be obtained for higher eigenvalues.

4. Non-linear problems

Exercise 4.1 (Local structure of the space of metrics, Ebin 1968)**.** If G is a compact Lie group and N a smooth closed manifold, the slice theorem describes the local structure of the orbits of points on N under the action of G. The purpose of this exercise is to generalize this description to an infinite-dimensional setting.

Let M be a smooth closed manifold, g_0 an arbitrary metric of regularity $C^{k,\alpha}$ for $k \in \mathbb{Z}_{\geq 0}, \alpha \in (0,1)$. We aim to describe locally the *space of metrics* near g_0.

Orbit under the diffeomorphism group. The group of diffeomorphisms $\mathrm{Diff}^{k+1,\alpha}(M)$ of regularity $(k+1,\alpha)$ acts on $C^{k,\alpha}$-regular metrics by pullback, namely $(\phi, g) \mapsto \phi^* g$.

(1) Show that the action $\mathrm{Diff}^{k+1,\alpha}(M) \times C^{k,\alpha}(M, S^2T^*M) \to C^{k,\alpha}(M, S^2T^*M)$ given by $(\phi, g) \mapsto \phi^* g$ is indeed well-defined and continuous. Is it better than continuous?

(2) Show that, for fixed $\phi \in \mathrm{Diff}^{k+1,\alpha}(M)$, the map
$$C^{k,\alpha}(M, S^2T^*M) \to C^{k,\alpha}(M, S^2T^*M), \qquad g \mapsto \phi^*g$$
is smooth.

(3) Show that, if $g_0 \in C^\infty(M, S^2T^*M)$ is smooth, then the map
$$\mathrm{Diff}^{k+1,\alpha}(M) \to C^{k,\alpha}(M, S^2T^*M), \qquad \phi \mapsto \phi^*g_0$$
is smooth.

(4) Compute the tangent space (at g_0) of the orbit of g_0 under the action of $\mathrm{Diff}^{k+1,\alpha}(M)$, and express it using the operator D_{g_0} (Exercise 2.5).

Slice theorem. In what follows, we will always assume that g_0 is smooth. Our aim is to show the following slice theorem, due to Ebin in 1968 [Ebi68].

Theorem (Slice theorem (Ebin, 1968))**.** *Assume that* $\ker D_{g_0} = \{0\}$ *(no Killing vector fields, or, equivalently, the isometry group is finite). Then, there exists* $\varepsilon > 0$ *such that for all metrics* g *such that* $\|g - g_0\|_{C^{k,\alpha}} < \varepsilon$*, there exists a unique* $\phi \in \mathrm{Diff}^{k+1,\alpha}(M)$ *close to the identity such that* $D^*_{g_0}(\phi^*g) = 0$.

(5) Is the map
$$\mathrm{Diff}^{k+1,\alpha}(M) \times C^{k,\alpha}(M, S^2T^*M) \to C^{k-1,\alpha}(M, T^*M), \quad (\phi, g) \mapsto D^*_{g_0}(\phi^*g)$$
smooth?

(6) Explain quickly why $(\phi^{-1})^*D^*_{g_0}\phi^* = D^*_{(\phi^{-1})^*g_0}$.

(7) Define $F(\phi, g) := D^*_{(\phi^{-1})^*g_0}$. Compute the derivative of F with respect to ϕ at the point $(\mathbf{1}, g_0)$.

(8) Show that $\partial_1 F(\mathbf{1}, g_0) : C^{k+1,\alpha}(M, TM) \to C^{k-1,\alpha}(M, T^*M)$ is an isomorphism.

(9) Conclude.

Exercise 4.2 (Prescribing the scalar curvature)**.** Let (M, g) be a smooth closed Riemannian manifold and denote by s_g the scalar curvature. Assume that $L_g := (n-1)\Delta_g - s_g$ has trivial kernel.

Show that the following holds: for all $m \geq 0$ and $0 < \alpha < 1$, there exists C, ε such that for all $h \in C^{m,\alpha}(M)$ with $\|h\|_{C^{m,\alpha}} < \varepsilon$, there exists $f \in C^{m+2,\alpha}(M)$ such that $s_{e^{2f}g} = s_g + h$ on M, and $\|f\|_{C^{m+2,\alpha}} \leq C\|h\|_{C^{m,\alpha}}$.

5. Miscellaneous

Exercise 5.1 (Reminders on connections)**.** Let M be a closed oriented manifold, $L \to M$ be a complex line bundle equipped with a connection ∇, and $F_\nabla \in C^\infty(M, \Lambda^2T^*M)$ be its curvature. Let $\Sigma \subset M$ be a closed surface.

(1) Show that $\int_\Sigma F_\nabla$ is independent of ∇ and only depends on $[\Sigma] \in H_2(M, \mathbb{Z})$.

(2) Show that $\int_\Sigma F_\nabla \in 2\pi i\mathbb{Z}$.

Exercise 5.2 (Gauss-Bonnet formula)**.** The purpose of this exercise is to prove the Gauss-Bonnet formula on closed surfaces: if (M, g) is a Riemannian surface of genus g, then

$$\int_M K_g \,\mathrm{vol}_g = 2\pi(2-2g). \tag{5.1}$$

(1) Show that $K_g \,\mathrm{vol}_g = iF_\nabla$ for some appropriate connection ∇ on a line bundle.
(2) Deduce that
$$\int_M K_g \,\mathrm{vol}_g$$
is independent of the metric g.
(3) A pair of pants S is a surface with 3 boundary components and no hole. There exists (many) hyperbolic metrics (constant curvature -1) on S. Compute the volume of (S, g) where g is hyperbolic.
(4) Deduce the Gauss-Bonnet formula (5.1).

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