GEOMETRIC ANALYSIS ON MANIFOLDS

The lecture notes from my webpage and personal notes taken during the classes are allowed. Other material such as books or online material is prohibited. The exam lasts three hours.

Problem: Evolution equations

Introduction. Throughout the problem, M is a closed (i.e. compact without boundary) oriented manifold equipped with a nowhere vanishing smooth volume form μ such that $\int_M \mu > 0$. We let $E \to M$ be a smooth Hermitian bundle with metric g_E. The space $L^2(M,E)$ is defined as the completion of $C^\infty(M,E)$ with respect to the norm

$$\|\varphi\|^2_{L^2(M,E)} := \int_M g_E(\varphi(x),\varphi(x))\ \mu(x).$$

We denote by $\Lambda_s \in \Psi^s(M,E)$ (for $s \in \mathbb{R}$) an invertible elliptic operator of order s with principal symbol $\sigma_{\Lambda_s}(x,\xi) = \langle\xi\rangle^s \mathbf{1}_{E_x}$ and such that $\Lambda_s^{-1} = \Lambda_{-s}$. The space $H^s(M,E)$ is then defined as the completion of $C^\infty(M,E)$ with respect to the norm

$$\|\varphi\|_{H^s(M,E)} := \|\Lambda_s\varphi\|_{L^2(M,E)}.$$

We will **admit** that for all $m \in \mathbb{R}$, the spaces $\Psi^m(M,E)$ are Fréchet spaces and that if $\mathbb{R} \ni t \mapsto P(t) \in \Psi^m(M,E)$ is continuous, then for all $s \in \mathbb{R}$, $t \mapsto P(t)$ is continuous as an operator in $\mathcal{L}(H^{s+m}(M,E), H^s(M,E))$ (operator norm).

Given $P \in \Psi^1(M,E)$, a pseudodifferential operator of order 1, we denote by

$$\sigma_P \in S^1(T^*M, \mathrm{End}(E))/S^0(T^*M,\mathrm{End}(E))$$

its principal symbol (or, more precisely, a representative of its principal symbol). We say that P satisfies the *symmetric hyperbolic condition* if the following holds:

$$\sigma_P + \sigma_P^* \in S^0(T^*M,\mathrm{End}(E)). \tag{0.1}$$

The aim of this problem is to study evolution equations of the form

$$\partial_t u(t) = P(t)u(t), \qquad u(0) = f \in H^s(M,E), s \in \mathbb{R}, \tag{0.2}$$

where $\mathbb{R} \ni t \mapsto P(t) \in \Psi^1(M,E)$ is a smooth family of operators satisfying the symmetric hyperbolic condition (0.1). We want to show that, for all $T > 0$, under the condition (0.1), the equation (0.2) admits a unique solution

$$u \in C^0([0,T], H^s(M,E)) \cap C^1([0,T], H^{s-1}(M,E)). \tag{0.3}$$

Examples.

(1) Verify that the symmetric hyperbolic condition (0.1) is independent of the choice of representative for the principal symbol $\sigma_P \in S^1(T^*M, \mathrm{End}(E))/S^0(T^*M, \mathrm{End}(E))$.

We now let $E = M \times \mathbb{C} \to M$ be the trivial line bundle and define $P := X$, where $X \in C^\infty(M, TM)$ is a vector field on M seen as a differential operator of order 1 acting on functions. Denote by $(\varphi_t)_{t\in\mathbb{R}}$ the flow on M generated by X, that is, such that

$$\frac{d}{dt}\varphi_t(x) = X(\varphi_t(x)), \qquad \forall t \in \mathbb{R}, x \in M.$$

(2) Show that P satisfies the symmetric hyperbolic condition (0.1).

(3) Given an initial data $f \in C^\infty(M)$, show that $u(t) := \varphi_t^* f$ (defined by $\varphi_t^* f(x) := f(\varphi_t x)$) solves the evolution equation (0.2).

Preliminary questions: the Cauchy problem in Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space.

(4) Let $A : \mathcal{H} \to \mathcal{H}$ a bounded linear operator. Show that the equation

$$\partial_t u(t) = Au(t), \qquad u(0) = f \in \mathcal{H},$$

has a unique solution $u \in C^1([0,T], \mathcal{H})$.

(5) Let $\mathbb{R} \ni t \mapsto A(t) \in \mathcal{L}(\mathcal{H})$ be a smooth family of bounded linear operators. Show that the equation

$$\partial_t u(t) = A(t)u(t), \qquad u(0) = f \in \mathcal{H},$$

has a unique solution $u \in C^1([0,T], \mathcal{H})$. *Hint: for $T_0 := (2\sup_{t\in[0,T]} \|A(t)\|)^{-1}$, you may consider the operator*

$$\Phi : C^0([0,T_0], \mathcal{H}) \to C^0([0,T_0], \mathcal{H})$$

defined by

$$\Phi u(t) := f + \int_0^t A(s)u(s)\mathrm{d}s,$$

and apply a fixed point theorem.

Uniqueness of the solution. We first show that, if it exists, a solution

$$u \in C^0([0,T], H^s(M,E)) \cap C^1([0,T], H^{s-1}(M,E))$$

to the evolution equation (0.2) with initial data $u(0) = f \in H^s(M,E)$ is necessarily unique.

(6) Show that such a solution u satisfies

$$\partial_t \|\Lambda_s u(t)\|^2_{L^2} = \langle B(t)\Lambda_s u(t), \Lambda_s u(t)\rangle + \langle [\Lambda_s, P(t)]u(t), \Lambda_s u(t)\rangle + \langle \Lambda_s u(t), [\Lambda_s, P(t)]u(t)\rangle,$$

where $B(t) = P(t) + P^*(t) \in \Psi^0(M,E)$ and $[\Lambda_s, P(t)] := \Lambda_s P(t) - P(t)\Lambda_s$.

(7) Let $A \in \Psi^{m_1}(M,E), B \in \Psi^{m_2}(M,E)$. Show that $[A,B] := AB - BA \in \Psi^{m_1+m_2-1}(M,E)$ if and only if the principal symbols of A and B commute, that is, $[\sigma_A, \sigma_B] := \sigma_A\sigma_B - \sigma_B\sigma_A = 0$.

(8) Show that $[\Lambda_s, P]\Lambda_s^{-1} \in \Psi^0(M,E)$.

(9) Deduce that there exists a constant $C > 0$ such that for all $t \in [0, T]$,

$$\partial_t \|\Lambda_s u(t)\|^2_{L^2} \leq C\|\Lambda_s u(t)\|^2_{L^2}.$$

(10) Conclude that for all $t \in [0, T]$

$$\|u(t)\|^2_{H^s} \leq e^{Ct}\|f\|^2_{H^s},$$

and deduce that a solution to the evolution equation (0.2) in

$$C^0([0, T], H^s(M, E)) \cap C^1([0, T], H^{s-1}(M, E))$$

is necessarily unique.

Mollified solution to the evolution equation. We now aim to construct a solution to the evolution equation. In what follows, we will take the initial data $f \in L^2(M, E)$.

We first study the regularization of the evolution equation (0.2). Let $\chi \in C^\infty_{\mathrm{comp}}(T^*M)$ be a nonnegative cutoff function equal to 1 near the zero section in T^*M, and define $\chi_\varepsilon(x, \xi) := \chi(x, \varepsilon\xi)$. We define $E_\varepsilon := \mathrm{Op}(\chi_\varepsilon \cdot \mathbf{1}_E)^* \,\mathrm{Op}(\chi_\varepsilon \cdot \mathbf{1}_E) \in \Psi^{-\infty}(M, E)$. We will **admit** that the family $(E_\varepsilon)_{\varepsilon>0}$ satisfies the following properties:

- $E^*_\varepsilon = E_\varepsilon$;
- For all $s \in \mathbb{R}$, for all $\psi \in H^s(M, E)$, $E_\varepsilon\psi \to_{\varepsilon\to 0} \psi$ in H^s;
- For all $s \in \mathbb{R}$, there exists $C > 0$ such that for all $\varepsilon > 0$, $\|E_\varepsilon\|_{H^s\to H^s} \leq C$.

(11) We fix $\varepsilon, T > 0$. Show that

$$\partial_t u_\varepsilon(t) = E_\varepsilon P(t)E_\varepsilon u_\varepsilon(t), \qquad u_\varepsilon(0) = f \in L^2(M, E),$$

admits a unique solution $u_\varepsilon \in C^1([0, T], L^2(M, E))$.

(12) Show that

$$\partial_t \|u_\varepsilon(t)\|^2_{L^2} = \langle E_\varepsilon B(t)E_\varepsilon u_\varepsilon(t), u_\varepsilon(t)\rangle_{L^2},$$

where $B(t) := P(t) + P^*(t) \in \Psi^0(M, E)$.

(13) Deduce that there exists a constant $C > 0$ such that for all $\varepsilon > 0$, for all $0 \leq t \leq T$:

$$\partial_t \|u_\varepsilon(t)\|^2_{L^2} \leq C\|u_\varepsilon(t)\|^2_{L^2},$$

and then that for all $0 \leq t \leq T$:

$$\|u_\varepsilon(t)\|^2_{L^2} \leq e^{Ct}\|f\|^2_{L^2}.$$

Solution to the evolution equation in weaker spaces. We still assume that the initial data $u(0) = f$ is in $L^2(M, E)$. Using the previous section, we now want to construct a first solution u to the evolution equation (0.2) but in weaker spaces, namely,

$$u \in C^0([0, T], H^{-1}(M, E)) \cap C^1([0, T], H^{-2}(M, E)).$$

We fix $T > 0$. Denote by $(t_n)_{n\geq 0}$ the rational numbers in $[0, T]$ (this is a countable set).

(14) Show that there exists a sequence $(\varepsilon_k)_{k\geq 0}$ converging to 0 as $k \to +\infty$ such that for all $n \geq 0$, $u_{\varepsilon_k}(t_n)$ converges in $H^{-1}(M, E)$ as $k \to +\infty$. We call the limit $u(t_n) \in H^{-1}(M, E)$.

This defines a function $u : [0, T] \cap \mathbb{Q} \to H^{-1}(M, E)$.

(15) Show that there exists a constant $C > 0$ such that for all $\varepsilon > 0$, for all $t, t' \in [0, T]$,

$$\|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{-1}} \leq C\|f\|_{L^2}|t - t'|.$$

(16) Deduce that $u : [0, T] \cap \mathbb{Q} \to H^{-1}(M, E)$ extends to a Lipschitz continuous function $u : [0, T] \to H^{-1}(M, E)$ such that for all $t, t' \in [0, T]$,

$$\|u(t) - u(t')\|_{H^{-1}} \leq C\|f\|_{L^2}|t - t'|.$$

(17) Show that $u_{\varepsilon_k} \to_{k\to+\infty} u$ in $C^0([0, T], H^{-1}(M, E))$, that is,

$$\lim_{k\to+\infty} \sup_{t\in[0,T]} \|u_{\varepsilon_k}(t) - u(t)\|_{H^{-1}} = 0$$

(18) Show that

$$u \in C^0([0, T], H^{-1}(M, E)) \cap C^1([0, T], H^{-2}(M, E))$$

and u solves (0.2). *Hint: you may start with the identity*

$$u_{\varepsilon_k}(t) = f + \int_0^t E_{\varepsilon_k} P(s) E_{\varepsilon_k} u_{\varepsilon_k}(s) ds,$$

and pass to the limit as $k \to +\infty$ in the adequate topology.

(19) More generally, assuming the initial data $u(0) = f$ is in $H^s(M, E)$ (for $s \in \mathbb{R}$), construct a solution

$$u \in C^0([0, T], H^{s-1}(M, E)) \cap C^1([0, T], H^{s-2}(M, E))$$

solving (0.2). *Hint: You may look for a solution of the form $u(t) = \Lambda_s^{-1}\widetilde{u}(t)$, where $t \mapsto \widetilde{u}(t)$ solves another evolution equation.*

Solution to the evolution equation. We eventually prove the existence of a solution to the evolution equation in the right spaces.

(20) Let $(u_n)_{n\geq 0}$ be a sequence in $L^2(M, E)$ such that $\|u_n\|_{L^2} \leq 1$ and assume that there exists $u \in \mathcal{D}'(M, E)$ such that $u_n \to u$ in $\mathcal{D}'(M, E)$. Show that $u \in L^2(M, E)$ and $\|u\|_{L^2} \leq 1$.

(21) In the previous question, is there always convergence $u_n \to u$ in $L^2(M, E)$? If not, provide a counter-example.

(22) Deduce that, if the initial data f is in $L^2(M, E)$, then for all $t \in [0, T]$,

$$u(t) \in L^2(M, E), \partial_t u(t) \in H^{-1}(M, E)$$

and that there exists a constant $C > 0$ such that for all $t \in [0, T]$,

$$\|u(t)\|_{L^2} \leq C\|f\|_{L^2}, \qquad \|\partial_t u(t)\|_{H^{-1}} \leq C\|f\|_{L^2}.$$

(23) We now consider a sequence $(f_j)_{j\geq 0}$ such that $f_j \in C^\infty(M, E)$ and $f_j \to f$ in $L^2(M, E)$. Let u_j be the solution to the evolution equation (0.2) with initial data $u_j(0) = f_j$. Show that $(u_j)_{j\geq 0}$ is a Cauchy sequence in

$$C^0([0, T], L^2(M, E)) \cap C^1([0, T], H^{-1}(M, E)),$$

and conclude on the existence of a solution

$$u \in C^0([0, T], L^2(M, E)) \cap C^1([0, T], H^{-1}(M, E))$$

to the evolution equation (0.2).

(24) Eventually, prove the existence of a solution

$$u \in C^0([0,T], H^s(M,E)) \cap C^1([0,T], H^{s-1}(M,E))$$

to the evolution equation (0.2) if the initial data $u(0) = f$ is in $H^s(M,E)$.

In the case where $P(t) = P$ is independent of t, the existence and uniqueness of the solution to the evolution equation (0.2) can be obtained by the Hille-Yosida theorem which guarantees that, under the symmetric hyperbolic condition (0.1), P is the generator of a strongly continuous semi-group of operators $S(t) \in \mathcal{L}(H^s(M,E))$ (for all $s \in \mathbb{R}$) such that $u(t) = S(t)u_0$.

Exercise

We define the following distributions in $\mathbb{R}^2$: for $s \in \mathbb{R}$, for all $\varphi \in C^\infty_{\text{comp}}(\mathbb{R}^2)$,

$$(\Gamma, \varphi) := \int_0^{2\pi} \varphi(\cos\theta, \sin\theta)\mathrm{d}\theta, \qquad (\delta_s, \varphi) := \int_{\mathbb{R}} \varphi(s,y)\mathrm{d}y.$$

For which values of $s \in \mathbb{R}$ is the product $\Gamma \times \delta_s$ well-defined in the sense of distributions? Compute $\Gamma \times \delta_s$ when it is well-defined.

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