Microlocal analysis in hyperbolic dynamics and geometry

Habilitation à diriger des recherches

Thibault Lefeuvre

December 7, 2022

À mes parents, à Marie.

Acknowledgement

I want to start by expressing my gratitude to all my collaborators and friends who have made this journey possible and more than enjoyable: Jan Bohr, Yann Chaubet, Sébastien Gouëzel, Yannick Guedes Bonthonneau, Colin Guillarmou, Gerhard Knieper, Andrei Moroianu, Gabriel Paternain, Uwe Semmelmann and Leo Tzou.

I am grateful to my colleagues in Jussieu in the group *Analyse complexe et géométrie* for their warm welcome in Fall 2020: Olivier Biquard, Gilles Courtois, Viet Dang, Alix Deruelle, Eleonora Di Nezza, Pascal Dingoyan, Elisha Falbel, Julien Marché, Vincent Michel, Frédéric Naud, Bram Petri, Andres Sambarino, Tat-dat To and Maxime Wolff. I also thank Viviane Baladi, Frédéric Faure, Gabriel Rivière, Mikko Salo, Gunther Uhlmann, András Vasy, Maciej Zworski, for their support in my career and many important maths discussion we had.

I want to give special thanks to Mihajlo Cekić for the fruitful collaboration that we started during the first Covid lockdown in Spring 2020 and which has already produced several exciting results. We discovered together in March 2021 that homoclinic orbits of Anosov flows admit an important monoidal structure – which we coined "Parry's free monoid" in tribute to William Parry – and this fundamental tool paved the way to most of our joint results described in this monograph: it led to the ergodicity of frame flows on pinched negatively-curved Riemannian manifolds (obtained in collaboration with Andrei Moroianu and Uwe Semmelmann), to the introduction of the "geodesic Wilson loop operator" which is a key tool in solving Kac's isospectral problem for connections, and to the discovery of the link between algebraic geometry and the ergodicity of partially hyperbolic dynamical systems obtained as isometric extensions of the geodesic flow over negatively-curved Riemannian manifolds. I am looking forward to even more exciting results!

Last but not least, I would like thank my family and friends for their important encouragement. And I thank Marie – who shares my life – with all my heart: none of this would have been possible without your support.

Thibault Lefeuvre, December 2022.

Contents

Introduction

This *Habilitation à diriger des recherches* is an overview of some of the works that I have been doing over the years 2020–2022 in the field of (partially) hyperbolic dynamical systems and geometry.

Chapters 1–3 deal with frame flow ergodicity and related topics. The main achievement is Theorem 3.2 and shows that, on even-dimensional negatively-curved Riemannian manifolds with quasi 1/4-pinched negative sectional curvature, the *frame flow* is ergodic. This almost answers a long-standing conjecture of Brin from the 70s-80s. Generalizations of this result are discussed: first, to the case of Kähler manifolds (see Theorem 3.6), second to general frame flows (see Theorem 3.10). In particular, we establish an important connection between algebraic geometry (namely, the classification of algebraic maps between spheres) and the study of certain partially hyperbolic dynamical systems obtained as frame flow extensions of the geodesic flow over a negatively-curved Riemannian manifold. We also study and prove the injectivity of the *geodesic Wilson loop operator* under a *low-rank assumption* in Theorem 2.3 – this operator is the analogue of the *marked length spectrum*, where metrics are replaced by connections. In turn, under the same low-rank assumption, this solves Kac's isospectral problem *Can one hear the shape of a drum?* for connections: the spectrum of the Bochner Laplacian *does* determine the connection up to gauge on a negatively-curved Riemannian manifold for vector bundles of low rank, see Theorem 2.8.

Chapters 4–5 are concerned with applications of microlocal techniques to two specific problems in geometry and probability. Chapter 4 deals with *lens rigidity* of Riemannian manifolds, that is, to what extent does the scattering map and the length of geodesics between pair of points on the boundary determine a Riemannian manifold with boundary? We give a positive answer to this rigidity problem in Theorem 4.8 in a neighborhood of a metric with negative sectional curvature and strictly convex boundary. In Chapter 5, we investigate the *narrow capture problem* for Lévy flights. These stochastic processes are quite similar to the Brownian motion but are generated by the fractional Laplacian and (roughly speaking) tend to "jump" more along geodesics. We compute the asymptotics of the expected time to find a small target the size of a geodesic ball of radius ε as $\varepsilon \to 0$ under a pure Lévy jump process, see Theorem 5.7. This is a well-known topic in the field of biology as such processes model predators hunting preys and is known as the *Lévy flight foraging hypothesis*.

An important tool that is consistently applied throughout the manuscript (although hidden sometimes ...) is microlocal analysis. Hence, this monograph is at the crossroad of many fields: microlocal analysis and PDEs, (partially) hyperbolic dynamical systems, Riemannian manifolds, algebraic topology and geometry, stochastic processes. The proofs of the main results are usually skipped but the main ingredients are given. When the arguments are not too long nor technical, they are written with full details. By this, we hope to make this monograph more pleasant to read.

(Pre)published work

All my publications are freely available on my webpage[1] or on the arXiv.

Ongoing work

(1) *On the ergodicity of the geodesic flow on non-positively curved surfaces.*
(2) *Microlocal analysis in hyperbolic dynamics and geometry*, book in progress.

CNRS

(3) *On the transport twistor of closed surfaces*, with Jan Bohr and Gabriel Paternain.
(4) *On the ergodicity of unitary frame flows on Kähler manifolds*, with Mihajlo Cekić, Andrei Moroianu and Uwe Semmelmann.
(5) *Geodesic Lévy flights and expected stopping time for random searches*, with Yann Chaubet, Yannick Guedes Bonthonneau and Leo Tzou.
(6) *On polynomial structures over spheres*, with Mihajlo Cekić.
(7) *Towards Brin's conjecture on frame flow ergodicity: new progress and perspectives*, with Mihajlo Cekić, Andrei Moroianu and Uwe Semmelmann, **Mathematics Research Reports, Volume 3 (2022), pp. 21-34.**
(8) *Local lens rigidity for manifolds of Anosov type*, with Mihajlo Cekić and Colin Guillarmou.
(9) *On the ergodicity of the frame flow on even-dimensional manifolds*, with Mihajlo Cekić, Andrei Moroianu and Uwe Semmelmann.
(10) *Isometric extensions of Anosov flows via microlocal analysis*, to appear in **Communications in Mathematical Physics.**
(11) *Generic injectivity of the X-ray transform*, with Mihajlo Cekić.
(12) *The Holonomy Inverse Problem*, with Mihajlo Cekić, to appear in **Journal of the European Mathematical Society.**
(13) *Radial source estimates in Hölder-Zygmund spaces for hyperbolic dynamics*, with Yannick Guedes Bonthonneau, to appear in **Annales Henri Lebesgue.**
(14) *Generic dynamical properties of connections on vector bundles*, avec Mihajlo Cekić, **Int. Math. Res. Not.**, Volume 2022, Issue 14, July 2022, Pages 10649–10703.

PhD. thesis

[1] https://thibaultlefeuvre.blog/

(15) *Local rigidity of manifolds with hyperbolic cusps II. Nonlinear theory*, with Yannick Guedes Bonthonneau.
(16) *Geodesic stretch and marked length spectrum rigidity*, with Colin Guillarmou and Gerhard Knieper, **Ergodic Theory Dynam. Systems** 42 (2022), no. 3, 974–1022.
(17) *Local rigidity of manifolds with hyperbolic cusps I. Linear theory and pseudodifferential calculus*, with Yannick Guedes Bonthonneau, to appear in **Annales de l'Institut Fourier**.
(18) *Classical and microlocal analysis of the X-ray transform on Anosov manifolds*, with Sébastien Gouëzel, **Anal. PDE 14 (2021), no. 1, 301–322.**
(19) *Local marked boundary rigidity under hyperbolic trapping assumptions*, **J. Geom. Anal. 30 (2020), no. 1, 448–465.**
(20) *Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces*, **Comment. Math. Helv. 95 (2020), no. 1, 129–166.**
(21) *The marked length spectrum of Anosov manifolds*, with Colin Guillarmou, **Ann. of Math. (2) 190 (2019), no. 1, 321–344.**
(22) *On the s-injectivity of the X-ray transform for manifolds with hyperbolic trapped set*, **Nonlinearity 32 (2019), no. 4, 1275–1295.**

Master thesis

(23) *On the genericity of the shadowing property for conservative homeomorphisms*, avec Pierre-Antoine Guihéneuf, **Proc. Amer. Math. Soc.** 146 (2018), no. 10, 4225–4237

1. Isometric extensions of Anosov flows

1.1. **Ergodicity in dynamical systems.** The theory of dynamical systems originates from the study of classical mechanics and the description of solutions to differential equations governing the evolution of points in phase space. A well-known historical example pioneered by Kepler and Newton in the XVII century and later enriched with a modern mathematical language by Poincaré in the late XIX century is our solar system, where planets are identified with points and their motion is governed by the law of gravitation. These mechanical systems, in the absence of a dissipative correction, share the property that they preserve a natural volume form on the phase space.

While it is usually impossible to predict the evolution of a single trajectory due to an inherent sensitivity to initial conditions, it is tempting to adopt a statistical approach and describe the long-time behaviour of *almost all* points, measured with respect to this natural flow-invariant form. This is the slant of ergodic theory. From this perspective, a natural property one may investigate on a given dynamical system is the *equidistribution* of a generic point in phase space, namely, whether it will spend in each region of the phase space an average time proportional to its volume. Phrased in mathematical language, ergodicity is the property that any measurable subset that is invariant by the dynamical transformation must have zero or full measure.

These physical considerations paved the way for a more systematic search of ergodic dynamical systems in mathematics. In a seminal article [Hop36], using what is now known as the classical *Hopf argument*, Hopf proved that geodesic flows on closed negatively-curved surfaces are ergodic with respect to a natural smooth measure called the Liouville measure, providing one of the first rigorous examples of chaotic systems of geometric flavour. Later, Anosov [Ano67] introduced in his thesis the notion of *uniformly hyperbolic flows* (also known as *Anosov flows* nowadays) and showed that they are ergodic whenever they preserve a smooth measure. Moreover, he proved that all geodesic flows on negatively-curved Riemannian manifolds are uniformly hyperbolic and thus ergodic. From a statistical perspective, these flows are now well understood and finer properties such as *mixing* or even *exponential mixing* are (almost) completely settled, see Liverani [Liv04] and Tsujii-Zhang [TZ], among other references on this question.

Shortly after Anosov's work, Brin-Pesin [BP74], Pugh-Shub [PS00], and others, investigated more general systems exhibiting a weaker form of hyperbolic behaviour, known as *partially hyperbolic systems*. While these dynamics still preserve some expanding and contracting directions, they also come with other *neutral* or *central* directions, in which the map/flow may behave infinitesimally as an isometry for instance. Historical examples of partially hyperbolic dynamics are provided by *frame flows* over closed negatively-curved Riemannian manifolds, and they will be extensively discussed in §3.

1.2. **Isometric extensions of Anosov flows.** Let $\mathcal{M}$ be a smooth closed manifold. We recall that a vector field $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ generates an *Anosov flow* $(\varphi_t)_{t\in\mathbb{R}}$ if there exists a continuous flow-invariant splitting of the tangent bundle $T\mathcal{M} = \mathbb{R}X \oplus E^s_{\mathcal{M}} \oplus E^u_{\mathcal{M}}$ into flow-direction, stable and unstable bundles, and uniform constants $C, \lambda > 0$ such that for all $t \geq 0$:

$$\|d\varphi_t v\| \leq Ce^{-\lambda t}\|v\|, \ \forall v \in E^s_{\mathcal{M}}, \qquad \|d\varphi_{-t} v\| \leq Ce^{-\lambda t}\|v\|, \ \forall v \in E^u_{\mathcal{M}}, \tag{1.1}$$

where $\| \bullet \|$ is the norm induced by an arbitrary Riemannian metric on $\mathcal{M}$. In the following, we will further assume that X preserves a *smooth measure* μ. In particular, by a standard result of Anosov [Ano67], this implies that $(\varphi_t)_{t\in\mathbb{R}}$ is ergodic with respect to μ. The goal of this paragraph is to study the ergodic properties of some specific *extensions* of $(\varphi_t)_{t\in\mathbb{R}}$ which we now describe.

We let (F, g_F) be a smooth closed Riemannian manifold. A fiber bundle $P \to \mathcal{M}$ is said to be a *Riemannian fiber bundle* with fiber (F, g_F) is $P \to \mathcal{M}$ is a F-fiber bundle over $\mathcal{M}$ which admits a reduction of its structure group $\mathrm{Diff}(F)$ to $\mathrm{Isom}(F, g_F)$. In particular, every fiber of P is a smooth Riemannian manifold which is isometric to (F, g_F). Note, however, that the total space P does not carry *a priori* a global metric whose restriction to the fibers is isometric to g_F. Given a Riemannian fiber bundle $p : P \to \mathcal{M}$ over $\mathcal{M}$, we say that a flow $(\Phi_t)_{t\in\mathbb{R}}$ on P is an *extension* of $(\varphi_t)_{t\in\mathbb{R}}$ on M if the following holds:

$$\forall t \in \mathbb{R}, \qquad p \circ \Phi_t = \varphi_t \circ p. \tag{1.2}$$

We will further say that it is an *isometric extension* if the maps

$$\Phi_t|_{P_x} \ : P_x \to P_{\varphi_t x},$$

are isometries for all $x \in \mathcal{M}, t \in \mathbb{R}$.

Example 1.1. The two main examples are provided by principal bundles and (the unit sphere of) vector bundles over $\mathcal{M}$. Indeed, if P is a principal G-bundle, where G is a compact Lie group, then every fiber is naturally isomorphic to G and thus any choice of a bi-invariant metric on G provides a metric on the fibers of P. Similarly, if $\mathcal{E} \to \mathcal{M}$ is a

Euclidean or Hermitian vector bundle over $\mathcal{M}$ (with metric h), then the unit sphere bundle

$$S\mathcal{E} := \{(x, f) \in \mathcal{E} \mid x \in \mathcal{M}, h_x(f, f) = 1\}$$

is a Riemannian fiber bundle whose fiber is isometric to $\mathbb{S}^{r-1}$ equipped with the round metric, where r denotes the real rank of $\mathcal{E}$.

If $(\Phi_t)_{t\in\mathbb{R}}$ is an isometric extension, then P carries a natural flow-invariant smooth measure ω obtained locally as the product of (the pullback of) μ wedged with the smooth Riemannian measure in the fibers (isometric to (F, g_F)). Understanding the ergodicity of $(\Phi_t)_{t\in\mathbb{R}}$ with respect to ω is a very natural question in order to describe the long-time statistical properties of the extended flow. As mentioned in §1.1, an archetypal example fitting in this framework is the frame flow over a negatively-curved Riemannian manifold (M, g), which will be further discussed in §3 (in this case $P = FM$ is the frame bundle and $\mathcal{M} = SM$).

Isometric extensions of Anosov flows are intrinsically more complicated to study due to their lack of uniform hyperbolicity. Indeed, the *vertical direction* $\mathbb{V} := \ker dp$ (where $p : P \to \mathcal{M}$ is the projection) now becomes a *neutral* or *central* direction, in the sense that the differential of the flow $(\Phi_t)_{t\in\mathbb{R}}$ acts as a linear isometry on $\mathbb{V}$ and the tangent bundle to P then splits as

$$TP = \mathbb{R}X_P \oplus E_P^s \oplus E_P^u \oplus \mathbb{V}, \tag{1.3}$$

where X_P is the vector field generating $(\Phi_t)_{t\in\mathbb{R}}$ and $E_P^{s,u}$ satisfy an expanding/contracting property similar to (1.1). Note that the sub-bundles $E_P^{s,u}$ also integrate to produce a (Hölder-continuous) foliation on P by strong stable and unstable manifolds $W_P^{s,u}$, see Pesin [Pes04] or Hasselblatt-Pesin [HP06] for the related diffeomorphism case.

The study of $(\Phi_t)_{t\in\mathbb{R}}$ fits into the theory of *partially hyperbolic dynamical systems*, which is still a very active field of research within the theory of dynamical systems. Such systems are usually defined as those admitting a similar splitting to (1.3) with expanding, contracting and neutral directions, although it is not necessarily required that the flow acts as an isometry on the central bundle but rather that its expansion (resp. contraction) rate is weaker than that of the unstable (resp. stable) bundle. We refer to [HP06] for an overview.

1.3. **Structural results.** We now detail some important results describing the ergodic components of the flow $(\Phi_t)_{t\in\mathbb{R}}$.

1.3.1. *Transitivity group. Parry's free monoid.* Following Hopf's argument in the Anosov case, it is natural to expect (at least heuristically) that the ergodic component of an arbitrary point $z \in P$ consists of all the other points $z' \in P$ that one can reach from z by following a concatenation of flow- and so-called us-paths, namely, paths that are either fully contained in a flowline of $(\Phi_t)_{t\in\mathbb{R}}$ or in a strong stable/unstable leaf $W_P^{s,u}$. The *full accessibility* of a flow is the property that any other point $z' \in P$ can be reached from z by such a concatenation of paths and it is expected that volume-preserving partially hyperbolic dynamical systems are ergodic whenever they are accessible[2]: this is known as the Pugh-Shub conjecture [PS00]. Under a certain additional center bunching assumption, the Pugh-Shub conjecture was proved by Burns-Wilkinson [BW10]. Taking advantage of the very algebraic structure of principal G-bundle extensions of Anosov flows, Brin [Bri75b] translated the accessibility property into a key algebraic notion, called the *transitivity group*: this is a subgroup $H \leqslant G$ (well-defined up to conjugacy in the structure group G) describing all the points in a fiber that are reachable by flow- and us-paths. We will now introduce this notion in the slightly more general context of isometric extensions of Anosov flows, as discussed in §1.2.

It will be convenient to fix an arbitrary periodic point $x_\star \in \mathcal{M}$ for the flow $(\varphi_t)_{t\in\mathbb{R}}$, generating a periodic orbit $\gamma_\star \subset \mathcal{M}$ of period $T_\star$. Denote by $\mathcal{H}$ the set of orbits of $(\varphi_t)_{t\in\mathbb{R}}$ that are *homoclinic* to $\gamma_\star$, namely, which accumulate in the past and in the future to $\gamma_\star$. Volume preserving (and more generally, transitive) Anosov flows satisfy that $\mathcal{H}$ is dense in $\mathcal{M}$: this can be easily proved by using the shadowing lemma and the density of periodic orbits in $\mathcal{M}$. Given $\gamma \in \mathcal{H}$ and a point w in the fiber $P_\star := P_{x_\star}$ over $x_\star$, there is a natural way to "parallel transport" w *along* γ (even though γ has infinite length!) in order to produce another point, denoted by $\rho(\gamma)w$. This construction goes as follows (see Figure 1, and [CLb] for more details):

(1) One picks an arbitrary point $x_- \in \gamma \cap W^u_{\mathcal{M}}(x_\star)$ (i.e. such that $d_{\mathcal{M}}(\varphi_{-t}x_\star, \varphi_{-t}x_-) \to 0$ as $t \to +\infty$, and this convergence is exponentially fast); then, in the fiber P_{x_-} over x_-, there exists a *unique* point $w_- \in W^u_P(w)$ such that $d_P(\Phi_{-t}w, \Phi_{-t}w_-) \to 0$ as $t \to +\infty$. The map $P_\star \to P_{x_-}, w \mapsto w_-$ is called the *unstable holonomy.*

[2] A refinement of this notion is the *essential accessiblity*, that is, accessibility up to measure zero, but this will not be needed here.

(2) One then "pushes" w_- by the flow $(\Phi_t)_{t\in\mathbb{R}}$ until it reaches a point $w_+ := \Phi_T(w_-)$ over $x_+ \in \gamma \cap W^s_{\mathcal{M}}(x_\star)$, where $T > 0$ is the unique time such that $x_+ = \varphi_T(x_-)$;
(3) Eventually, applying a similar (but stable this time) holonomy to (1), one can produce an element $\rho(\gamma)w \in P_\star$.

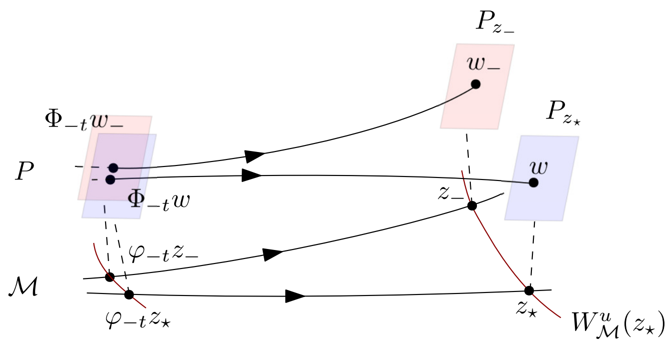


FIGURE 1. Bottom: black lines represent flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathcal{M}$; red lines represent strong unstable leaves. Top: black lines represent flowlines of $(\Phi_t)_{t\in\mathbb{R}}$ on P; blue and red parallelograms represent fibers of P.

For the sake of clarity, it is important to have in mind that the above-mentioned holonomies are actually parallel transports in strong stable, unstable, flow leaves of $(\varphi_t)_{t\in\mathbb{R}}$ with respect to a *dynamical* Ehresmann connection on the Riemannian fiber bundle $P \to \mathcal{M}$ induced by the lifted flow $(\Phi_t)_{t\in\mathbb{R}}$. We do not properly introduce this connection as it will not be used in what follows. The element $\rho(\gamma)$ is intrinsically defined as an element in $\mathrm{Isom}(P_\star)$. If we identify[3] isometrically the fiber $P_\star \simeq F$, then $\rho(\gamma)$ can be identified with left multiplication by an element of the group $\mathrm{Isom}(F)$ itself, that is, $\rho(\gamma) \in \mathrm{Isom}(F)$. Hence, introducing *Parry's free monoid*[4] $\mathbf{G}$ as the formal set of words

$$\mathbf{G} := \left\{\gamma_1^{k_1}...\gamma_p^{k_p} \mid p \in \mathbb{N}, k_j \in \mathbb{N}, \gamma_j \in \mathcal{H}, j = 1, ..., p\right\},$$

we see that the above-mentioned construction produces a natural representation

$$\rho : \mathbf{G} \to \mathrm{Isom}(F), \tag{1.4}$$

[3]This requires a certain choice at this stage but what follows we only depend on this choice up to conjugacy within $\mathrm{Isom}(F)$.

[4]A monoid is a set endowed with an associative product, a neutral element, but no inverse.

whose image $H := \overline{\rho(\mathbf{G})} \leqslant \mathrm{Isom}(F)$ is called the *transitivity group*. Note that this subgroup is only well-defined up to conjugacy as it requires to choose a (non-canonical) identification $P_\star \simeq \mathrm{Isom}(F)$. Moreover, it can be checked that H is nothing but the *holonomy group* of the dynamical Ehresmann connection on P.

The group $\mathrm{Isom}(F)$ is a closed Lie group [MS39] and thus the closure of the image of the representation $H := \overline{\rho(\mathbf{G})}$ is a closed Lie subgroup [Hel01, Theorem 2.3] (well-defined up to conjugacy in $\mathrm{Isom}(F)$). Since P may not be a principal bundle, the orbit space $H\backslash P$ may not be a smooth manifold but it is still a Hausdorff topological space endowed with a natural measure $\nu := \mathrm{pr}_* \mu_F$, where $\mathrm{pr} : F \to H\backslash F$ is the projection and μ_F is the Riemannian measure on F induced by the metric g.

1.3.2. *Structural result in the general case.* For the sake of simplicity, we now assume that the flow-invariant measure μ on $\mathcal{M}$ is a probability measure. The transitivity group turns out to be crucial in understanding the ergodicity of the flow $(\Phi_t)_{t\in\mathbb{R}}$ as illustrated by the following result we obtained in [Lef]:

Theorem 1.2 (L. '21)**.** *Under the above assumptions, the followings holds:*

(i) Ergodicity: *There exists an open H-invariant subset $F_0 \subset F$ of full measure (with respect to μ_F) such that for all $\mathrm{x} \in H\backslash F_0$, there exists an associated flow-invariant smooth submanifold $Q(\mathrm{x}) \subset P$ which is a smooth Riemannian fiber bundle over $\mathcal{M}$ with fiber diffeomorphic to a closed manifold Q_0 (independent of x) and such that the restriction of $(\Phi_t)_{t\in\mathbb{R}}$ to $Q(\mathrm{x})$ is ergodic (with respect to the flow-invariant smooth measure induced by μ_E on $Q(\mathrm{x})$). Moreover, there exists a natural isometry*

$$\Psi : L^2(H\backslash F, \nu) \xrightarrow{\sim} \ker_{L^2}(X_P). \tag{1.5}$$

In particular, H acts transitively on the fiber F if and only if the flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on P.

(ii) Mixing: *If the fiber of $Q(\mathrm{x})$ is not the total space of a Riemannian submersion over the circle and $(\varphi_t)_{t\in\mathbb{R}}$ is mixing, then the restriction of $(\Phi_t)_{t\in\mathbb{R}}$ to $Q(\mathrm{x})$ is also mixing. In particular, if H acts transitively on the fiber F and F is not the total space of a fiber bundle over the circle, then the flow $(\Phi_t)_{t\in\mathbb{R}}$ is mixing on P.*

The extension isomorphism Ψ in (1.5) is simply defined by "pushing" an H-invariant L^2-function, defined on a certain fiber $E_{x_\star} \simeq F$, by the flow $(\Phi_t)_{t\in\mathbb{R}}$ in order to obtain a well-defined invariant function in $\ker_{L^2}(X_E)$.

The manifold Q_0 is obtained as a *principal orbit* for the H-action on the fiber F, in the sense of isometric actions on Riemannian manifolds (see [Lef, Section 2.3] and [DK00] for the standard terminology in Riemann group actions). The condition that Q_0 does not fiber over the circle is sufficient but obviously not necessary for mixing as the frame flow over a 3-dimensional hyperbolic manifold (the frame bundle is then an $\mathbb{S}^1$-bundle over the 5-dimensional unit tangent bundle of the manifold) has fiber isometric to the circle and is nevertheless mixing, and even exponentially mixing [HM79, Moo87, GK21]. Yet, it is simple to construct an example of an extension to a trivial $\mathbb{S}^1$-bundle that is ergodic and not mixing: this is satisfied by the flow

$$\Phi_t(x, \theta) := (\varphi_t(x), \theta + t \bmod 2\pi),$$

on $P := \mathcal{M} \times \mathbb{S}^1$ for instance, see Lemma 1.3 below for a proof. This condition can be refined a lot: for instance, if the fiber is isometric to $\mathrm{U}(r)$ (which obviously fibers over $\mathbb{S}^1$ via the determinant map), a much more precise sufficient condition can be given, see [CLMS22, Section 5.3].

Observe that, when a manifold is connected, a necessary condition for it to fiber over the circle is that its fundamental group surjects onto $\mathbb{Z}$. Compact semisimple Lie groups have finite fundamental group (see [DK00, Corollary 3.9.4]) so they never fiber over $\mathbb{S}^1$, which easily implies that the extension of a mixing volume-preserving Anosov flow to a principal G-bundle, where G is a compact semisimple compact Lie group, is ergodic if and only if it is mixing, see the next paragraph §1.3.3 for further details.

Let us eventually mention that some of the results of Theorem 1.2 are already contained in the literature. For instance, Brin showed that ergodicity is equivalent to $H = G$ in the case of principal G-bundles [BP74, Bri75b, Bri75a] and Dolgopyat [Dol02, Corollary 4.8] had already noticed (in the case of Anosov diffeomorphisms extensions) that semisimplicity of G implies that ergodicity is equivalent to mixing. However, the precise structure of $\ker_{L^2} X_E$ described in (1.5) seems to be new. More generally, results on ergodicity for isometric extensions of hyperbolic dynamics are spread out in the literature, hard to locate, and usually not written in a modern way.

Theorem 1.2 is also quite far from recent considerations on partially hyperbolic dynamics [HP06, Wil10], where dynamical systems may not arise from such a geometric framework. Let us also stress that, since our description of the transitivity group via Parry's free monoid has a more representation-theoretic flavour, it makes it also clearer that the (non-)ergodicity of the extended flow on principal bundles is intimately connected to the (non-)existence of reductions of the structure group of the bundle, or to flow-invariant sections on certain associated vector bundles. This will be extensively discussed in §1.4 when developing the non-Abelian Livšic theory, see Theorem 1.5 for instance. These questions (of (non-)existence of flow-invariant sections) may then be addressed by means of geometric identities such as the twisted Pestov/Weitzenböck identities, and this will play a crucial role later in our study of the ergodicity of the frame flow, see §3.

We will not attempt to prove Theorem 1.2 here and refer the interested reader to [Lef]. The proof heavily relies on microlocal analysis and the theory of anisotropic Sobolev spaces.

We end this paragraph with the following Lemma showing the importance of the condition that Q_0 does not fiber over the circle:

Lemma 1.3. *Let $P := \mathcal{M} \times \mathbb{S}^1$ equipped with the product flow $(\Phi_t)_{t\in\mathbb{R}} := (\varphi_t)_{t\in\mathbb{R}} \otimes (R_t)_{t\in\mathbb{R}}$, where $(R_t)_{t\in\mathbb{R}}$ denotes the rotation in the circle. Then, $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on P and, in this case, all these sets are equal: $F = \mathbb{S}^1 = Q_\star = H$. However, $(\Phi_t)_{t\in\mathbb{R}}$ is not mixing.*

Proof. First of all, the flow $(\Phi_t)_{t\in\mathbb{R}}$ cannot be mixing since the smooth function $f(x,\theta) := e^{i\theta}$ satisfies $X_P f = if$, $\langle f, \mathbf{1}_P\rangle_{L^2} = 0$ and thus the correlation

$$C_t(f,f) := \int_P f \circ \Phi_t \cdot \overline{f} \, \mathrm{d}\mu_P = e^{it}\|f\|^2_{L^2},$$

does not converge to 0. Let us now show ergodicity. Let $H \leqslant \mathbb{S}^1$ be the transitivity group. Then H is either equal to $\mathbb{S}^1$, in which case the flow is ergodic, or H is finite. Let us show that the latter is impossible. Indeed, if it were the case, then by the first part of Theorem 1.2 we would get a $\mathbb{Z}_k$-bundle over M for some integer $k \in \mathbb{Z}_{\geq 0}$ and the holonomy along every closed orbit γ in M would be given by $e^{2i\pi p_\gamma/k}$ for some $p_\gamma \in \{0, ..., k-1\}$. Now, if T is the period of a closed orbit, the holonomy is given by e^{iT} so it suffices to find a closed orbit with length T such that $T \notin (2\pi/k)\mathbb{Z}$. By [PP90], the number of closed orbits in the window $[2\pi n + 1/(3k), 2\pi n + 2/(3k)]$ grows exponentially in n, so there is a n_0 large enough such that there exists at least one closed orbit whose length is contained in that interval. This is a contradiction. □

1.3.3. *Structural result for principal bundles.* We now discuss the particular case where P is a principal G-bundle over $\mathcal{M}$, with G being a compact Lie group. We will say that a flow $(\Phi_t)_{t\in\mathbb{R}}$ is a *principal extension* of $(\varphi_t)_{t\in\mathbb{R}}$ to the bundle P if it satisfies the following two conditions:

$$\varphi_t \circ p = p \circ \Phi_t, \qquad R_g \circ \Phi_t = \Phi_t \circ R_g, \qquad \forall t \in \mathbb{R},\ \forall g \in G, \quad (1.6)$$

where $R_g : P \to P$ denotes the fiberwise right-action of the group. This is a particular case of isometric extensions. Such a flow then preserves a natural smooth measure ω which can be locally written as the (pullback of) measure on $\mathcal{M}$ wedged with the Haar measure on the group.

We let $P_\star := P_{x_\star}$ be the fiber over an arbitrary periodic point $x_\star$ used to define homoclinic orbits. By definition, the transitivity group H is a subgroup of the isometry group $\mathrm{Isom}(P_\star)$ of the fiber $P_\star$. Nevertheless, it is easier in practice to identify the fiber $P_\star$ with the group G itself: for that, we fix an arbitrary element $w_\star \in P_\star$ and then consider the map $\Psi : G \to P_\star, g \mapsto R_g w_\star$. By this identification, for $\gamma \in \mathcal{H}$, $\Psi^{-1}\rho(\gamma)\Psi$ acts as an isometry of G and commutes with the right action on G (by itself) so it is a left action on G (by itself) and can thus be identified with an element of the group G, namely $\Psi^{-1}\rho(\gamma)\Psi = L_g$ for some $g \in G$. We then define

$$H_{w_\star} := \overline{\{g \in G \mid \exists \gamma \in \mathbf{G}, L_g = \Psi^{-1}\rho(\gamma)\Psi\}}.$$

In other words, the groups $H_{w_\star} \leqslant G$ and $H \leqslant \mathrm{Isom}(P_\star)$ are simply conjugate by the map Ψ. The group $H_{w_\star}$ is a closed subgroup of the compact Lie group G, hence a Lie group. Note that changing the point $w_\star \in P_\star$ by another point $w'_\star$, one gets another subgroup $H_{w'_\star}$ that is conjugate to $H_{w_\star}$ in G. In order to simplify notations, we will simply write $H_\star := H_{w_\star} \leqslant G$. Also note that, in the case of a principal bundle, the set of non-principal (or *singular*) points is empty and the quotient space $H_\star\backslash G$ is a smooth manifold.

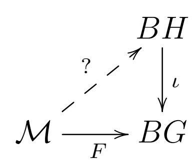
Corollary 1.4. *Under the above assumptions, there exists a smooth principal $H_\star$-bundle $Q \to M$ such that $w_\star \in Q$, $Q \subset P$ is a flow-invariant subbundle and the restriction of $(\Phi_t)_{t\in\mathbb{R}}$ to Q is ergodic. More generally, one has:*

$$\ker_{L^2} X_P \xrightarrow{\sim} L^2(H_\star\backslash G).$$

In particular, the principal bundle P admits a reduction of the structure group to $H_\star$. If $H_\star = G$, then the flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic. If $(\varphi_t)_{t\in\mathbb{R}}$ is mixing and G is semisimple, then $(\Phi_t)_{t\in\mathbb{R}}$ is also mixing.

Proof. The proof is a straightforward consequence of Theorem 1.2. For the last part of the statement relative to mixing, it suffices to observe that G cannot fiber over $\mathbb{S}^1$ when G is semisimple. Indeed, since G is connected, this would imply by the exact homotopy sequence that $\pi_1(G)$ surjects onto $\mathbb{Z}$ but compacntess and semisimplicity implies that its fundamental group is finite [DK00, Corollary 3.9.4]. □

Following Corollary 1.4, a sound strategy to prove ergodicity is therefore to assume that $H \subsetneq G$ is a strict subgroup and to seek a contradiction. There is obviously a first topological constraint provided by the fact that P must admit a reduction of its structure group to H. Recall that, a principal can be seen as the data of a (homotopy class of) map $F : \mathcal{M} \to BG$, where G denotes the classifying space of G. Then P is obtained as $P = F^*(EG)$, where EG is the weakly contractible space endowed with a proper free action of G such that $BG = EG/G$. The functor B being covariant, there is a natural map $\iota : BH \to BG$ and a *reduction of the structure group* is then simply a factorization of F by ι according to the following diagram:



This strategy was successfully used by Brin-Gromov [BG80] in order to show that the frame flow is ergodic on all odd-dimensional (with dimension $\neq 7$) negatively-curved Riemannian manifolds.

However, this topological argument is not always sufficient. The key idea then is to show that whenever $H \neq G$, one can produce additional flow-invariant geometric structures (sections of certain associated bundles) over $\mathcal{M}$ using representation theory and to prove by means of geometric arguments that such structures cannot actually exist.

1.4. **Non-Abelian Livšic theory.** Throughout this paragraph, $\mathcal{M}$ is a smooth closed manifold endowed with a flow $(\varphi_t)_{t\in\mathbb{R}}$ with infinitesimal generator $X \in C^\infty(\mathcal{M}, T\mathcal{M})$. We assume that the flow is Anosov and that it is transitive, namely, it admits a dense orbit but we do not make any volume-preserving assumption. Let $(\mathcal{E}, \nabla^{\mathcal{E}})$ be a smooth Hermitian (or Euclidean) vector bundle of rank r equipped with a unitary connection $\nabla^{\mathcal{E}}$ and let $F\mathcal{E} \to \mathcal{M}$ be the orthonormal frame bundle of $\mathcal{E}$ over $\mathcal{M}$. As before, we let $x_\star \in \mathcal{M}$ be an arbitrary periodic point and set $\mathcal{E}_\star := \mathcal{E}_{x_\star}$. Parallel transport of sections of $\mathcal{E}$ along flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ yields a partially hyperbolic flow of orthonormal frames $(\Phi_t)_{t\in\mathbb{R}}$ on $F\mathcal{E}$. Hence, by the previous paragraphs, we obtain a transitivity group $H \leqslant \mathrm{U}(\mathcal{E}_\star) \simeq \mathrm{U}(r)$ (or $H \leqslant \mathrm{SO}(r)$ in the real case) describing completely the dynamics of $(\Phi_t)_{t\in\mathbb{R}}$. Since the transitivity group H acts on $\mathcal{E}_\star \simeq \mathbb{F}^r$ (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$), we can define the set of vectors that are H-invariant, namely:

$$(\mathbb{F}^r)^H := \{v \in \mathbb{F}^r \mid \forall h \in H, hv = v\}.$$

The following non-Abelian Livšic Theorem relates H-invariant objects to flow-invariant objects:

Theorem 1.5 (Non-Abelian Livšic Theorem, Cekić-L. '21)**.** *The evaluation map*

$$\mathrm{ev} : C^\infty(SM, \mathcal{E}) \cap \ker \mathbf{X} \to (\mathbb{F}^r)^H, \qquad \mathrm{ev}(f) := f(z_\star) \in \mathbb{F}^r$$

is an isomorphism.

Remark 1.6. More generally, if $\mathfrak{o}$: Vect → Vect is one of the natural operations on the category of finite-dimensional vector spaces (symmetric, exterior power, tensor product, dual), one gets an induced representation $\rho_{\mathfrak{o}} : \mathbf{G} \to \mathrm{End}(\mathfrak{o}(\mathbb{F}^r))$ and the same result holds, that is,

$$\mathrm{ev} : C^\infty(SM, \mathfrak{o}(\mathcal{E})) \cap \ker \mathbf{X} \to (\mathfrak{o}(\mathbb{F}^r))^H$$

is an isomorphism.

Idea of proof. It is straightforward to check that the map is well-defined. Injectivity is also easy to obtain since, if $u \in \ker \mathbf{X} \cap C^\infty(M, \mathcal{E})$, one has $X_M|u|^2 = 0$ and thus $|u|$ is constant by ergodicity of the flow $(\varphi_t)_{t\in\mathbb{R}}$. Hence, if $\mathrm{ev}_\star(u) = u(x_\star) = 0$, we deduce that $u = 0$. Surjectivity is less easy to obtain and we refer to [CLb, Lemma 3.6] for a detailed proof. The idea is that, given $u_\star \in \mathcal{E}^\rho_{x_\star}$, one can construct by hand a Lipschitz-continuous section u on M such that $u(x_\star) = u_\star$ (by "pushing" $u_\star$ by the flow along homoclinic orbits). Using our regularity result with Bonthonneau [GL20, Theorem 1.4], we can then bootstrap this Lipschitz section to a smooth section. ☐

We now further assume that $\mathbb{F} = \mathbb{C}$ in order to simplify the discussion. It applies to the real case $\mathbb{F} = \mathbb{R}$ with the obvious modifications and also probably, more generally, to all vector bundles over a field of characteristic 0. The representation $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_\star) \simeq \mathrm{U}(r)$ can be decomposed into a sum of irreducible representations

$$\mathcal{E}_\star = \oplus_{i=1}^K \mathcal{E}_{\star,i}^{\oplus n_i}, \tag{1.7}$$

where $\mathcal{E}_{\star,i} \subset \mathcal{E}_\star$ and $n_i \geq 1$, each factor $\mathcal{E}_{\star,i}$ is $\mathbf{G}$-invariant and the induced representation on each factor is irreducible, and for $i \neq j$, the induced representations on $\mathcal{E}_{\star,i}$ and $\mathcal{E}_{\star,j}$ are not isomorphic.

We let $\mathbb{C}[\mathbf{G}]$ be the formal algebra generated by $\mathbf{G}$ over $\mathbb{C}$ and let $\mathbf{R} := \rho\left(\mathbb{C}[\mathbf{G}]\right)$. By Burnside's Theorem (see [Lan02, Corollary 3.3] for instance), one has that:

$$\mathbf{R} = \oplus_{i=1}^{K} \Delta_{n_i} \mathrm{End}(\mathcal{E}_i),$$

where $\Delta_{n_i} u = u \oplus ... \oplus u$ for $u \in \mathrm{End}(\mathcal{E}_i)$, the sum being repeated n_i-times. We introduce the *commutant* $\mathbf{R}'$ of $\mathbf{R}$, defined as:

$$\mathbf{R}' := \left\{ u \in \mathrm{End}(\mathcal{E}_\star) \mid \forall v \in \mathbf{R}, v^{-1} u v = u \right\}.$$

We then have:

Corollary 1.7. *The evaluation map*

$$\mathrm{ev} : C^\infty(SM, \mathrm{End}(\mathcal{E})) \cap \ker \mathbf{X} \to \mathbf{R}'$$

is an isomorphism. In particular these spaces have same dimension, that is

$$\dim \left(\ker \nabla_X^{\mathrm{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \mathrm{End}(\mathcal{E}))} \right) = \dim(\mathbf{R}') = \sum_{i=1}^{K} n_i^2.$$

Moreover, the vector bundle $\mathcal{E}$ breaks up as a direct sum

$$\mathcal{E} = \oplus_{i=1}^{K} \mathcal{E}_i^{\oplus n_i},$$

where $(\mathcal{E}_i)_{x_\star} = \mathcal{E}_{\star,i}$ as in (1.7) and each $\mathcal{E}_i$ is a flow-invariant vector bundle.

Proof. It suffices to observe that $\mathbf{R}'$ is precisely the set of H-invariant endomorphisms on $\mathbb{C}^r$, where the H-action is by conjugacy (the natural induced action on endomorphisms), and to apply Theorem 1.5 with Remark 1.6 and the functor $\mathfrak{o} : E \mapsto E \otimes E^* = \mathrm{End}(E)$. ☐

A connection $\nabla^{\mathcal{E}}$ is said to be *opaque* with respect to the flow $(\varphi_t)_{t\in\mathbb{R}}$ if it does not preserve any non-trivial subbundle $\mathcal{F} \subset \mathcal{E}$ by parallel transport along the flowlines of $(\varphi_t)_{t\in\mathbb{R}}$. We then obtain the following equivalent statements, whose proof is immediate:

Corollary 1.8. *The following statements are equivalent:*

(i) *The connection $\nabla^{\mathcal{E}}$ is opaque;*
(ii) $\ker(\nabla_X^{\mathrm{End}(\mathcal{E})}|_{C^\infty(\mathcal{M}, \mathrm{End}(\mathcal{E}))}) = \mathbb{C} \cdot \mathbf{1}_{\mathcal{E}}$*;*
(iii) *The representation $\rho : \mathbf{G} \to \mathrm{U}(\mathcal{E}_\star)$ is irreducible.*

Parallel transport with respect to $\nabla^{\mathcal{E}}$ along flowlines of $(\varphi_t)_{t\in\mathbb{R}}$ generates a cocycle C over $\mathcal{M}$ such that

$$C(x,t) : \mathcal{E}_x \to \mathcal{E}_{\varphi_t(x)}, \qquad (1.8)$$

is the parallel transport along the segment $(\varphi_s(x))_{s\in[0,t]}$. In a more general setting, we may consider $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$, two Hermitian vector bundles, equipped with two respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. Recall that if $\nabla^{\mathcal{E}_2} = p^*\nabla^{\mathcal{E}_1}$ are gauge-equivalent, for some unitary map $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$[5], then the two induced cocycles (1.8) satisfy the commutation relation:

$$\forall x \in \mathcal{M}, \forall t \in \mathbb{R} \qquad C_1(x,t) = p(\varphi_t x) C_2(x,t) p(x)^{-1}.$$

We say that such cocycles are *cohomologous*. In particular, given a closed orbit $\gamma = (\varphi_t x_0)_{t\in[0,T]}$ of the flow, one has

$$C_1(x_0, T) = p(x_0) C_2(x_0, T) p(x_0)^{-1},$$

i.e. the parallel transport maps are conjugate.

Definition 1.9. We say that the connections $\nabla^{\mathcal{E}_{1,2}}$ are *trace-equivalent* if for all *primitive* closed orbits $\gamma \in \mathcal{G}^\sharp$, we have:

$$\mathrm{Tr}(C_1(x_\gamma, \ell(\gamma))) = \mathrm{Tr}(C_2(x_\gamma, \ell(\gamma))), \tag{1.9}$$

where $x_\gamma \in \gamma$ is arbitrary and $\ell(\gamma)$ is the primitive period of γ.

This condition could be *a priori* obtained with $\mathrm{rank}(\mathcal{E}_1) \neq \mathrm{rank}(\mathcal{E}_2)$. The following result we obtained in [CLb] asserts that it is not the case and that trace-equivalence actually implies that the cocycles are cohomologous. It improves known results in Livšic cocycle theory (in particular [Par99, Sch99]).

Theorem 1.10 (Cekić-L., '21)**.** *Assume $\mathcal{M}$ is endowed with a smooth transitive Anosov flow. Let $\mathcal{E}_1, \mathcal{E}_2 \to \mathcal{M}$ be two Hermitian vector bundles over $\mathcal{M}$ equipped with respective unitary connections $\nabla^{\mathcal{E}_1}$ and $\nabla^{\mathcal{E}_2}$. If the connections are* trace-equivalent *in the sense of Definition 1.9, then there exists a fiberwise isometry $p \in C^\infty(\mathcal{M}, \mathrm{U}(\mathcal{E}_2, \mathcal{E}_1))$ such that:*

$$\forall x \in \mathcal{M}, \forall t \in \mathbb{R}, \qquad C_1(x,t) = p(\varphi_t x) C_2(x,t) p(x)^{-1}, \tag{1.10}$$

i.e. the cocycles induced by parallel transport are cohomologous. Moreover, $\mathcal{E}_2 \simeq \mathcal{E}_1$ are isomorphic.

Actually, for any given $L > 0$, it suffices to assume that the trace-equivalent holonomy condition (1.9) holds for all primitive periodic orbits of length $\geq L$ in order to get the conclusion of the theorem. Surprisingly, the rather weak condition (1.9) implies in particular that the bundles are isomorphic and the trace of the holonomy of unitary connections along closed orbits should allow one in practice to classify

[5]Here, we denote by $\mathrm{U}(\mathcal{E}_2, \mathcal{E}_1) \to \mathcal{M}$ the bundle of unitary maps from $\mathcal{E}_2 \to \mathcal{E}_1$. Of course, it may be empty if the bundles are not isomorphic.

vector bundles over manifolds carrying Anosov flows. Even more surprisingly, the rank of $\mathcal{E}_1$ and $\mathcal{E}_2$ might be *a priori* different and Theorem 1.10 actually shows that the ranks have to coincide.

Theorem 1.10 has an interesting straightforward corollary. A unitary connection is said to be *transparent* if the holonomy along all closed orbits is trivial.

Corollary 1.11. *Assume $\mathcal{M}$ is endowed with a smooth transitive Anosov flow. Let $\mathcal{E} \to \mathcal{M}$ be a Hermitian vector bundle over $\mathcal{M}$ of rank r equipped with a unitary connection $\nabla^{\mathcal{E}}$. Then, the following statements are equivalent:*

(i) $\nabla^{\mathcal{E}}$ *is transparent.*

(ii) $\mathrm{Tr}(C(x_\gamma, \ell(\gamma^\sharp))) = \mathrm{rank}(\mathcal{E})$ *for all primitive closed orbits* $\gamma^\sharp \in \mathcal{G}^\sharp$.

(iii) $\mathcal{E}$ *is trivial and trivialized by a smooth orthonormal family* $e_1, \ldots, e_r \in C^\infty(\mathcal{M}, \mathcal{E})$ *such that* $\nabla^{\mathcal{E}}_X e_i = 0$.

2. Isospectral connections, Geodesic Wilson loop operator

2.1. **Geodesic Wilson loop operator.**
Throughout this paragraph, (M, g) is a closed Riemannian manifold with negative sectional curvature (or whose geodesic flow on its unit tangent bundle is Anosov).

2.1.1. *Definition. General result.* For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we let $\mathbf{A}^{\mathbb{F}}$ denote the moduli space of connections on finite-dimensional Euclidean (resp. Hermitian) vector bundles over M. In other words, a point $a \in \mathbf{A}$ is a pair $a = ([E], [\nabla^E])$, where $[E] \to M$ is an isomorphism class of vector bundles and $[\nabla^E]$ is a class of connections up to gauge. A point a will be called a *virtual connection*. Let us also recall that the gauge group for unitary connections on a vector bundle E is $C^\infty(M, \mathrm{U}(E))$ and an element $p \in C^\infty(M, \mathrm{U}(E))$ acts by pullback $p^*\nabla^E := p\nabla^E(p^{-1}\bullet)$. As usual, the connection $p^*\nabla^E$ and ∇^E are said to be gauge-equivalent.

Under the negatively-curved assumption, the set of (primitive) free homotopy classes $\mathcal{C}^\sharp = \{c_1, c_2, ...\}$ is in 1-to-1 correspondance with the set of (primitive) closed geodesics. Given $a = ([E], [\nabla^E])$ and $c \in \mathcal{C}^\sharp$, we introduce the notation:

$$\mathbf{W}_c(a) := \mathrm{Tr}(C(x_\gamma, \ell(\gamma))), \tag{2.1}$$

where γ is the unique (primitive) closed geodesic in the class c, $x_\gamma \in \gamma$ is arbitrary, $\ell(\gamma)$ is the length of γ and, following (1.8), $C(x_\gamma, \ell(\gamma))$ denotes the holonomy of the connection ∇^E along γ. Note that (2.1) does not depend on the choice of representative in the class a, nor does it depend on the choice of point x_γ on γ.

Definition 2.1 (Geodesic Wilson Loop operator)**.** The operator

$$\mathbf{W} : \mathbf{A}^{\mathbb{F}} \to \ell^\infty(\mathcal{C}^\sharp), \qquad a \mapsto (\mathbf{W}_{c_1}(a), \mathbf{W}_{c_2}(a), ...), \tag{2.2}$$

is called the Geodesic Wilson Loop operator (GWL operator in short)

This operator is very similar to the *marked length spectrum* in the metric case, which assigns to a class $c \in \mathcal{C}^\sharp$ the length of the unique closed geodesic in this class. In the metric case, the marked length spectrum map is conjectured to be injective on all Anosov manifolds: this is known as the Burns-Katok conjecture [BK85] and was partially solved in some cases [Cro90, Ota90a, Ham99, GL19].

However, as we shall see below in Proposition 2.22, the GWL operator is never injective on even-dimensional manifolds. For surfaces, this is quite elementary as one can take $a_1 := (\mathbb{C}, d)$, the trivial line bundle equipped with the flat connection, and $a_2 := (\kappa, \nabla^{\mathrm{LC}})$, the canonical line bundle equipped with the Levi-Civita connection. It is then immediate to check that $\mathbf{W}(a_1) = \mathbf{W}(a_2) = \mathbf{1}$ but $a_1 \neq a_2$ since κ is not trivial. Nevertheless, we expect the following to be true:

Conjecture 2.2 (Cekić-L. '21)**.** *The Geodesic Wilson loop operator is injective on odd-dimensional negatively-curved Riemannian manifolds.*

We are still very far from a complete understanding of Conjecture 2.2. However, we are able to show that the geodesic Wilson loop operator is injective under a *low-rank* assumption on the vector bundle. We proved the following result in [CLa]:

Theorem 2.3 (Cekić-L. '22)**.** *There exists a integer $q_{\mathbb{F}}(n)$ (with $q_{\mathbb{R}}(n)$ even) satisfying*

$$q_{\mathbb{R}}(2^p) = 2^p, \qquad 2^p \leq q_{\mathbb{R}}(n) \leq n < 2^{p+1}, \qquad q_{\mathbb{C}}(n) = q_{\mathbb{R}}(n)/2, \quad (2.3)$$

such that for all smooth closed connected negatively-curved Riemannian manifolds (M^{n+1}, g), the geodesic Wilson loop operator

$$\mathbf{W} : \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)} \to \ell^\infty(\mathcal{C}^\sharp)$$

is injective.

The fact that $q_{\mathbb{C}}(n) = q_{\mathbb{R}}(n)/2$ simply means that complex vector bundles of rank r should be considered as real vector bundles of rank $2r$. The precise definition of $q_{\mathbb{F}}(n)$ is postponed to §2.3.1 but for now, let us mention that its exact value is unknown for $n \geq 48$, unless $n = 2^p$. For $n \leq 47$, one has $q_{\mathbb{R}}(n) = 2^p$, where p is defined by (2.3), that is:

$$q(2) = q(3) = 2, q(4) = ... = q(7) = 4, ..., q(32) = ... = q(47) = 32.$$

We note that the gauge class of a connection is uniquely determined from the holonomies along *all* closed loops [Bar91, Kob54] and that in mathematical physics our operator $\mathbf{W}$ is indeed known as the *Wilson loop* operator [Bea13, Gil81, Lol94, Wil74]. In stark contrast, Theorem 2.3 says that the *restriction to closed geodesics* of this operator already determines the gauge class of the connection. The ideas of proof of Theorem 2.3 will be explained in §2.5.1.

2.1.2. *Generic local injectivity.* There are also some particular cases for which we know that the GWL operator is injective. The following result shows the local injectivity in a neighborhood of a generic connection and was obtained in [CLb]:

Theorem 2.4 (Cekić-L. '21)**.** *Let (M, g) be a smooth Anosov Riemannian manifold of dimension ≥ 3 and let $E \to M$ be a smooth Hermitian vector bundle. Let $a_0 \in \mathbf{A}_E$ be a* generic *point. Then, the geodesic Wilson loop operator* (2.2) *is* locally injective *near a_0.*

By *local injectivity*, we mean the following: there exists $N \in \mathbb{N}$ (independent of a_0) such that $\mathbf{W}$ is locally injective in the C^N-quotient topology on $\mathbf{A}_E$. In other words, for any element $\nabla_0^E \in a_0$, there exists $\varepsilon > 0$ such that the following holds; if $\nabla_{1,2}^E$ are two smooth unitary connections such that $\|p_i^* \nabla_i^E - \nabla_0^E\|_{C^N} < \varepsilon$ for some $p_i \in C^\infty(M, \mathrm{U}(E))$, and $\mathbf{W}(\nabla_1^E) = \mathbf{W}(\nabla_2^E)$, then ∇_1^E and ∇_2^E are gauge-equivalent.

We say that a point a is *generic* if it enjoys the following two features:

(A) a is **opaque**. Following the terminology of §1.4, this means that for all $\nabla^E \in a$, the parallel transport map along geodesics does not preserve any non-trivial subbundle $F \subset E$ (i.e. F is preserved by parallel transport along geodesics if and only if $F = \{0\}$ or $F = E$). This was proved in Corollary 1.8 to be equivalent to the fact that $\mathbf{X} := \pi^* \nabla_X^{\mathrm{End}(E)}$ has 1-dimensional kernel $\mathbb{C} \cdot \mathbf{1}_E$ (on smooth sections). Here $\pi : SM \to M$ is the projection, $\nabla^{\mathrm{End}(E)}$ is the induced connection on the endomorphism bundle.

(B) a has **solenoidally injective generalized X-ray transform** $\Pi_1^{\mathrm{End}(E)}$ on twisted 1-forms with values in $\mathrm{End}(E)$. This last assumption is less easy to describe in simple geometric terms: roughly speaking, the X-ray transform is an operator of integration of symmetric m-tensors along closed geodesics. For vector-valued symmetric m-tensors, this might not be well-defined, and one needs a more general (hence, more abstract) definition involving the residue at $z = 0$ of the meromorphic extension of the family $\mathbb{C} \ni z \mapsto (-\mathbf{X} - z)^{-1}$, see §2.4.

We proved in [CL21a, CL21b] that in dimension $n \geq 3$, properties **(A)** and **(B)** are satisfied on an open dense subset $\omega \subset \mathbf{A}_E$ with respect to the C^N-quotient topology.[6].

The ideas of proof of Theorem 2.4 will be explained in §2.5.3. Although we do not explain it, we also proved in the same article [CLb]

[6]More precisely, there exists $N \in \mathbb{N}$ and a subset $\Omega \subset \mathcal{A}_E$ of the (affine) Fréchet space of smooth affine connections on E such that $\omega = \pi_E(\Omega)$ (where $\pi_E : \mathcal{A}_E \to \mathbf{A}_E$ is the projection) and

- Ω is invariant by the action of the gauge-group, namely $p^*\Omega = \Omega$ for all $p \in C^\infty(M, \mathrm{U}(E))$;
- Ω is open, namely for all $\nabla_0^E \in \Omega$, there exists $\varepsilon > 0$ such that if $\nabla^E \in \mathcal{A}_E$ and $\|\nabla^E - \nabla_0^E\|_{C^N} < \varepsilon$, then $\nabla^E \in \Omega$;
- Ω is dense, namely for all $\nabla_0^E \in \mathcal{A}_E$, for all $\varepsilon > 0$, there exists $\nabla^E \in \Omega$ such that $\|\nabla^E - \nabla_0^E\|_{C^N} < \varepsilon$;
- Connections in Ω satisfy properties **(A)** and **(B)**.

that the Geodesic Wilson loop Operator is injective on *almost-flat* connections, that is, connections whose curvature tensor is small compared to the minimum of the absolute value of the sectional curvature on the Riemannian manifold.

2.1.3. *Sums of line bundles.* Let $\mathbf{W}_1$ be the restriction of the Geodesic Wilson loop operator to line bundles. The moduli space $\mathbf{A}_1$ of all connections on line bundles carries a natural Abelian group structure using the tensor product. Recall that the topology of a line bundle is determined by its first Chern class that is

$$\mathrm{Vect}_1(M) \ni [L] \mapsto c_1([L]) \in H^2(M, \mathbb{Z})$$

is a bijection, where $[L]$ stands for a class of isomorphic line bundles, see [Bry93, Theorems 2.2.14 and 2.2.15]. When restricted to line bundles, the GWL operator $\mathbf{W}_1$ takes value in $\ell(\mathcal{C}^\sharp, \mathrm{U}(1))$, namely the set of sequences indexed by primitive free homotopy classes and it is clear that

$$\mathbf{W}_1 : \mathbf{A}_1 \to \ell^\infty(\mathcal{C}^\sharp, \mathrm{U}(1))$$

is a multiplicative group homomorphism. We have the following result, mainly due to Paternain [Pat09]:

Proposition 2.5 (Paternain)**.** *Let (M, g) be a smooth n-dimensional Anosov Riemannian manifold. If $n \geq 3$, then the restriction of the GWL operator to line bundles*

$$\mathbf{W}_1 : \mathbf{A}_1 \longrightarrow \ell^\infty(\mathcal{C}^\sharp), \tag{2.4}$$

is globally injective. Moreover, if $n = 2$ then:

$$\ker \mathbf{W}_1 = \left\{ ([\kappa^{\otimes k}], [\nabla^{\mathrm{LC}^{\otimes k}}]), k \in \mathbb{Z} \right\},$$

where $\kappa \to M$ denotes the canonical line bundle and ∇^{LC} is connection induced on κ by the Levi-Civita connection.

Proof. We start with a preliminary observation. Let (M, g) be a smooth closed Riemannian manifold of dimension ≥ 3 and let $\pi : SM \to M$ be the projection. Let $L_1 \to M$ and $L_2 \to M$ be two Hermitian line bundles. If $\pi^* L_1 \simeq \pi^* L_2$ are isomorphic, then $L_1 \simeq L_2$ are isomorphic. Indeed, the topology of line bundles is determined by their first Chern class. As a consequence, it suffices to show that $c_1(L_1) = c_1(L_2)$. By assumption, we have $c_1(\pi^* L_1) = \pi^* c_1(L_1) = c_1(\pi^* L_2) = \pi^* c_1(L_2)$ and thus it suffices to show that $\pi^* : H^2(M, \mathbb{Z}) \to H^2(SM, \mathbb{Z})$ is injective when $\dim(M) \geq 3$. But this is then a mere consequence of the Gysin exact sequence [BT82, Proposition 14.33].

Now, assume that $\mathbf{W}_1(a_1) = \mathbf{W}_1(a_2)$, where $a_1 \in \mathbf{A}_{L_1}$ and $a_2 \in \mathbf{A}_{L_2}$ are two classes of connections defined on two (classes of) line bundles. By Theorem 1.10, we obtain that the pullback bundles $\pi^* L_1$ and $\pi^* L_2$ are isomorphic, hence $L_1 \simeq L_2$ are isomorphic by the previous observation. Up to composing by a first bundle (unitary) isomorphism, we can therefore assume that $L_1 = L_2 =: L$. Let $\nabla_1^L \in a_1$ and $\nabla_2^L \in a_2$ be two representatives of these classes. They satisfy $\mathbf{W}(\nabla_1^L) = \mathbf{W}(\nabla_2^L)$. Combing Theorem 1.10 with [Pat09, Theorem 3.2], the GWL operator $\mathbf{W}_L$ is known to be globally injective for connections on the same fixed bundle. Hence ∇_1^L and ∇_2^L are gauge-equivalent.

For the second claim, $\mathrm{x} = ([L], a)$. If $\mathbf{W}_1(\mathrm{x}) = (1, 1, ...)$ (i.e. the connection is transparent), then by Theorem 1.10, one has that $\pi^* L \to SM$ is trivial. By the Gysin sequence [BT82, Proposition 14.33], this implies that $c_1(L)$ is divisible by $2g-2$, where g is the genus of M (see [Pat09, Theorem 3.1]), hence $[L] = [\kappa^{\otimes k}]$ for some $k \in \mathbb{Z}$. Moreover, the Levi-Civita connection on $\kappa^{\otimes k}$ is transparent and by uniqueness (see [Pat09, Theorem 3.2]), this implies that $a = ([\kappa^{\otimes k}], [\nabla^{\mathrm{LC}^{\otimes k}}])$. □

Remark 2.6. The target space in (2.4) is actually $\ell^\infty(\mathcal{C}^\sharp, \mathrm{U}(1))$ (sequences indexed by $\mathcal{C}^\sharp$ and taking values in $\mathrm{U}(1)$) which can be seen as a subset of $\mathrm{U}(\ell^\infty(\mathcal{C}^\sharp))$, the group of unitary operators of the Banach space $\ell^\infty(\mathcal{C}^\sharp)$ (equipped with the sup norm). Then $\mathbf{W}_1$ is a group homomorphism and Proposition 2.5 asserts that

$$\mathbf{W}_1 : \mathbf{A}_1 \to \mathrm{U}(\ell^\infty(\mathcal{C}^\sharp))$$

is a faithful unitary representation of the Abelian group $\mathbf{A}_1$.

We end this paragraph with a generalization of Proposition 2.5. There is a natural submonoid $\mathbf{A}' \subset \mathbf{A}$ which is obtained by considering sums of lines bundles equipped with unitary connections, that is:

$$\mathbf{A}' := \{a_1 \oplus ... \oplus a_k \mid k \in \mathbb{N}, a_i \in \mathbf{A}_1\}.$$

We then have the following:

Theorem 2.7 (Cekić-L., '21)**.** *Let (M, g) be a smooth Anosov Riemannian manifold of dimension ≥ 3. Then the restriction of the primitive trace map to $\mathbf{A}'$:*

$$\mathbf{W} : \mathbf{A}' \longrightarrow \ell^\infty(\mathcal{C}^\sharp)$$

is globally injective.

The proof is a combination of the non-Abelian Lišic theory (see Theorem 1.5) and Proposition 2.5.

2.2. **Application to isospectral connections.** Theorem 2.3 has an important corollary which we now state. Given $a \in \mathbf{A}^{\mathbb{F}}$, we can form the *Bochner Laplacian* (or *connection Laplacian*) $\Delta_E := \nabla_E^* \nabla_E$[7]. It is a self-adjoint operator on $L^2(M, E)$ with discrete non-negative spectrum. We define the *spectrum map* as

$$\mathbf{S} : \mathbf{A}^{\mathbb{F}} \ni a \mapsto \mathrm{spec}_{L^2}(\Delta_E) \subset \mathbb{R}_+^{\mathbb{Z}_{\geq 0}}. \tag{2.5}$$

Note that this is a well-defined map on the moduli space, that is, it does not depend on a choice of gauge for the connection. Following the celebrated paper of Kac [Kac66], *Can one hear the shape of a drum?* one can ask the following question: is the spectrum map (2.5) injective? In other words, does the spectrum of the Bochner Laplacian determine the connection up to gauge-equivalence?

This is the analogous question to the standard inverse spectral problem of recovering a metric g from the knowledge of the spectrum of the usual Hodge Laplacian Δ_g acting on functions. Among hyperbolic surfaces, it is known that the spectrum of the Hodge Laplacian does not determine the metric up to isometries by a result of Vigneras [Vig80]. Nevertheless, Sharafutdinov [Sha09] proved that the spectrum map is locally injective in a neighbourhood of a locally symmetric Riemannian space of negative curvature. Apart from negatively-curved spaces, other counterexamples were provided (earlier) by Milnor [Mil64], Sunada [Sun85] using covering spaces, and counterexamples to Kac's isospectral question also exist for piecewise smooth planar domains [GWW92] (for the Dirichlet Laplacian). The infinitesimal isospectral problem was also studied by various authors and turns out to be injective in negative curvature [GK80, CS98, PSU14a]. It is deeply connected to the *marked length spectrum* conjecture (also known as the Burns-Katok [BK85] conjecture), see [Cro90, Ota90a, GL19]. We refer to [Zel04, Zel14] for further details about Kac's standard isospectral problem for metrics.

Recall that a metric is said to have *simple length spectrum* if all closed geodesics have different lengths. This is a generic condition with respect to the metric, see [Abr70, Ano82]. We will derive an injectivity result for the spectrum map (2.5) on low-rank vector bundles whenever the underlying metric has simple length spectrum.

Theorem 2.8 (Cekić-L. '22). *For all smooth closed connected negatively-curved Riemannian manifold (M^{n+1}, g) with simple length spectrum,*

[7]Formally, this operator is only well-defined up to conjugacy if $a \in \mathbf{A}^{\mathbb{F}}$ is a virtual connection.

the spectrum map

$$\mathbf{S} : \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)} \to \mathbb{R}_+^{\mathbb{Z}_{\geq 0}}$$

is injective.

Theorem 2.8 seems to be the first result where Kac's inverse spectral problem can be fully solved with such an infinite-dimensional moduli space of geometric objects (here, connections on low-rank vector bundles). In particular, the situation seems to be very different from the metric case, where counterexamples are known to exist as discussed above. Also note that, in the general case, counter-examples to the injectivity of the spectrum map (2.5) were constructed by Kuwabara [Kuw90] using the Sunada method [Sun85] but on coverings of a given Riemannian manifold (the simple length spectrum condition is thus violated). As we shall see, it is natural to conjecture that the map (2.5) is injective on odd-dimensional manifolds, whenever the length spectrum of (M, g) is simple.

Similarly to Theorem 2.4, the injectivity of the spectrum maps (2.5) also holds near a generic point $a_0 \in \mathbf{A}$ in the moduli space, for sums of line bundles, and for almost-flat connections.

Proof of Theorem 2.8. Let (M^{n+1}, g) be a smooth closed connected Riemannian manifold with Anosov geodesic flow and simple length spectrum. Consider $a_1, a_2 \in \mathbf{A}$, two connections with same spectrum $\mathbf{S}(a_1) = \mathbf{S}(a_2)$ (note that we do not require that the vector bundles are the same; they could have different ranks *a priori*). The trace formula of Duistermaat-Guillemin [DG75a, Gui73] applied to the connection Laplacian Δ_{a_i} reads (when the length spectrum is simple):

$$\lim_{t \to \ell(\gamma_g(c))} (t - \ell(\gamma_g(c))) \sum_{j \geq 0} e^{-i\sqrt{\lambda_j^{(i)}(a_i)}t} = \frac{\ell(\gamma_g(c^\sharp))\mathbf{W}_c(a_i)}{2\pi |\det(\mathbb{1} - P_{\gamma_g(c)})|^{1/2}}, \quad (2.6)$$

where

- $\lambda_j^{(i)}$ are the eigenvalues of the connection Laplacian Δ_{a_i},
- $\mathcal{C}$ is the set of free homotopy classes of M (in negative curvature, this set is in one-to-one correspondance with the set of closed geodesics, that is, there exists exactly one closed geodesic $\gamma_g(c)$ in each free homotopy class $c \in \mathcal{C}$),
- $\mathcal{C}^\sharp$ is the set of primitive orbits (i.e. going twice around the same geodesic orbit is excluded),
- $\sharp : \mathcal{C} \to \mathcal{C}^\sharp$ is the operator giving the primitive orbit associated to an orbit,
- P_γ is the Poincaré map associated to the orbit γ and $\ell(\gamma)$ its length.

As a consequence, when the connection Laplacians are isospectral, the left-hand side of (2.6) is the same for both connections a_1 and a_2, and one obtains that

$$\mathbf{W}(a_1) = \mathbf{W}(a_2).$$

It then suffices to conclude by Theorem 2.3. □

2.3. **Fiberwise algebraic sections.** The key tool in the proof of Theorem 2.3 is the study of the (fiberwise) algebraicity of flow-invariant sections.

2.3.1. *Polynomial structures over spheres.* Let $n \geq 1$. We call *polynomial structure* on the sphere $\mathbb{S}^n$ an algebraic map $\mathbb{S}^n \to G(r)$, where $G(r)$ is a real algebraic variety depending on some parameter $r \in \mathbb{Z}_{\geq 0}$. The terminology will be justified later as these maps will naturally appear in the context of vector bundles over the sphere. We will be interested in the cases where $G = \mathbb{S}^r$, $\mathrm{SO}(r)$, $\mathrm{U}(r)$, $\mathrm{SU}(r)$ or $\mathrm{Gr}_{\mathbb{F}}(k, r)$, the Grassmannian of k-planes in $\mathbb{F}^r$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. By *algebraic* (or *polynomial*), we mean that the coordinates of $F : \mathbb{S}^n \to G(r)$ are polynomials in the v-variable for $v \in \mathbb{R}^{n+1}$, that is, the map is the restriction of a finite collection of polynomials (not necessarily homogeneous) on $\mathbb{R}^{n+1}$ to the sphere $\mathbb{S}^n$. For instance, a map $\mathbb{S}^n \to \mathrm{SO}(r)$ is algebraic if all the entries of the matrices defined over $\mathbb{S}^n$ are polynomials. Note that the Grassmannian can always be identified with the subset of $\mathrm{End}(\mathbb{F}^r)$ given by orthogonal projectors of rank k. We define $q_G(n)$ as the *least integer* $r \in \mathbb{Z}_{\geq 0}$ for which there exists a non-constant algebraic map $\mathbb{S}^n \to G(r)$.

When r is large, it is not difficult to construct non-constant polynomial mappings. For instance, the identity map $\mathbb{S}^n \to \mathbb{S}^n$ is an algebraic map of polynomial degree 1. Thus, the (non-)existence of such algebraic maps only becomes interesting when r is smaller than n. We shall see below, that the general notation $q_G(n)$ is actually irrelevant and that theses integers can be determined in terms of the single number $q(n) := q_{\mathbb{S}}(n)$. Computing the precise value of $q(n)$ is still an open question but an important result was obtained by Wood [Woo68]:

Theorem 2.9 (Wood '68)**.** *Let $n \in \mathbb{Z}_{\geq 0}$, $0 \leq r \leq n-1$. Assume that there exists a power of 2 among $\{r+1, ..., n\}$. Then, there exists no non-constant polynomial mapping $\mathbb{S}^n \to \mathbb{S}^r$.*

From Theorem 2.9, we have the following bounds:

$$n/2 < q(n) \leq n, \qquad q(2^k) = 2^k. \tag{2.7}$$

We point out that the classification of quadratic polynomial mappings was completely settled by Yiu [Yiu94]. In particular, this provides an *explicit* upper bound $q(n) \leq q_2(n)$, where $q_2(n)$ is the least integer such that there exists a quadratic polynomial mapping $\mathbb{S}^n \to \mathbb{S}^r$ (see [Yiu94, Theorem 4] for the precise value of $q_2(n)$). But the general explicit determination of $q(n)$ seems out of reach for the moment. Using the three Hopf fibrations $\mathbb{S}^3 \to \mathbb{S}^2, \mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ (which are quadratic polynomial maps), the first values of $q(n)$ can be easily computed and one gets

$$q(2) = q(3) = 2, \quad q(4) = ... = q(7) = 4, \quad q(8) = ... = q(15) = 15.$$

Although less obvious, it can also be proved that

$$q(16) = ... = q(31) = 16, \qquad q(47) = ... = q(32) = 32. \tag{2.8}$$

The proof relies on the Hopf construction which is defined as follows: given a bilinear map $F : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^t$ such that $|F(x,y)|^2 = |x|^2|y|^2$, one defines $H : \mathbb{R}^r \times \mathbb{R}^s \to \mathbb{R}^{t+1}$ by $H(x,y) := (|x|^2 - |y|^2, 2F(x,y))$ which yields a natural quadratic map $\mathbb{S}^{r+s-1} \to \mathbb{S}^t$ as $|H(x,y)| = 1$ for $|x|^2 + |y|^2 = 1$. Given n, a fixed odd integer, the Radon-Hurwitz number $\rho(n+1) - 1$ determines the maximal number of linearly independent vector fields on $\mathbb{S}^n$. We can thus construct a Hopf map $\mathbb{S}^{n+\rho(n+1)} \to \mathbb{S}^{n+1}$ by taking $F : \mathbb{R}^{n+1} \times \mathbb{R}^{\rho(n+1)} \to \mathbb{R}^{n+1}$ such that

$$F(x,y) = y_0 x + y_1 J_1 x + ... + y_{\rho(n+1)-1} J_{\rho(n+1)-1} x,$$

where $J_1, ..., J_{\rho(n+1)-1}$ are orthogonal almost-complex structures on $\mathbb{R}^{n+1}$ (this corresponds to the representation of the Clifford algebra $\mathrm{C}\ell_{\rho(n+1)-1}$ on $\mathbb{R}^{n+1}$). Of course, taking $n = 1, 3, 7$, one recovers the usual three Hopf fibrations $\mathbb{S}^3 \to \mathbb{S}^2, \mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$.

From the Hopf construction, one can easily cook up a quadratic map $\mathbb{S}^{31} \to \mathbb{S}^{24}$ and another quadratic map $\mathbb{S}^{24} \to \mathbb{S}^{16}$, thus giving a map $\mathbb{S}^{31} \to \mathbb{S}^{16}$ of polynomial degree 4, hence proving (2.8). The same construction works up to $n = 47$ but the first unknown value seems to be $q(48)$, that is, it is not known whether there exists a polynomial mapping $\mathbb{S}^{48} \to \mathbb{S}^{47}$ (necessarily of degree ≥ 3 by [Yiu94]), see [Tot07, Top of page 6] where this is discussed. We now derive some elementary properties on the numbers $q_G(n), q(n)$. The following holds:

Lemma 2.10. *$n \mapsto q(n)$ is non-decreasing.*

Proof. Let $n \leq m$ and assume that $q(n) > q(m)$. Denote by $F : \mathbb{S}^m \to \mathbb{S}^{q(m)}$ the non-constant polynomial mapping. As F is non-constant, we can find $v \in \mathbb{S}^{q(m)}$ such that $Z := F^{-1}(\{v\}) \subset \mathbb{S}^m$ is non-empty and not equal to $\mathbb{S}^m$. The set Z is a real affine variety so we can find a non-singular point $x_0 \in Z$. Then, there exists a geodesic circle $\mathcal{C} \subset \mathbb{S}^m$ passing through x_0 and locally (near x_0) not contained in Z. We can then take a linear embedding $\iota : \mathbb{S}^n \to \mathbb{S}^m$ such that $\mathcal{C} \subset \iota(\mathbb{S}^n)$. The map $F \circ \iota : \mathbb{S}^n \to \mathbb{S}^{q(m)}$ is non-constant algebraic so this contradicts the definition of $q(n)$. ☐

Lemma 2.11. *$q(n)$ is even.*

Proof. The Hopf construction always provides a quadratic mapping $\mathbb{S}^{2k+1} \to \mathbb{S}^{2k}$ by taking $r = 2k, s = 2, t = 2k$, $J \in \Lambda^2\mathbb{R}^{2k}$, an almost-complex structure and by setting for $x \in \mathbb{R}^{2k}, y \in \mathbb{R}^2$, $F(x,y) = y_1 x + y_2 Jx$. By construction, $|F(x,y)|^2 = |x|^2|y|^2$, and we thus get a quadratic map $\mathbb{S}^{2k+2-1} = \mathbb{S}^{2k+1} \to \mathbb{S}^{2k}$.

Now, we claim that if there is a non-constant algebraic mapping $F : \mathbb{S}^n \to \mathbb{S}^{2k+1}$, then the composition with the Hopf construction still provides a non-constant algebraic mapping $\mathbb{S}^n \to \mathbb{S}^{2k+1} \to \mathbb{S}^{2k}$. There are two cases: if $F(\mathbb{S}^n) \subset \mathbb{S}^{2k+1}$ is contained in a great sphere of $\mathbb{S}^{2k+1}$, then we actually have a non-constant algebraic map $F : \mathbb{S}^n \to \mathbb{S}^{2k}$. If not, it suffices to use that the preimage of a point $v \in \mathbb{S}^{2k}$ under the Hopf mapping $H : \mathbb{S}^{2k+1} \to \mathbb{S}^{2k}$ is the intersection of $\mathbb{S}^{2k+1}$ with a linear subspace of $\mathbb{R}^{2k+2}$ (in particular, it is contained in a great sphere), see [Yiu86, Theorem 1.4] for instance. ☐

Lemma 2.12. *Assume that $n \geq 2$. Then, one has:*

$$q_{\mathrm{SO}}(n) = q_{\mathrm{Gr}_{\mathbb{R}}(k)}(n) = 1 + q(n),$$
$$q_{\mathrm{U}}(n) = q_{\mathrm{SU}}(n) = q_{\mathrm{Gr}_{\mathbb{C}}(k)}(n) = 1 + q(n)/2.$$

Proof. We start with the following observation: for all $n \geq 2$, there are natural algebraic (quadratic) mappings

$$\mathbb{S}^{n-1} \to \mathrm{SO}(n), \qquad \mathbb{S}^{2n-1} \to \mathrm{SU}(n), \tag{2.9}$$

obtained by taking the symmetry with respect to the $\mathbb{R}$-span or $\mathbb{C}$-span of v, that is $2\pi_v - \mathbb{1}_{\mathbb{F}^n}$, where $\pi_v := \langle v, \bullet\rangle v$ is the orthogonal projection onto the real (resp. complex) line spanned by v, $\langle \bullet, \bullet \rangle$ is the standard Euclidean or Hermitian metric on $\mathbb{F}^n$. (Note that there is also a quadratic map $\mathbb{S}^{4n-1} \to \mathrm{Sp}(n)$ in the quaternionic case). (In the real case, the symmetry maps to $\mathrm{O}(n)$ instead of $\mathrm{SO}(n)$ when n is even but then it suffices to multiply by a constant coefficient matrix $\varepsilon(n) \in \mathrm{O}(n)$ with determinant -1 to get an element in $\mathrm{SO}(n)$. When n is odd, we just write $\varepsilon(n) = \mathbb{1}$.)

We first deal with the real case. Assume that we have a non-constant algebraic map $F : \mathbb{S}^n \to \mathbb{S}^{q(n)}$. Then, we can use (2.9) to produce an algebraic map $\mathbb{S}^n \to \mathrm{SO}(n)$ given by $v \mapsto \varepsilon(n)(2\pi_{F(v)} - \mathbb{1}_{\mathbb{R}^n})$. We claim that this is non-constant: indeed, if it were constant, then $\pi_{F(v)} = \langle \bullet, F(v)\rangle F(v)$ would be constant which is absurd. This gives

$q_{\mathrm{SO}}(n) \leq q(n) + 1$. On the other hand, assume that we have a non-constant algebraic map $F : \mathbb{S}^n \to \mathrm{SO}(q_{\mathrm{SO}}(n))$. Then, we can cook up an algebraic map $\mathbb{S}^n \to \mathrm{SO}(q_{\mathrm{SO}}(n)) \to \mathbb{S}^{q_{\mathrm{SO}}(n)-1}$, where the second arrow is obtained by taking one of the columns of the matrix. (This map is non-constant either, otherwise the whole matrix would be constant, which contradicts the assumption.) This gives $q_{\mathrm{SO}}(n) - 1 \geq q(n)$, and thus $q_{\mathrm{SO}}(n) = 1+q(n)$. More generally, the same argument works for $\mathrm{Gr}_{\mathbb{R}}(k)$ since there are natural maps $\mathrm{SO}(r) \to \mathrm{Gr}_{\mathbb{R}}(k,r)$ and $\mathrm{Gr}_{\mathbb{R}}(k,r) \to \mathrm{SO}(r)$ obtained by taking respectively the projection onto the linear span of the first k-th columns of the matrix, and the orthogonal symmetry with respect to the k-plane.

We now deal with the complex case. Observe that, given an algebraic map $p : \mathbb{S}^n \to \mathrm{U}(r)$, $\det p : \mathbb{S}^n \to \mathrm{U}(1) \simeq \mathbb{S}^1$ is also algebraic, hence constant since $n \geq 2$. Thus, up to rescaling by a constant factor, an algebraic map $\mathbb{S}^n \to \mathrm{U}(r)$ is the same as an algebraic map $\mathbb{S}^n \to \mathrm{SU}(r)$, which gives $q_{\mathrm{U}}(n) = q_{\mathrm{SU}}(n)$. As in the complex case, the same argument also gives the equality with $q_{\mathrm{Gr}_{\mathbb{C}}(k)}(n)$ by taking orthogonal projectors. Eventually, we observe that there are algebraic mappings

$$\mathbb{S}^n \to \mathrm{U}(q_{\mathrm{U}}(n)) \to \mathrm{SO}(2q_{\mathrm{U}}(n)) \to \mathbb{S}^{2q_{\mathrm{U}}(n)-1},$$

giving $2q_{\mathrm{U}}(n) - 1 \geq 2q_{\mathrm{U}}(n) - 2 \geq q(n)$ since $q(n)$ is even by Lemma 2.11. Moreover, we also have

$$\mathbb{S}^n \to \mathbb{S}^{q(n)} \to \mathrm{U}(1 + q(n)/2),$$

where the second arrow is given by the natural mapping (2.9) sending v to the orthogonal symmetry with respect to the $\mathbb{C}$-span of v. This gives $q(n) = 2q_{\mathrm{U}}(n) - 2$. □

As we shall see below in §2.5, polynomial maps $\mathbb{S}^n \to G(r)$ will naturally appear as maps all of whose coordinates are finite sums of spherical harmonics. Recall that the space of L^2 functions on $\mathbb{S}^n$ breaks up as

$$L^2(\mathbb{S}^n) = \oplus_{k\geq 0}\Omega_k(\mathbb{S}^n), \qquad \Omega_k(\mathbb{S}^n) := \ker(\Delta_{\mathbb{S}^n} - k(k+n-1)), \tag{2.10}$$

where $\Delta_{\mathbb{S}^n}$ is the Laplacian induced by the round metric on the sphere. Writing $\mathbf{H}_\ell(\mathbb{R}^{n+1})$ for the space of homogeneous harmonic polynomials of degree $\ell \geq 0$ in $\mathbb{R}^{n+1}$, the restriction map

$$r : C^\infty(\mathbb{R}^{n+1}) \ni f \mapsto f|_{\mathbb{S}^n} \in C^\infty(\mathbb{S}^n),$$

yields an isomorphism

$$r : \mathbf{H}_\ell(\mathbb{R}^{n+1}) \to \Omega_\ell(\mathbb{S}^n). \tag{2.11}$$

Hence, there is a natural identification

$$\oplus_{k=0}^{\ell} \Omega_k(\mathbb{S}^n) \simeq \oplus_{k=0}^{\ell} \mathbf{H}_k(\mathbb{R}^{n+1}) \subset \mathbb{R}_\ell[v], \tag{2.12}$$

where $\mathbb{R}_\ell[v]$ denotes the space of polynomials in the $v \in \mathbb{R}^{n+1}$ variable of degree $\leq \ell$. We will say that a function $f \in C^\infty(\mathbb{S}^n)$ has *finite Fourier content* if its L^2-decomposition (2.10) is a finite sum of spherical harmonics. By (2.12), a map $\mathbb{S}^n \to G(r)$ whose coordinates have finite Fourier content is in particular a polynomial map.

2.3.2. *Fourier degree of sections.* The decomposition (2.10) of smooth functions over $\mathbb{S}^n$ as a sum of spherical harmonics also applies to functions defined on the sphere bundle SM of a Riemannian manifold (M, g), or more generally to sections of the pull-back to SM of vector bundles over M.

More precisely, if $\mathcal{E} := \pi^* E$ denotes the pull-back to SM of a vector bundle E over M, its restriction to any fiber $S_xM \simeq \mathbb{S}^{n-1}$ is trivial, so the restriction of any section $f \in C^\infty(SM, \mathcal{E})$ to S_xM can be identified with a vector-valued function $f|_{S_xM} : \mathbb{S}^{n-1} \to E_x$. The vertical Laplacian $\Delta_{\mathbb{V}}$ acts on sections of $\mathcal{E}$ and satisfies $(\Delta_{\mathbb{V}} f)|_{S_xM} = \Delta(f|_{S_xM})$ for every $x \in M$ (where Δ is the Laplacian of the round sphere). Correspondingly, setting for $x \in M$,

$$\Omega_k(\mathcal{E})_x := \{f \in C^\infty(S_xM, \mathcal{E}) \mid \Delta_{\mathbb{V}} f = k(n+k-2)f\},$$

we get a vector bundle $\Omega_k(\mathcal{E}) \to M$ and the decomposition of any section $f \in C^\infty(SM, \mathcal{E})$ as $f = \sum_{j\geq 0} f_j$, with $f_j \in C^\infty(M, \Omega_j(\mathcal{E}))$. If the above sum is finite, i.e. $f = \sum_{j=0}^{k} f_j$ with $f_k \neq 0$, we say that f has *finite degree* k. If the above sum only contains even (resp. odd) spherical harmonics, i.e. $f = \sum_{j\geq 0} f_{2j}$ (resp. $f = \sum_{j\geq 0} f_{2j+1}$), we say that f is *even* (resp. *odd*).

From the description of Ω_k as the set of harmonic homogeneous polynomials on $\mathbb{R}^n$ of degree k, it easily follows that $\Omega_k(\mathcal{E})_x$ can be identified with $S_0^k(T_x^*M) \otimes E_x$ (trace-free symmetric tensors) by the tautological map

$$\pi_k^* : S_0^k(T_x^*M) \otimes E_x \to \Omega_k(\mathcal{E})_x$$

defined as follows: if $K \in S_0^k(T_x^*M)$ is a trace-free symmetric tensor of degree k and $s \in E_x$, one defines

$$\pi_k^*(K \otimes s)_{(x,v)} := K(v, \ldots, v)s_x, \qquad \forall v \in S_xM. \tag{2.13}$$

More generally, (2.13) identifies $S^k(T^*M) \otimes E$ with $\oplus_{j\geq 0} \Omega_{k-2j}(\mathcal{E})$ with the convention that $\Omega_j(\mathcal{E}) = \{0\}$ for $j < 0$.

Whenever E is equipped with a metric connection ∇^E, we can consider the pull-back connection $\pi^*\nabla^E$ on $\mathcal{E} := \pi^* E$. We set $\mathbf{X} :=$

$(\pi^*\nabla^E)_X$, where X is the geodesic vector field on SM, which is nothing but the generator of the parallel transport of sections of E with respect to ∇^E along geodesics. This operator may be seen to have the mapping property

$$\mathbf{X} : C^\infty(M, \Omega_k(\mathcal{E})) \to C^\infty(M, \Omega_{k-1}(\mathcal{E})) \oplus C^\infty(M, \Omega_{k+1}(\mathcal{E})) \quad (2.14)$$

and can thus be decomposed as a sum $\mathbf{X} = \mathbf{X}_- + \mathbf{X}_+$ onto each summand of (2.14). The operator $\mathbf{X}_+$ is elliptic and has finite-dimensional kernel (when M is compact) whose elements are called *twisted conformal Killing tensors* (in short, twisted CKTs). Moreover, the mapping property (2.14) ensures that $\mathbf{X}$ maps even (resp. odd) sections to odd (resp. even) sections. Elements in the kernel of $\mathbf{X}$ are *flow-invariant*; equivalently, they have the property of invariance under parallel transport along geodesic flowlines.

2.3.3. *Twisted symmetric tensors.* Given a section $u \in C^\infty(M, S^mT^*M \otimes E)$, the connection ∇^E produces an element

$$\nabla^E u \in C^\infty(M, T^*M \otimes S^mT^*M \otimes E).$$

In coordinates, if $(\mathbf{e}_1, ..., \mathbf{e}_r)$ is a local orthonormal frame for E and $\nabla^E = d + \Gamma$, for some one-form with values in skew-Hermitian matrices Γ, such that $\nabla^E \mathbf{e}_k = \sum_{i=1}^n \sum_{l=1}^r \Gamma_{ik}^l dx_i \otimes \mathbf{e}_l$, we have:

$$\begin{aligned} \nabla^E(\sum_{k=1}^r u_k \otimes \mathbf{e}_k) &= \sum_{k=1}^r \left(\nabla u_k \otimes \mathbf{e}_k + u_k \otimes \nabla^E \mathbf{e}_k\right) \\ &= \sum_{k=1}^r \left(\nabla u_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k u_l \otimes dx_i\right) \otimes \mathbf{e}_k, \end{aligned} \quad (2.15)$$

where $u_k \in C^\infty(M, S^mT^*M)$ and ∇ is the Levi-Civita connection. The symmetrization operator

$$\mathcal{S}^E : C^\infty(M, \otimes^m T^*M \otimes E) \to C^\infty(M, S^mT^*M \otimes E)$$

is defined by:

$$\mathcal{S}^E \left(\sum_{k=1}^r u_k \otimes \mathbf{e}_k\right) = \sum_{k=1}^r \mathcal{S}(u_k) \otimes \mathbf{e}_k,$$

where $u_k \in C^\infty(M, S^mT^*M)$ and in coordinates, writing

$$u_k = \sum_{i_1,...,i_m=1}^n u_{i_1...i_m}^{(k)} dx_{i_1} \otimes ... \otimes dx_{i_m},$$

we have

$$\mathcal{S}(dx_{i_1} \otimes ... \otimes dx_{i_m}) = \frac{1}{m!} \sum_{\pi \in \mathfrak{S}_m} dx_{\pi(i_1)} \otimes ... \otimes dx_{\pi(i_m)},$$

where $\mathfrak{S}_m$ denotes the group of permutations of order m. For the sake of simplicity, we will write $\mathcal{S}$ instead of $\mathcal{S}^E$. We can symmetrize (2.15) to produce an element $D^E := \mathcal{S}\nabla^E u \in C^\infty(M, S^{m+1}T^*M \otimes E)$ given in coordinates by:

$$D^E\left(\sum_{k=1}^r u_k \otimes e_k\right) = \sum_{k=1}^r \left(Du_k + \sum_{l=1}^r \sum_{i=1}^n \Gamma_{il}^k \sigma(u_l \otimes dx_i)\right) \otimes \mathbf{e}_k, \tag{2.16}$$

where $D := \mathcal{S}\nabla$ is the usual symmetric derivative of symmetric tensors. Elements of the form $Du \in C^\infty(M, S^{m+1}T^*M)$ are called *potential tensors*. By comparison, we will call elements of the form $D^E f \in C^\infty(M, S^{m+1}T^*M \otimes E)$ *twisted potential tensors*. The operator D^E is a first order differential operator and the expression of its principal symbol can be read off from (2.16), namely:

$$\begin{aligned}\sigma_{\text{princ}}(D^E)(x,\xi)\cdot\left(\sum_{k=1}^r u_k(x) \otimes \mathbf{e}_k(x)\right)& \\ &= \sum_{k=1}^r (\sigma_{\text{princ}}(D)(x,\xi)\cdot u_k(x)) \otimes \mathbf{e}_k(x) \\ &= i\sum_{k=1}^r \sigma(\xi \otimes u_k(x)) \otimes \mathbf{e}_k(x),\end{aligned}$$

where $\mathbf{e}_k(x) \in \mathcal{E}_x, u_k(x) \in S^m T_x^* M$ and the basis $(\mathbf{e}_1(x), ..., \mathbf{e}_r(x))$ is assumed to be orthonormal for the metric h on E. One can check that this is an injective map, which means that D^E is a left-elliptic operator and can be inverted on the left modulo a smoothing remainder. Its kernel is finite-dimensional and consists of smooth elements.

We recall the notation $(\pi^*\nabla^E)_X := \mathbf{X}$. The following remarkable commutation property holds (see [CL21a, Section 2]):

$$\forall m \in \mathbb{Z}_{\geq 0}, \quad \pi_{m+1}^* D^E = \mathbf{X}\pi_m^*. \tag{2.17}$$

The vector bundle $S^mT^*M \otimes E$ is naturally endowed with a canonical fiberwise metric induced by the metrics g and h which allows to define a natural L^2 scalar product. The L^2 formal adjoint $(D^E)^*$ of D^E is of divergence type (in the sense that its principal symbol is surjective for every $(x,\xi) \in T^*M \setminus \{0\}$, see [CL21a, Definition 3.1] for further details). We call *twisted solenoidal tensors* the elements in its kernel.

By ellipticity of D^E, for any twisted m-tensor f there exists a unique $p \in (\ker D^E)^\perp \cap C^\infty(M, S^{m-1}T^*M \otimes E), h \in C^\infty(M, S^mT^*M \otimes E)$ such that:

$$f = D^E p + h, \quad (D^E)^* h = 0. \tag{2.18}$$

This decomposition bears resemblance with the Hodge decomposition of differential forms; we also note that (2.18) could be extended to other regularities. We define $\pi_{\ker(D^E)^*} f := h$ as the L^2-orthogonal projection on twisted solenoidal tensors. This can be expressed as:

$$\pi_{\ker(D^E)^*} = \mathbb{1} - D^E[(D^E)^* D^E]^{-1}(D^E)^*, \tag{2.19}$$

where $[(D^E)^* D^E]^{-1}$ is the resolvent of the operator $(D^E)^* D^E$ (defined as follows: $[(D^E)^* D^E]^{-1} = 0$ on $\ker(D^E)^* D^E$, and on the L^2-orthogonal of $\ker(D^E)^* D^E$ it is genuinely given by the inverse of $(D^E)^* D^E$, well defined by Fredholm theory of elliptic operators).

2.3.4. *Twisted Pestov identity. Finiteness of the Fourier degree of flow-invariant sections.* The key identity for that is the twisted Pestov identity: it is a twisted version of an energy identity on the unit tangent bundle, first introduced by Mukhometov [Muk75, Muk81] and Amirov [Ami86], then in its classical form by Pestov and Sharafutdinov [PS88, Sha94] and finally stated in full generality by Guillarmou-Paternain-Salo-Uhlmann [GPSU16]. This identity has found several applications in the past twenty years, be it in the study of inverse spectral problems, see Croke-Sharafutdinov [CS98], or in tensor tomography [PSU13].

If (E, ∇^E) is a vector bundle with an orthogonal connection (over M), we write $\mathrm{End}_{\mathrm{sk}}(E)$ for skew-symmetric endomorphisms of E and

$$F_\nabla = F_{\nabla^E} = (\nabla^E)^{\circ 2} \in C^\infty(M, \Lambda^2 T^*M \otimes \mathrm{End}_{\mathrm{sk}}(E)),$$

for the curvature. Following [GPSU16, Section 3], we define $\mathcal{V} \to SM$ by $\mathcal{V}_{(x,v)} := v^\perp$. Let $\mathcal{F}^E \in C^\infty(SM, \mathcal{V} \otimes \mathrm{End}_{\mathrm{sk}}(E))$ be defined by the identity:

$$\langle \mathcal{F}^E(x,v)e, w \otimes e' \rangle := \langle (F_\nabla)_x(v,w)e, e' \rangle, \tag{2.20}$$

where $(x,v) \in SM, e, e' \in E_x, w \in \mathcal{V}_{(x,v)} = v^\perp$, and the metric on the right-hand side is the tensor product metric on $\mathcal{V}_{(x,v)} \otimes E_x$. Similarly, we will view the Riemannian curvature tensor as an operator on $\mathcal{V} \otimes E$, defined by the relation:

$$R(x,v)(w \otimes e) = (R_x(w,v)v) \otimes e, \quad w \in \mathcal{V}(x,v), e \in E_x.$$

Lemma 2.13. *We have, for any orthonormal basis* $\mathbf{e}_1, \dots, \mathbf{e}_n \in T_xM$, *and* $v \in S_xM$:

$$\mathcal{F}^E(x,v) = \sum_{i=1}^{n} \mathbf{e}_i \otimes F_{\nabla}(v, \mathbf{e}_i). \tag{2.21}$$

All the norms below are the L^2-norms. In order to avoid repetitions, we suppress the subscript L^2. We call the following identity, the *twisted Pestov identity*. It is slightly different from what [GPSU16] call a twisted identity but the following lemma can be easily recovered from [GPSU16, Proposition 3.5].

Lemma 2.14 (Localized Pestov identity). *Let* (M,g) *be a Riemannian manifold of dimension* n. *The following identity holds: for all* $k \in \mathbb{Z}_{\geq 0}$, $u \in C^\infty(M, \Omega_k \otimes \mathcal{E})$,

$$\begin{aligned}\frac{(n+k-2)(n+2k-4)}{n+k-3}\|\mathbf{X}_- u\|^2 - \frac{k(n+2k)}{k+1}\|\mathbf{X}_+ u\|^2 + \|Z(u)\|^2 \\ = \langle R\nabla^E_{\mathbb{V}} u, \nabla^E_{\mathbb{V}} u\rangle + \langle \mathcal{F}^E u, \nabla^E_{\mathbb{V}} u\rangle,\end{aligned} \tag{2.22}$$

where Z *is a first order differential operator which we do not make explicit.*

We set

$$Q^E_k(u,u) := \langle R\nabla^E_{\mathbb{V}} u, \nabla^E_{\mathbb{V}} u\rangle + \langle \mathcal{F}^E u, \nabla^E_{\mathbb{V}} u\rangle. \tag{2.23}$$

The following inequality holds:

Lemma 2.15. *Assume that* (M,g) *has sectional curvature bounded from above by* $-\delta < 0$. *Then:*

$$Q^E_k(u,u) \leq (-\delta k^2 + kq(E))\|u\|^2_{L^2}, \tag{2.24}$$

where $q(E)$ *only depends on the curvature tensor of* ∇^E. *In particular, there exists an integer* k_0 *such that* $Q^E_k(u,u) \leq 0$ *for every* $k \geq k_0$.

Proof. The proof is based on the Cauchy-Schwarz inequality. Given $u \in C^\infty(M, \Omega_k \otimes E)$, one has:

$$\begin{aligned}\langle R\nabla^E_{\mathbb{V}} u, \nabla^E_{\mathbb{V}} u\rangle &\leq -\delta\langle \nabla^E_{\mathbb{V}} u, \nabla^E_{\mathbb{V}} u\rangle \\ &\leq -\delta\langle \Delta^E_{\mathbb{V}} u, u\rangle \leq -\delta k(n+k-2)\|u\|^2.\end{aligned}$$

□

Now, if $f \in C^\infty(SM, \mathcal{E})$ satisfies $\mathbf{X}f = 0$, we write $f = \sum_{k\geq 0} f_k$ with $f_k \in C^\infty(M, \Omega_k(\mathcal{E}))$ which satisfy $\|f_k\|_{H^1} \to 0$ as $k \to \infty$. Moreover, by (2.14) writing $\mathbf{X} = \mathbf{X}_+ + \mathbf{X}_-$ gives

$$\mathbf{X}_+ f_k + \mathbf{X}_- f_{k+2} = 0, \qquad \text{for every } k \geq 0. \tag{2.25}$$

Applying (2.22) to $u = f_k$ we get for every $k > k_0$:

$$\frac{(n+k-2)(n+2k-4)}{n+k-3}\|\mathbf{X}_- f_k\|^2_{L^2} \leq \frac{k(n+2k)}{k+1}\|\mathbf{X}_- f_{k+2}\|^2_{L^2}.$$

In particular, this implies $\|\mathbf{X}_- f_k\|^2_{L^2} \leq \|\mathbf{X}_- f_{k+2}\|^2_{L^2}$ for every $k \geq k_0$. On the other hand, $\|\mathbf{X}_- f_k\|^2_{L^2}$ tends to 0 as $k \to \infty$ because $\|\mathbf{X}_- f_k\|^2_{L^2} \leq \|\mathbf{X} f_k\|^2_{L^2} \leq \|f_k\|^2_{H^1} \to 0$ by smoothness of f, so $\mathbf{X}_- f_k = 0$ for $k \geq k_0$. By (2.25) we then also have $\mathbf{X}_+ f_k = 0$ for $k \geq 2 + k_0$, so using (2.22) and (2.24) for $u := f_k$ shows that $f_k = 0$ for $k \geq 2 + k_0$. This gives the following result, originally proved in [GPSU16, Theorem 4.1]:

Corollary 2.16. *Every $f \in C^\infty(SM, \mathcal{E})$ with $\mathbf{X} f = 0$ has finite degree.*

This finiteness result on the Fourier degree will play an important role in what follows. In particular, in every fiber $S_x M$ (for all $x \in M$), such a flow invariant section f yields a *fiberwise polynomial structure*.

2.4. **Generalized twisted X-ray transform.** We explain the link between the widely studied notion of Pollicott-Ruelle resonances (see for instance [Liv04, GL06, BL07, FRS08, FS11, FT13, DZ16]) and the notion of (twisted) Conformal Killing Tensors introduced in the last paragraph. We also refer to [CL21a] for an extensive discussion about this.

2.4.1. *Definition of the resolvents.* Let $\mathcal{M}$ be a smooth manifold endowed with a vector field $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ generating an Anosov flow in the sense of (1.1). Throughout this paragraph, we will always assume that the flow is volume-preserving. It will be important to consider the dual decomposition to (1.1), namely

$$T^*(\mathcal{M}) = \mathbb{R}E_0^* \oplus E_s^* \oplus E_u^*,$$

where $E_0^*(E_s \oplus E_u) = 0, E_s^*(E_s \oplus \mathbb{R}X) = 0, E_u^*(E_u \oplus \mathbb{R}X) = 0$. As before, we consider a vector bundle $\mathcal{E} \to \mathcal{M}$ equipped with a unitary connection $\nabla^{\mathcal{E}}$ and set $\mathbf{X} := \nabla^{\mathcal{E}}_X$. Since X preserves a smooth measure $\mathrm{d}\mu$ and $\nabla^{\mathcal{E}}$ is unitary, the operator $\mathbf{X}$ is skew-adjoint on $L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu)$, with dense domain

$$\mathcal{D}_{L^2} := \left\{ u \in L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu) \mid \mathbf{X} u \in L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu) \right\}. \tag{2.26}$$

Its L^2-spectrum consists of absolutely continuous spectrum on $i\mathbb{R}$ and of embedded eigenvalues. We introduce the resolvents

$$\begin{aligned}\mathbf{R}_+(z) &:= (-\mathbf{X}-z)^{-1} = -\int_0^{+\infty} e^{-tz}e^{-t\mathbf{X}}\mathrm{d}t,\\ \mathbf{R}_-(z) &:= (\mathbf{X}-z)^{-1} = -\int_{-\infty}^{0} e^{zt}e^{-t\mathbf{X}}\mathrm{d}t,\end{aligned} \tag{2.27}$$

initially defined for $\Re(z) > 0$. (Let us stress on the conventions here: $-\mathbf{X}$ is associated to the positive resolvent $\mathbf{R}_+(z)$ whereas $\mathbf{X}$ is associated to the negative one $\mathbf{R}_-(z)$.) Here $e^{-t\mathbf{X}}$ denotes the propagator of $\mathbf{X}$, namely the parallel transport by $\nabla^{\mathcal{E}}$ along the flowlines of X. If $\mathbf{X} = X$ is simply the vector field acting on functions (i.e. $\mathcal{E}$ is the trivial line bundle), then $e^{-tX}f(x) = f(\varphi_{-t}(x))$ is nothing but the composition with the flow.

There exists a family $\mathcal{H}^s_\pm$ of Hilbert spaces called *anisotropic Sobolev spaces*, indexed by $s > 0$, such that the resolvents can be meromorphically extended to the whole complex plane by making $\mathbf{X}$ act on $\mathcal{H}^s_\pm$. The poles of the resolvents are called the *Pollicott-Ruelle resonances* and have been widely studied in the aforementioned literature [Liv04, GL06, BL07, FRS08, FS11, FT13, DZ16]. Note that the resonances and the resonant states associated to them are intrinsic to the flow and do not depend on any choice of construction of the anisotropic Sobolev spaces. More precisely, there exists a constant $c > 0$ such that $\mathbf{R}_\pm(z) \in \mathcal{L}(\mathcal{H}^s_\pm)$ are meromorphic in $\{\Re(z) > -cs\}$. For $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$), the space $\mathcal{H}^s_+$ (resp. $\mathcal{H}^s_-$) consists of distributions which are microlocally H^s in a neighborhood of E_s^* (resp. microlocally H^s in a neighborhood of E_u^*) and microlocally H^{-s} in a neighborhood of E_u^* (resp. microlocally H^{-s} in a neighborhood of E_s^*), see [FS11, DZ16]. These spaces also satisfy $(\mathcal{H}^s_+)' = \mathcal{H}^s_-$ (where one identifies the spaces using the L^2-pairing).

From now on, we will assume that s is fixed and small enough, and set $\mathcal{H}_\pm := \mathcal{H}^s_\pm$. We have

$$H^s \subset \mathcal{H}_\pm \subset H^{-s}. \tag{2.28}$$

and there is a certain strip $\{\Re(z) > -\varepsilon_{\mathrm{strip}}\}$ (for some $\varepsilon_{\mathrm{strip}} > 0$) on which $z \mapsto \mathbf{R}_\pm(z) \in \mathcal{L}(\mathcal{H}_\pm)$ is meromorphic (and the same holds for small perturbations of $\mathbf{X}$).

These resolvents satisfy the following equalities on $\mathcal{H}_\pm$, for z not a resonance:

$$\mathbf{R}_\pm(z)(\mp\mathbf{X}-z) = (\mp\mathbf{X}-z)\mathbf{R}_\pm(z) = \mathbb{1}_{\mathcal{E}}. \tag{2.29}$$

Given $z \in \mathbb{C}$ which not a resonance, we have:

$$\mathbf{R}_+(z)^* = \mathbf{R}_-(\overline{z}), \tag{2.30}$$

where this is understood in the following way: given $f_1, f_2 \in C^\infty(\mathcal{M}, \mathcal{E})$, we have

$$\langle \mathbf{R}_+(z) f_1, f_2 \rangle_{L^2} = \langle f_1, \mathbf{R}_-(\overline{z}) f_2 \rangle_{L^2}.$$

We will always use this convention for the definition of the adjoint. Since the connections are unitary and the flow preserves a smooth measure, the propagators $e^{-t\mathbf{X}}$ preserve the norm in $L^2(\mathcal{M}, \mathcal{E}; \mathrm{d}\mu)$. As a consequence, the formulas (2.27) converge when $\Re(z) > 0$ and thus we obtain the following statement that we record for future purposes:

$$\text{the resonance spectrum of } \pm\mathbf{X} \text{ is contained in } \{z \in \mathbb{C} \mid \Re(z) \leq 0\}. \tag{2.31}$$

A point $z_0 \in \mathbb{C}$ is a resonance for $-\mathbf{X}$ (resp. $\mathbf{X}$) i.e. is a pole of $z \mapsto \mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$) if and only if there exists a non-zero $u \in \mathcal{H}^s_+$ (resp. $\mathcal{H}^s_-$) for some $s > 0$ such that $-\mathbf{X}u = z_0 u$ (resp. $\mathbf{X}u = z_0 u$). If γ is a small counter clock-wise oriented circle around z_0, then the spectral projector onto the resonant states is

$$\Pi^\pm_{z_0} = -\frac{1}{2\pi i}\int_\gamma \mathbf{R}_\pm(z)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma (z \pm \mathbf{X})^{-1}\mathrm{d}z,$$

where we use the abuse of notation that $-(\mathbf{X}+z)^{-1}$ (resp. $(\mathbf{X}-z)^{-1}$) to denote the meromorphic extension of $\mathbf{R}_+(z)$ (resp. $\mathbf{R}_-(z)$).

We now briefly describe the resonances at $z = 0$. We can write in a neighborhood of $z = 0$ the following Laurent expansion (beware the conventions):

$$\mathbf{R}_+(z) = -\mathbf{R}_0^+ - \frac{\Pi_0^+}{z} + \mathcal{O}(z).$$

(Or in other words, using our abuse of notations, $(\mathbf{X}+z)^{-1} = \mathbf{R}_0^+ + \Pi_0^+/z + \mathcal{O}(z)$.) And:

$$\mathbf{R}_-(z) = -\mathbf{R}_0^- - \frac{\Pi_0^-}{z} + \mathcal{O}(z).$$

(Or in other words, $(z-\mathbf{X})^{-1} = \mathbf{R}_0^- + \Pi_0^-/z + \mathcal{O}(z)$.) As a consequence, these equalities define the two operators $\mathbf{R}_0^\pm$ as the holomorphic part (at $z = 0$) of the resolvents $-\mathbf{R}_\pm(z)$. We introduce:

$$\Pi := \mathbf{R}_0^+ + \mathbf{R}_0^-. \tag{2.32}$$

We have:

Lemma 2.17. *We have* $(\mathbf{R}_0^+)^* = \mathbf{R}_0^-, (\Pi_0^+)^* = \Pi_0^- = \Pi_0^+$. *Thus* Π *is formally self-adjoint. Moreover, it is nonnegative in the sense that for all* $f \in C^\infty(\mathcal{M}, \mathcal{E})$, $\langle \Pi f, f\rangle_{L^2} = \langle f, \Pi f\rangle_{L^2} \geq 0$. *Also,* $\langle \Pi f, f\rangle = 0$ *if and only if* $\Pi f = 0$ *if and only if* $f = \mathbf{X}u + v$, *for some* $u \in C^\infty(\mathcal{M}, \mathcal{E}), v \in \ker(\mathbf{X}|_{\mathcal{H}_\pm})$.

Proof. See [CL21a, Lemma 5.1]. □

We also record here for the sake of clarity the following identities:

$$\begin{aligned}
&\Pi_0^\pm \mathbf{R}_0^+ = \mathbf{R}_0^+ \Pi_0^\pm = 0, \Pi_0^\pm \mathbf{R}_0^- = \mathbf{R}_0^- \Pi_0^\pm = 0,\\
&\mathbf{X}\Pi_0^\pm = \Pi_0^\pm \mathbf{X} = 0, \mathbf{X}\mathbf{R}_0^+ = \mathbf{R}_0^+ \mathbf{X} = \mathbb{1} - \Pi_0^+,\\
&-\mathbf{X}\mathbf{R}_0^- = -\mathbf{R}_0^- \mathbf{X} = \mathbb{1} - \Pi_0^-.
\end{aligned} \tag{2.33}$$

2.4.2. *Generalized X-ray transform.* The discussion is carried out here in the closed case, but could also be generalized to the case of a manifold with boundary. We introduce the operator

$$\Pi := \mathbf{R}_0^+ + \mathbf{R}_0^-,$$

where $\mathbf{R}_0^+$ (resp. $\mathbf{R}_0^-$) denotes the holomorphic part at 0 of $-\mathbf{R}_+(z)$ (resp. $-\mathbf{R}_-(z)$) and Π_0^+ is the L^2-orthogonal projection on the (smooth) resonant states at 0. Such an operator was first introduced in the non-twisted case by Guillarmou [Gui17a]. The operator $\Pi + \Pi_0^+$ is the derivative of the (total) L^2-spectral measure at 0 of the skew-adjoint operator $\mathbf{X}$.

Definition 2.18 (Generalized X-ray transform of twisted symmetric tensors)**.** We define the generalized X-ray transform of twisted symmetric tensors as the operator:

$$\Pi_m := \pi_{m*}\left(\Pi + \Pi_0^+\right)\pi_m^*.$$

In what follows, we will mostly use this operator with $m = 1$. In this case, the operator Π_1 takes a one-form valued in some bundle $\mathcal{E}$, pulls it back on the unit tangent bundle to a spherical harmonic of degree 1 twisted by some bundle (π_1^*-operator), then "averages" this spherical harmonic along the geodesic flowlines ($(\Pi + \Pi_0^+)$-operator) and then selects the first spherical harmonic of this distribution in order to give a twisted one-form on the base manifold M (π_{1*}-operator). When we want to emphasize the dependence of Π_m on a connection ∇^E, we will write $\Pi_m^{\nabla^E}$.

Remark 2.19. This also allows to define a generalized (twisted) X-ray transform Π_m for an arbitrary unitary connection ∇^E on E. Indeed, it is not clear a priori if one sticks to the usual definition of the X-ray transform that one can find a "natural" candidate for the X-ray transform on twisted tensors. For instance, one could consider the map

$$\mathcal{C} \ni \gamma \mapsto I_m^{\nabla^E} f(\gamma) := \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} (e^{-t\mathbf{X}} f)(x_\gamma, v_\gamma) \mathrm{d}t \in E_{x_\gamma},$$

where $\gamma \in \mathcal{C}$ is a closed geodesic and $(x_\gamma, v_\gamma) \in \gamma$. However, this definition does depend on the choice of base point $(x_\gamma, v_\gamma) \in \gamma$ and it would no longer be true that $I_m^{\nabla^E}(D^E p)(\gamma) = 0$ unless the connection is transparent.

By (2.17) and (2.33), we have the equalities:

$$(D^E)^* \Pi_m = 0 = \Pi_m D^E, \tag{2.34}$$

showing that Π_m maps the set of twisted solenoidal tensors to itself. We say that the generalised X-ray transform is *solenoidally injective* (s-injective) on m-tensors, if for all $u \in C^\infty(SM, \mathcal{E})$ and $f \in C^\infty(M, S^m T^*M \otimes E)$

$$\mathbf{X}u = \pi_m^* f \implies \exists p \in C^\infty(M, S^{m-1}T^*M \otimes E) \text{ such that } f = D^E p. \tag{2.35}$$

We have the following:

Lemma 2.20. *The generalised X-ray transform is s-injective on m-tensors if and only if Π_m is injective on solenoidal tensors (if this holds, we say Π_m is* s-injective*).*

Proof. Assume that $\Pi_m f = 0$ and f is a twisted solenoidal m-tensor. Then

$$\langle \Pi_m f, f \rangle_{L^2} = \langle \Pi \pi_m^* f, \pi_m^* f \rangle_{L^2} + \langle \Pi_0^+ \pi_m^* f, \pi_m^* f \rangle_{L^2} = 0.$$

Both terms on the right hand side are non-negative by Lemma 2.17, hence both of them vanish, and the same Lemma implies that $\Pi \pi_m^* f = 0$ and $\Pi_0^+ \pi_m^* f = 0$. Thus $\mathbf{X}u = \pi_m^* f$ for some smooth u, so by the s-injectivity of generalised X-ray transform we obtain f is potential, which implies $f = 0$. The other direction is obvious by (2.33). □

Next, we show Π_m enjoys good analytical properties:

Lemma 2.21. *The operator*

$$\Pi_m : C^\infty(M, S^m T^*M \otimes E) \to C^\infty(M, S^m T^*M \otimes E)$$

is:

(i) *A pseudodifferential operator of order -1,*
(ii) *Formally self-adjoint and elliptic on twisted solenoidal tensors (its Fredholm index is thus equal to 0 and its kernel/cokernel are finite-dimensional),*

(iii) *Under the assumption that* Π_m *is* s*-injective, the following stability estimates hold:*

$$\forall s \in \mathbb{R},\ \forall f \in H^s(M, S^m T^*M \otimes E),\ \|\pi_{\ker(D^E)^*} f\|_{H^s} \leq C_s \|\Pi_m f\|_{H^{s+1}},$$

for some $C_s > 0$ *and for some* $C > 0$*:*

$$\forall f \in H^{-1/2}(M, S^m T^*M \otimes E),\ \langle \Pi_m f, f\rangle_{L^2} \geq C \|\pi_{\ker(D^E)^*} f\|^2_{H^{-1/2}}.$$

In particular, these estimates hold if (M, g) *has negative curvature and* ∇^E *has no twisted CKTs.*

Point (3) is a quantitative improvement of the statement: $\Pi_m f = 0, f \in \ker(D^E)^* \implies f = 0$, i.e. it provides a stability estimate for the X-ray transform (see Lemma 2.20 for the relation between Π_m and the X-ray transform).

Proof. The proof of the first two points follows from a rather straightforward adaptation of the proof of [Lef19b, Theorem 2.5.1] (see also [Gui17a] for the original arguments); we omit it. It remains to prove the third point.

The first estimate follows from (2), the elliptic estimate and the fact that Π_m is s-injective. The last estimate in the non-twisted case follows from [GKL19, Lemma 2.1] (or [Lef19b, Theorem 2.5.6]) and subsequent remarks; the twisted case follows by minor adaptations.

If (M, g) has negative curvature and ∇^E has no twisted CKTs, using Lemma 2.20 and by [GPSU16, Sections 4, 5] we get that Π_m is s-injective, proving the claim. □

2.5. **Strategy of proof.** In this paragraph, we explain Theorems 2.3, 2.4 and 2.7.

2.5.1. *Proof under low-rank assumption.* We first prove Theorem 2.3. Now that we have all the tools at our disposal, the proof is more a recollection of previous results.

Proof of Theorem 2.3. Given $a = ([E], [\nabla^E]) \in \mathbf{A}^{\mathbb{F}}$ over M, we can consider the lift $([\mathcal{E}], [\nabla^{\mathcal{E}}]) := \pi^*([E], [\nabla^E])$ to SM and the induced representation of Parry's free monoid

$$\rho : \mathbf{G} \to \mathrm{End}(\mathbb{F}^r)$$

given by parallel transport of sections along homoclinic orbits (for the geodesic flow on SM) with respect to $\nabla^{\mathcal{E}}$. Recall that the character of a representation is defined as $\chi_\rho := \mathrm{Tr}(\rho(\bullet))$. The key point is that equality of the Wilson loop operator $\mathbf{W}(a_1) = \mathbf{W}(a_2)$ implies that the representations (1.4) of Parry's free monoid $\rho_i : \mathbf{G} \to \mathrm{SO}(r_i) \simeq \mathrm{SO}(\mathcal{E}_{ix_\star})$ (or $\mathrm{U}(r_i)$ in the complex case) have the same character, see

[CLb, Proposition 3.18]. This is due to the fact that periodic orbits are dense and thus by the shadowing lemma for hyperbolic flows [FH19, Chapter 5], one can approximate homoclinic orbits by periodic ones. Hence, by standard algebra, these representations are isomorphic, see [Lan02, Chapter XVII, Corollary 3.8].

In other words, there is an isometry $p_\star : \mathcal{E}_{1z_\star} \to \mathcal{E}_{2z_\star}$ such that

$$\rho_1 = p_\star^{-1} \rho_2 p_\star. \tag{2.36}$$

This implies that the two bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ have same rank r and that both transitivity groups H_1 and H_2 are the same (up to conjugacy within $\mathrm{SO}(r)$ or $\mathrm{U}(r)$).

We can then apply the non-Abelian Livšic Theorem 1.5 with the bundle $\mathcal{E}_2 \otimes \mathcal{E}_1^*$ equipped with $\nabla^{\mathcal{E}_2} \otimes \nabla^{\mathcal{E}_1^*}$: indeed, using Remark 1.6, the induced representation

$$\rho_{\mathcal{E}_2 \otimes \mathcal{E}_1^*} : \mathbf{G} \to \mathrm{End}\left(\mathrm{Hom}(\mathcal{E}_{1z_\star}, \mathcal{E}_{2z_\star})\right)$$

given by $\rho_{\mathcal{E}_2\otimes\mathcal{E}_1^*}(g)p := \rho_2(g)^{-1} p \rho_1(g)$ admits $p_\star$ as a fixed point by (2.36). Hence, by Theorem 1.5, one obtains a well-defined flow-invariant isometry $p \in C^\infty(SM, \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))$ such that $p(z_\star) = p_\star$, that is, $\mathbf{X}p = 0$ where $\mathbf{X} = (\nabla^{\mathcal{E}_2} \otimes \nabla^{\mathcal{E}_1^*})_X$. In particular, this implies that $\mathcal{E}_1 = \pi^* E_1 \simeq \pi^* E_2 = \mathcal{E}_2$ are isomorphic over SM. If we further assume that (M, g) has negative sectional curvature, we can apply Corollary 2.16, to deduce that p has finite Fourier degree. Now, specifying to a point $x \in M$, the bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over S_xM are trivial and can thus be identified with $\mathbb{F}^r$. Hence p defines an algebraic map $p : \mathbb{S}^n \to \mathrm{SO}(r)$ (or $\mathrm{SU}(r)$ in the complex case) so this map has degree 0 by assumption on $r \leq q_{\mathbb{F}}(n)$ and thus p descends to a smooth isometry in $C^\infty(M, \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))$, parallel with respect to the induced connection on the homomorphism bundle, that is, $\nabla^{E_2 \otimes E_1^*} p = 0$. In turn, this is the same as the gauge-equivalence of the connections. □

2.5.2. *Counter-example on even-dimensional manifolds.* We now show that the Geodesic Wilson loop operator (2.2) is never injective on even-dimensional Riemannian manifolds. We denote by $\star$ the Hodge star operator on a Riemannian manifold (M^{2n}, g), and introduce $\Lambda^\pm := \{\alpha \in \Lambda^n T^*M \mid \star\alpha = \pm\alpha\}$ (if n is even) and $\Lambda^\pm := \{\alpha \in \Lambda^n T^*_{\mathbb{C}}M \mid \star\alpha = \pm i\alpha\}$ (if n is odd), the vector bundles of self-dual and anti self-dual n-forms equipped with the natural Levi-Civita connections $\nabla^\pm$, respectively. Set $a_+ := (\Lambda^+, \nabla^+)$ and $a_- := (\Lambda^-, \nabla^-)$.

Proposition 2.22. *Let (M^{2n}, g) be a negatively-curved Riemannian manifold. Then, the pairs a_+ and a_- are trace-equivalent along closed* *geodesics, that is,*

$$\mathbf{W}(a_+) = \mathbf{W}(a_-),$$

but $a_+ \neq a_-$.

Proof. We start by proving that a_+ and a_- are trace-equivalent along closed geodesics. We argue for n even: the proof for n odd is similar up to minor modifications. By Theorem 1.5, the equality $\mathbf{W}(\Lambda^+, \nabla^+) = \mathbf{W}(\Lambda^-, \nabla^-)$ is equivalent to the existence of an orthogonal map

$$p \in C^\infty(SM, \mathrm{Hom}(\pi^*\Lambda^+, \pi^*\Lambda^-))$$

that intertwines the operators $\pi^*\nabla^+_X$ and $\pi^*\nabla^-_X$, i.e.

$$\forall u \in C^\infty(SM, \pi^*\Lambda^+), \quad \pi^*\nabla^+_X u = p^{-1}\pi^*\nabla^-_X(pu), \tag{2.37}$$

so it suffices to exhibit this map. (Equivalently, (2.37) says that p is invariant with respect to the tensor product connection induced by $\pi^*\nabla^\pm$ on $\mathrm{Hom}(\pi^*\Lambda^+, \pi^*\Lambda^-)$ in the X-direction.) Introduce the commuting orthogonal projections $\Pi^\pm := \frac{\mathrm{id} \pm \star}{2}$ onto $\Lambda^\pm$, i.e. $(\Pi^\pm)^2 = \Pi^\pm$, $\Pi^+\Pi^- = \Pi^-\Pi^+ = 0$, and $\Lambda^\pm = \Pi^\pm\Lambda^n$. We claim that the map p defined as follows:

$$p(x, v)\alpha := 2\Pi^-(x)\big(v \wedge \star(\alpha \wedge v)\big), \quad \alpha \in \Lambda^+_x, \tag{2.38}$$

is flow-invariant and that for every $(x, v) \in SM$, $p(x, v) : \Lambda^+_x \to \Lambda^-_x$ is an isometry. (Note that in (2.38), we have implicitly identified v with a 1-form using the metric.)

In order to check the flow-invariance (2.37), it suffices to take a geodesic segment $\gamma \subset M$ tangent to v_1, v_2 at the points $x_1, x_2 \in M$, respectively, and observe that the parallel transport P_γ along γ satisfies for an arbitrary $\alpha \in \Lambda^+_{x_1}$:

$$\begin{aligned} p(x_2, v_2)P_\gamma\alpha &= 2\Pi^-(x_2)\big(v_2 \wedge \star(P_\gamma\alpha \wedge v_2)\big) \\ &= 2P_\gamma\Pi^-(x_1)\big(v_1 \wedge \star(\alpha \wedge v_1)\big) = P_\gamma p(x_1, v_1)\alpha. \end{aligned}$$

Here in the second equality we used that P_γ commutes with $\star$, Π^-, that it distributes over the wedge product, and that it satisfies $P_\gamma v_1 = v_2$. This completes the proof of the claim.

The isometry property follows from the following observation: take an orthonormal basis $(\mathbf{e}_i)_{i=1}^{2n}$ of T_xM such that $v = \mathbf{e}_1$. For an increasing $(n-1)$-tuple $I = (i_1, i_2, \ldots, i_{n-1}) \subset \{2, ..., 2n\}$, define $\alpha_I^\pm := \sqrt{2}\Pi^\pm\mathbf{e}_1 \wedge \mathbf{e}_I$; it is straightforward to check that $(\alpha_I^\pm)_I$ is an orthonormal basis of $\Lambda^\pm_x$. Moreover, a simple computation shows that

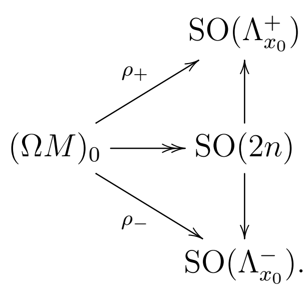
$$p(x, \mathbf{e}_1)\alpha_I^+ = \sqrt{2}\Pi^-(x)\big(\mathbf{e}_1 \wedge \star\big(\star(\mathbf{e}_1 \wedge \mathbf{e}_I) \wedge \mathbf{e}_1\big)\big) = \sqrt{2}\Pi^-(x)\big(\mathbf{e}_1 \wedge \mathbf{e}_I\big) = \alpha_I^-,$$

where in the second equality we used $\star\big(\star(\mathbf{e}_1 \wedge \mathbf{e}_I) \wedge \mathbf{e}_1\big) = \mathbf{e}_I$, so p is an isometry. This completes the proof that $\mathbf{W}(a_+) = \mathbf{W}(a_-)$.

We now show that $a_+ \neq a_-$. Actually, when $\dim M = 4$, $[\Lambda^+] \neq [\Lambda^-]$, that is, the bundles are not isomorphic: this follows from the fact that their first Pontryagin classes $p^\pm \in H^4(M,\mathbb{Z})$ satisfy the formula $p^+ - p^- = 4\mathbf{e}$, where $0 \neq \mathbf{e} \in H^4(M,\mathbb{Z})$ is the Euler class [Wal04, Chapter 6, Proposition 5.4(2)], which is strictly positive by the generalised Gauss-Bonnet theorem, see [Che55, Theorem 5].

However, we can always show that $a_+ \neq a_-$, that is, the connections are not gauge-equivalent. Indeed, the equality $a_+ = a_-$ would entail the existence of an isometry $p \in C^\infty(M, \mathrm{Hom}(\Lambda^+, \Lambda^-))$ such that $p^*\nabla^-(\bullet) = \nabla^+(\bullet) = p^{-1}\nabla^-(p\bullet)$. In particular, restricting to an arbitrary point $x_0 \in M$, we would obtain that the holonomy representations $\rho_\pm$ of the loop space ΩM (based at x_0) in $\mathrm{SO}(\Lambda^\pm_{x_0})$ are conjugate, that is $\rho_+(\gamma) = p^{-1}(x_0)\rho_-(\gamma)p(x_0)$ for every loop $\gamma \in \Omega M$ based at x_0. In particular, this holds for all $\gamma \in (\Omega M)_0$, that is, $\gamma \in \Omega M$ such that $[\gamma] = 0 \in \pi_1(M, x_0)$. In order to conclude, we then argue distinctly for each possible holonomy groups arising in negative curvature (see Berger's classification[8] [Ber53]):

(i) **Generic case**, $\mathrm{Hol}_0(M) = \mathrm{SO}(2n)$: in this case, the morphisms $\rho_\pm$ factor surjectively through $\mathrm{SO}(T_{x_0}M) \simeq \mathrm{SO}(2n)$, which is impossible since $\Lambda^\pm$ are irreducible non-isomorphic representations of $\mathrm{SO}(2n)$, see [BtD85, Chapter 6, Section 5.5]:



(ii) **Kähler case**, $\mathrm{Hol}_0(M) = \mathrm{U}(n) < \mathrm{SO}(2n)$: as in (1), it suffices to show that the restricted representations of $\mathrm{U}(n)$ on $\Lambda^\pm$ are not isomorphic and we will actually show that the restricted representations to the center $\mathrm{U}(1)$ of $\mathrm{U}(n)$ on $\Lambda^\pm$ are not isomorphic. Let $\rho : \mathrm{U}(1) \to \mathrm{SO}(2)$ be the standard representation; the standard representation $\mathrm{U}(1) \to \mathrm{End}(\mathbb{R}^{2n})$ is given by $\rho \oplus ... \oplus \rho$

[8] Berger's classification applies to simply connected manifolds, so we need to pass to the universal cover of (M, g), hence the restriction to the homotopically trivial loops $(\Omega M)_0$.

(n times). Recall that $R^+(G,\mathbb{R})$, the real representation semiring of a group G, is equipped with a (semi)ring homomorphism $\lambda_t : R^+(G,\mathbb{R}) \mapsto R(G,\mathbb{R})[[t]]$ given by (see [BtD85, Chapter 7])

$$\lambda_t(\rho) := 1 + \rho t + (\Lambda^2\rho)t^2 + ...$$

such that $\lambda_t(\rho_1 \oplus \rho_2) = \lambda_t(\rho_1) \cdot \lambda_t(\rho_2)$. Hence, in our case, for $\rho : \mathrm{U}(1) \to \mathrm{SO}(2)$ as above, we get

$$\begin{aligned}\lambda_t(\rho \oplus ... \oplus \rho) &= \lambda_t(\rho)^n \\ &= (1 + \rho t + t^2)^n = 1 + ... + c_n(\rho)t^n + ... + t^{2n},\end{aligned} \tag{2.39}$$

where $c_n(\rho)$ is a polynomial in the representation ρ such that

$$c_n(\rho) = \rho^{\otimes n} + \text{lower order terms.} \tag{2.40}$$

The representation of $\mathrm{U}(1)$ on $\Lambda^n\mathbb{R}^{2n}$ is precisely given by the coefficient $c_n(\rho)$ in the expansion (2.39) and (2.40) shows that this representation admits a weight n of multiplicity 1. As a consequence, when n is even, the two real representations $\Lambda^{\pm}$ cannot be isomorphic.

When n is odd, this argument needs to be slightly adapted since we need to complexify $\Lambda^n\mathbb{R}^{2n}$ in order to obtain the decomposition $\Lambda^n\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^+ \oplus \Lambda^-$. Hence, complexifying (2.40), we see that $\Lambda^n\mathbb{R}^{2n} \otimes_{\mathbb{R}} \mathbb{C}$ is given by $\rho^{\otimes n} \otimes_{\mathbb{R}} \mathbb{C} + \text{l.o.t.} = \rho_{\mathbb{C}}^{\otimes n} + \text{l.o.t.}$, where $\rho_{\mathbb{C}}$ stands for the complexification of ρ. Note that $\rho : \mathrm{U}(1) \to \mathrm{SO}(2)$ can already be seen as a 1-dimensional complex representation which we denote by η (in order to avoid confusion) and thus $\rho_{\mathbb{C}} = \eta \oplus \overline{\eta}$ (as complex representations), which yields:

$$\rho^{\otimes n} \otimes_{\mathbb{R}} \mathbb{C} = (\eta \oplus \overline{\eta})^{\otimes n} = \eta^{\otimes n} + \text{terms involving } \overline{\eta}.$$

(Here $\eta^{\otimes n}$ is the 1-dimensional complex representation associated to the character $z \mapsto z^n$.) Since $\eta^{\otimes n}$ appears with multiplicity 1, it implies that $\Lambda^+ \neq \Lambda^-$ in this case too.

(iii) **Quaternion Kähler case**, $\mathrm{Hol}_0(M) = \mathrm{Sp}(n).\mathrm{Sp}(1) < \mathrm{SO}(4n)$: same argument as in (2) (case n even) by restricting to $\mathrm{U}(1)$.

(iv) **Octonionic (or Cayley) hyperbolic plane**, that is, $\mathrm{Hol}_0(M) = \mathrm{Spin}(9) < \mathrm{SO}(16)$: the faithful representation $\mathrm{Spin}(9) \to \mathrm{SO}(16)$ is given by the spinor representation Δ_9 but by [Fri01, Theorem 1], it is known that the induced representations $\Lambda^p\Delta_9$ are multiplicity-free[9], so $\rho_+ \neq \rho_-$.

This completes the proof. ∎

[9] This can also be checked using the LiE program.

Remark 2.23. Set $\Lambda^{\text{odd/even}} := \Lambda^{\text{odd/even}}T^*M$, the bundle of odd/even differential forms, and consider $a_{\text{odd/even}} := (\Lambda^{\text{odd/even}}, \nabla^{\text{LC}}) \in \mathbf{A}^{\mathbb{R}}$, where ∇^{LC} is the Levi-Civita connection. Similarly to Proposition 2.22 it is straightforward to see $\mathbf{W}(a_{\text{odd}}) = \mathbf{W}(a_{\text{even}})$. If $n = 2$, then $\Lambda^{\text{odd}} \not\simeq \Lambda^{\text{even}}$ since $\Lambda^{\text{odd}} = \Lambda^1 T^*M$ and $\Lambda^{\text{even}} \simeq \mathbb{R}^2$ have distinct Euler classes; if $n > 2$, then $\Lambda^{\text{odd/even}}$ are in the 'stable regime' (they have rank $2^{n-1} > n$) so they are isomorphic if and only if stably isomorphic. Calculations indicate (private communication with O. Randal-Williams) that $\Lambda^{\text{odd/even}}$ are equal in K-theory and are hence isomorphic for $n > 2$. However, by a similar argument to Proposition 2.22, we expect that $a_{\text{odd}} \neq a_{\text{even}}$. Also, guided by the case of $\Lambda^{\text{odd/even}}$, we may expect that $\Lambda^+ \simeq \Lambda^-$ in higher dimensions.

The 4-dimensional case is enlightening and we detail here some computations. The two trace-equivalent pairs $(\Lambda^+, \nabla^+), (\Lambda^-, \nabla^-)$ provide a non-trivial example of a polynomial structure over $\mathbb{S}^3$ which turns out the be the usual Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.

Example 2.24. Assume that $\dim M = 4$. When restricted to the sphere S_xM over some $x \in M$ and identifying $\Lambda^\pm_x \simeq \mathbb{R}^3$, the map p defined by (2.38) provides a non-constant polynomial mapping $p : \mathbb{S}^3 \to \mathrm{SO}(3)$ which, when restricted to a column, then gives a quadratic polynomial map $\mathbb{S}^3 \to \mathbb{S}^2$. We claim that, in local coordinates, it is given by the standard Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ (as described in §2.3.1 for instance). Indeed, taking $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$, an orthonormal basis of T_xM, we can use the following orthonormal basis of $\Lambda^\pm_x$ (we freely identify vectors and 1-forms via the metric):

$$\alpha_1^\pm := \frac{\mathbf{e}_1 \wedge \mathbf{e}_2 \pm \mathbf{e}_3 \wedge \mathbf{e}_4}{\sqrt{2}}, \alpha_2^\pm := \frac{\mathbf{e}_1 \wedge \mathbf{e}_3 \mp \mathbf{e}_2 \wedge \mathbf{e}_4}{\sqrt{2}}, \alpha_3^\pm := \frac{\mathbf{e}_1 \wedge \mathbf{e}_4 \pm \mathbf{e}_2 \wedge \mathbf{e}_3}{\sqrt{2}}. \tag{2.41}$$

We are looking for an expression of $p(v)\left(\frac{\mathbf{e}_1\wedge\mathbf{e}_2+\mathbf{e}_3\wedge\mathbf{e}_4}{\sqrt{2}}\right)$ in the basis $(\alpha_1^-, \alpha_2^-, \alpha_3^-)$. A quick computation shows:

$$\begin{aligned}
&p(v)\left(\tfrac{\mathbf{e}_1\wedge\mathbf{e}_2+\mathbf{e}_3\wedge\mathbf{e}_4}{\sqrt{2}}\right)\\
&= 2\Pi^-\left(v \wedge \star\left(\tfrac{\mathbf{e}_1\wedge\mathbf{e}_2+\mathbf{e}_3\wedge\mathbf{e}_4}{\sqrt{2}} \wedge v\right)\right)\\
&= \sqrt{2}\Pi^-\big((v_1^2+v_2^2)\mathbf{e}_1\wedge\mathbf{e}_2 + (v_2v_3 - v_1v_4)\mathbf{e}_1\wedge\mathbf{e}_3 + (v_1v_3+v_2v_4)\mathbf{e}_1\wedge\mathbf{e}_4\\
&-(v_1v_3+v_2v_4)\mathbf{e}_2\wedge\mathbf{e}_3 + (v_2v_3 - v_1v_4)\mathbf{e}_2\wedge\mathbf{e}_4 + (v_3^2+v_4^2)\mathbf{e}_3\wedge\mathbf{e}_4\big)\\
&= \left(v_1^2+v_2^2-(v_3^2+v_4^2)\right)\alpha_1^- + 2\left(v_2v_3 - v_1v_4\right)\alpha_2^- + 2\left(v_1v_3+v_2v_4\right)\alpha_3^-.
\end{aligned}$$

Writing $z_0 := v_1 + iv_2 \in \mathbb{C}, z_1 = v_3 + iv_4 \in \mathbb{C}$, we thus get

$$p(v)\left(\tfrac{\mathbf{e}_1\wedge\mathbf{e}_2+\mathbf{e}_3\wedge\mathbf{e}_4}{\sqrt{2}}\right) = (|z_0|^2 - |z_1|^2, 2z_0z_1^*),$$

which is the usual expression for the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$.

2.5.3. *Proof under generic assumption.* We now explain the ideas behind the proof of Theorem 2.4.

Ideas of proof of Theorem 2.4. We assume that $a_0 \in \mathbf{A}_E$ is a generic point, that is, it is opaque and a has solenoidally injective generalized X-ray transform $\Pi_1^{\mathrm{End}(E)}$. We let a be a class of virtual connections close enough to a_0. The (virtual) connections a_0 and a on E induce a connection $a \otimes a_0^*$ (the *mixed* connection) on $\mathrm{End}(E) \simeq E \otimes E^*$ such that, if $a = a_0$, then $a_0 \otimes a_0^*$ is the usual connection induced by a_0 on $\mathrm{End}(E)$.

We let $\mathbf{X}_a$ be (a representative of) the operator induced on the bundle $\pi^*\mathrm{End}(E) \to SM$ by taking $\mathbf{X}_a := \pi^*\nabla_X^{\mathrm{End}(E)}$, where $\nabla^{\mathrm{End}(E)}$ is a representative of $a \otimes a_0^*$. Observe that for $a = a_0$, the opacity assumption implies that $\mathbf{X}_a$ has a unique simple Pollicott-Ruelle resonant states at $z = 0$ given by the span of the identity map $\mathbb{C} \cdot \mathbf{1}_{\mathcal{E}}$ (where $\mathcal{E} = \pi^*E \to SM$). Hence, by smoothness of simples resonances [Bon20], there is a well-defined map (for a close to a_0) $a \mapsto \lambda_a$, mapping a virtual connection a to the unique Pollicott-Ruelle resonance near $z = 0$ of $\mathbf{X}_a$. By construction, $\lambda_{a_0} = 0$ and $\lambda_a \leq 0$ by (2.31) (because the connection is unitary).

The key idea behind the proof of Theorem 2.4 is then to show the following: the map $a \mapsto \lambda_a$ is *strictly concave* near $a = a_0$. The proof of this fact relies on the following: by construction, $\partial_a \lambda_a|_{a=a_0} = 0$, and a computation shows that

$$\partial_a^2 \lambda_a|_{a=a_0}(h,h) = -\langle \Pi_1^{\mathrm{End}(E)} h, h\rangle_{L^2},$$

for all tangent vectors $h \in T_{a_0}\mathbf{A}_E$ (similarly to the metric case, these tangent vectors can be obtained as some divergence-free vectors). (The moduli space $\mathbf{A}_E$ can be shown to be a smooth Fréchet manifold near generic points.) Now, using the assumption that $\Pi_1^{\mathrm{End}(E)}$ is solenoidal injective for the connection a_0, we obtain by elliptic estimates that

$$\partial_a^2 \lambda_a|_{a=a_0}(h,h) \leq -C\|h\|_{H^{-1/2}}^2,$$

for some uniform constant $C > 0$. This is what proves the local strict concavity of the functional $a \mapsto \lambda_a$, and thus, we have a bound of the form $\|a - a_0\|^2 \leq C\lambda_a$, for some $C > 0$.

Then, the conclusion of the proof is immediate: if a is close to a_0 and is such that $\mathbf{W}(a_0) = \mathbf{W}(a)$, we know by the Livšic cocycle Theorem 1.10 that there exists a map $p \in C^\infty(SM, \mathrm{U}(\mathcal{E}))$ (where $\mathcal{E} = \pi^*E$) such that

$$\forall z \in SM, \forall t \in \mathbb{R}, \qquad C_0(z,t) = p(\varphi_t x) C(z,t) p(x)^{-1},$$

where C_0 and C denote the respective parallel transport along geodesic flowlines with respect to a_0 and a. But equivalently, this means that p is in the kernel of $\mathbf{X}_a$ and since p is smooth, it is a Pollicott-Ruelle resonant state for $\mathbf{X}_a$. Hence, $\lambda_a = 0$ and the above discussion involving the concavity argument yields $a = a_0$. This concludes the proof. $\square$

3. Frame flow ergodicity

3.1. **Statement of the problem. Main result.** Let (M^n, g) be a closed connected oriented n-dimensional Riemannian manifold with negative sectional curvature. Let $SM := \{(x,v) \in TM \mid |v|_g = 1\}$ be the unit tangent bundle over M and let

$$FM := \{(x, v, \mathbf{e}_2, ..., \mathbf{e}_n) \mid (x,v) \in SM, \\ (v, \mathbf{e}_2, ..., \mathbf{e}_n) \text{ oriented orthonormal basis of } T_xM\}$$

be the principal $\mathrm{SO}(n)$-bundle (over M) of oriented orthonormal frames.

We can also consider $\widehat{\pi} : FM \to SM, (x, v, \mathbf{e}_2, \ldots, \mathbf{e}_n) \mapsto (x, v)$ as a principal $\mathrm{SO}(n-1)$-bundle over SM, that is, a point $w \in FM$ over $(x,v) = \widehat{\pi}(w)$ corresponds to an orthonormal frame $(\mathbf{e}_2, \ldots, \mathbf{e}_n)$ of the orthogonal complement $v^\perp \subset T_xM$. Denoting by ∇ the Levi-Civita connection on M, the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is defined on SM by setting for $t \in \mathbb{R}, (x,v) \in SM$, $\varphi_t(x,v) := (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$, where $t \mapsto \gamma_{x,v}(t)$ is the unit-speed curve on M solving the geodesic equation $\nabla_{\dot{\gamma}_{x,v}}\dot{\gamma}_{x,v} = 0$ with initial conditions $\gamma_{x,v}(0) = x, \dot{\gamma}_{x,v}(0) = v$.

The *frame flow* $(\Phi_t)_{t\in\mathbb{R}}$ on FM is then defined as follows:

$$\Phi_t(x, v, \mathbf{e}_2, ..., \mathbf{e}_n) := (\varphi_t(x,v), \tau_{x,v}(t)\mathbf{e}_2, ..., \tau_{x,v}(t)\mathbf{e}_n),$$

where $\tau_{x,v}(t)$ denotes the parallel transport along the segment $\gamma_{x,v}([0,t])$ of geodesic with respect to the Levi-Civita connection ∇. Note that $(\Phi_t)_{t\in\mathbb{R}}$ is a typical exemple of an isometric extension over the geodesic flow as introduced in §1.2, so it is in particular a partially hyperbolic dynamical system.

Moreover, since the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ preserves the *Liouville measure* μ on SM, the frame flow $(\Phi_t)_{t\in\mathbb{R}}$ preserves a smooth measure ω on P induced by μ and the Haar measure on $\mathrm{SO}(n-1)$. It is therefore natural to study the ergodicity of the frame flow with respect to the smooth measure ω.

It was first shown by Brin [Bri75b] (for $n = 3$) and later by Brin-Gromov [BG80] (for n odd and different from 7) that negatively-curved n-dimensional manifolds have an ergodic frame flow. As already mentioned at the end of §1.3.3, once the dynamical framework is settled, the proof boils down to a (non-trivial) statement in algebraic topology on the classification of topological structures over even dimensional spheres. It is however hopeless to expect *all* negatively-curved manifolds to have an ergodic frame flow: indeed, it can be checked that Kähler manifolds of real dimension $n = 2m \geq 4$ such as compact quotients $\Gamma\backslash\mathbb{CH}^m$ of the complex hyperbolic space (where $\Gamma \leqslant \mathrm{Isom}(\mathbb{CH}^m)$

is a lattice) do not have an ergodic frame flow[10] due to the reduction of their holonomy group from $\mathrm{SO}(n)$ to $\mathrm{U}(m)$. This is discussed at length in §3.4.

Denoting by $\kappa_g(u \wedge v)$ the sectional curvature of the 2-plane spanned by $u, v \in TM$, we will say that (M,g) has *δ-pinched negative curvature* for some $\delta \in (0,1]$ if there exists a constant $C > 0$ such that the following inequalities hold:

$$-C \leq \kappa_g(u \wedge v) \leq -C\delta.$$

Note that, up to rescaling the metric, one can always assume that $C = 1$. Since Kähler manifolds are at most 0.25-pinched by a result of Berger [Ber60], Brin [Bri82] stated the following natural conjecture:

Conjecture 3.1 (Brin '82)**.** *If (M,g) is δ-pinched for some $\delta > 0.25$, then the frame flow is ergodic.*

More generally, Brin conjectures in the same article (see [Bri82, Conjecture 2.9]) that the frame flow should be ergodic as long as there is no reduction of the holonomy group of the manifold. However, up to now, ergodicity of the frame flow in dimension 7 and on even-dimensional manifolds was only known for nearly-hyperbolic manifolds, that is, manifolds with a pinching δ very close to 1: strictly greater than 0.8649... in even dimensions different from 8, due to Brin-Karcher [BK84], and strictly greater than 0.9805... in dimensions 7 and 8, due to Burns-Pollicott [BP03]. There has been no progress on Conjecture 3.1 in the past twenty years, until our result in [CLMS]. We introduce the number $\delta(n)$ given by:

$$\delta(4) = 0.2928..., \quad \delta(6) = 0.2823..., \quad \delta(7) = 0.4962...,$$
$$\delta(8) = 0.6212..., \quad \delta(134) = 0.6716...,$$

and for $n \geq 10, n \neq 134, n \equiv 2 \mod 4$:

$$\delta(n) = \frac{\frac{2}{3}\sqrt{3(n^2-1)}+\frac{1}{2}}{3(n+1)+\frac{2}{3}\sqrt{3(n^2-1)}-\frac{1}{2}}$$

and for $n \geq 12, n \equiv 0 \mod 4$:

$$\delta(n) = \frac{n+5+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}{3(n+1)+\frac{8}{3}\sqrt{(n-1)(n+2)}+\frac{2(n+2)(n+4)}{3(n+1)(n+6)}\left(5n+3+\frac{4}{3}\sqrt{3(n^2-1)}\right)}.$$

[10] This may be seen as follows: the complex structure J of a Kähler manifold commutes with parallel transport $\tau_{x,v}(t)$, so the set $\{(x, v, \mathbf{e}_2, ..., \mathbf{e}_n) \in FM \mid g_x(Jv, \mathbf{e}_2) \geq 0\}$ is invariant and has positive, but not full measure. In the even-dimensional case, the situation therefore requires additional care.

Asymptotically, $\delta(4\ell+2) \to_{\ell\to\infty} 0.2779...$ and $\delta(4\ell) \to_{\ell\to\infty} 0.5572....$ Moreover, the sequence $(\delta(4\ell+2))_{\ell\geq 2}$ is increasing and $\delta(10) = 0.2725...$, while $(\delta(4\ell))_{\ell\geq 3}$ is decreasing and $\delta(12) = 0.5948....$

Theorem 3.2 (Cekić-L.-Moroianu-Semmelmann, '21)**.** *Let (M^n, g) be a closed n-dimensional negatively-curved oriented Riemannian manifold with δ-pinched curvature and $n \geq 3$. Then the frame flow is ergodic if:*

(i) *n is odd and $n \neq 7$ [BG80],*
(ii) *n is even or $n = 7$ and $\delta > \delta(n)$.*

The numerical value of $\delta(n)$ is depicted in Figure 3 for $n \in \{4, ..., 150\}$.

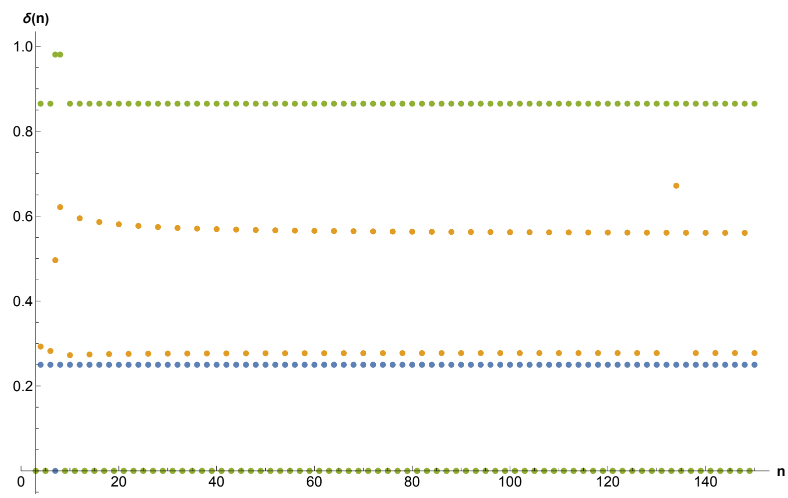


FIGURE 2. In green: the bounds existing in the literature [BG80, BK84, BP03]. In orange: the bounds provided by Theorem 3.2. In blue: the conjectural 0.25 threshold.

While former attempts to prove Conjecture 3.1 were mostly based on algebraic topology or on the geometry of the universal cover of the manifold, the strategy of proof for Theorem 3.2 is different and relies on the introduction of new ideas from Riemannian geometry. More precisely, we make systematic use of the twisted Pestov identity (2.22). Our argument consists of three distinct parts, each of them belonging to a different area of mathematics:

(i) **Hyperbolic dynamical systems:** the frame bundle $\widehat{\pi} : FM \to SM$ is a principal $\mathrm{SO}(n-1)$-bundle; by Theorem 1.2, using the hyperbolic structure of the geodesic flow, the non-ergodicity of the frame flow entails the existence of a transitivity group

$H \subsetneq \mathrm{SO}(n-1)$ and an H-principal subbundle $FM \supset Q \to SM$ which is an ergodic component of the frame flow, see §1.

(ii) **Algebraic topology:** in particular, restricting to an arbitrary point $x_0 \in M$, one obtains an H-subbundle $Q_{x_0} \to S_{x_0}M \simeq \mathbb{S}^{n-1}$ of the frame bundle of $\mathbb{S}^{n-1}$; this is already a strong topological constraint, called *reduction of the structure group* of the sphere, which rules out most possible cases for the subgroup H, see Corollary 1.4.

(iii) **Riemannian geometry:** when topology is not sufficient to rule out the existence of a structure group reduction, we show that, using the non-Abelian Livšic Theorem 1.5, and according to the possible values of H, one can produce a smooth flow-invariant section $f \in C^\infty(SM, \mathcal{E})$, where $\mathcal{E} = \pi^*\Lambda^p TM$ (with $p = 1, 2, 3$) or $\mathcal{E} = \pi^* S^2 TM$, and $\pi : SM \to M$ is the projection. In turn, the existence of such object can be ruled out by means of the twisted Pestov identity (2.22) whenever the pinching δ is sufficiently large, see §3.3.2.

Three exotic dimensions appear in this setting: $n = 7, 8$, and 134. They correspond to special topological structures possibly carried by the spheres $\mathbb{S}^6, \mathbb{S}^7$, and $\mathbb{S}^{133}$, respectively. The induced flow-invariant sections obtained in point (3) above then take values in $\pi^*\Lambda^p TM$ for $p = 2, 3$. It is more difficult to rule out the existence of such objects and our method requires a larger pinching than in other cases, see Figure 3 (for instance, the orange dot on the right-hand side corresponds to the case $n = 134$).

On the other hand, when $n \equiv 2 \mod 4$, the maximal number of linearly independent vector fields on the sphere $\mathbb{S}^{n-1}$ is 1, which simplifies our analysis and eventually yields a flow-invariant section of π^*TM, whereas in the case $n \equiv 0 \mod 4$, it is at least 3 and we get a flow-invariant orthogonal projector which is a section of $\pi^* S^2 TM$.

3.2. **Topological reduction.** Assuming that the frame flow is not ergodic, the transitivity group H is a strict subgroup of $\mathrm{SO}(n-1)$ and by Theorem 1.2 (and Corollary 1.4), there exists an H-principal bundle $Q \subset FM$ over SM. Restricting Q to the fiber $S_xM \simeq \mathbb{S}^{n-1}$ over some $x \in M$ defines an H-principal bundle $Q_x \subset F(\mathbb{S}^{n-1})$ over S_xM, i.e. a *structure group reduction* of the orthonormal frame bundle of the round sphere to H. We already described briefly reduction of structure groups in §1.3.3.

Alternatively, this can be seen as follows: since the two hemispheres of $\mathbb{S}^{n-1}$ are contractible (hence, any bundle over these is trivial), an $\mathrm{SO}(n-1)$-principal bundle over $\mathbb{S}^{n-1}$ is simply given by the data of a *clutching function* c at the equator $c : \mathbb{S}^{n-2} \to \mathrm{SO}(n-1)$, taking values in $\mathrm{SO}(n-1)$, and defined up to homotopy (equivalently, c is an element of the homotopy group $\pi_{n-2}(\mathrm{SO}(n-1))$). The bundle admits a reduction of its structure group to $H \leqslant \mathrm{SO}(n-1)$ if c can be factored through H (here $\iota : H \to \mathrm{SO}(n-1)$ is the embedding):

$$\begin{array}{ccc} & & H \\ & \overset{c_0}{\nearrow} & \big\downarrow \iota \\ \mathbb{S}^{n-2} & \xrightarrow[c]{} & \mathrm{SO}(n-1) \end{array}$$

This fact alone has already strong topological consequences. Indeed, using the work of Adams [Ada62], Leonard [Leo71], and Čadek–Crabb [ČC06], one can prove:

Proposition 3.3. *The following holds:*

(i) *If $n \geq 3$ is odd and $n \neq 7$, there is no reduction of the structure group of $\mathbb{S}^{n-1}$ to a strict subgroup of $\mathrm{SO}(n-1)$.*

(ii) *For all other $n \geq 4$, if there exists a reduction of the structure group of $\mathbb{S}^{n-1}$ to a strict subgroup H of $\mathrm{SO}(n-1)$, then up to conjugation, H is contained in one of the following subgroups K of $\mathrm{SO}(n-1)$:*
- *If $n = 7$, $K = \mathrm{U}(3) \subset \mathrm{SO}(6)$;*
- *If $n = 8$, $K = \mathrm{G}_2$ or $K = \mathrm{SO}(p) \times \mathrm{SO}(7-p) \subset \mathrm{SO}(7)$ with $p = 1, 2, 3$;*
- *If $n = 134$, $K = \mathrm{E}_7 \subset \mathrm{SO}(133)$ or $K = \mathrm{SO}(132) \subset \mathrm{SO}(133)$;*
- *If $n \equiv 2 \mod 4$, $n \neq 134$, $K = \mathrm{SO}(n-2) \subset \mathrm{SO}(n-1)$;*
- *If $n \equiv 0 \mod 4$, $n \neq 8$, $K = \mathrm{SO}(p) \times \mathrm{SO}(n-1-p) \subset \mathrm{SO}(n-1)$ with $1 \leq p \leq (n-2)/2$.*

Note that the Brin–Gromov result [BG80] about the ergodicity of the frame flow on negatively curved compact Riemannian manifolds in odd dimensions different from 7 is a direct consequence of Corollary 1.4 and Proposition 3.3 (1) (which was proved by Leonard [Leo71]).

Remark 3.4. Let us also point out that, unlike other topological reductions which *do* appear, we actually do not know whether the E_7-structure on $\mathbb{S}^{133}$ exists or not. This is still an open question.

We will now use the discussion of §1.4 in order to produce new flow-invariant geometric objects whenever the flow is not ergodic, that is, whenever the transitivity group H is strictly contained in $\mathrm{SO}(n-1)$. There is a natural associated vector bundle $\mathcal{V} = FM \times_\rho \mathbb{R}^{n-1} \to SM$, given by the canonical representation $\rho : \mathrm{SO}(n-1) \to \mathrm{Aut}(\mathbb{R}^{n-1})$, called the *normal bundle*. This bundle is also isomorphic to the *vertical* bundle of the spherical fibration $SM \to M$, that is, the vector bundle whose fiber at $(x, v) \in SM$ is the $(n-1)$-dimensional space $v^\perp \subset T_xM$. There is a natural way to parallel transport sections of this bundle along geodesic flowlines with respect to the (lift of the) Levi-Civita connection and, therefore, it makes sense to talk about flow-invariant sections.

The key point is then that for each group K occurring in Proposition 3.3 (2), one can find non-zero K-invariant vectors in some tensorial representations. More precisely:

- $\mathrm{U}(3) \subset \mathrm{SO}(6)$ preserves a non-zero 2-form in $\Lambda^2\mathbb{R}^6$;
- G_2 preserves a non-zero 3-form in $\Lambda^3\mathbb{R}^7$;
- $\mathrm{E}_7 \subset \mathrm{SO}(133)$ preserves a non-zero 3-form[11] in $\Lambda^3\mathbb{R}^{133}$;
- $\mathrm{SO}(n-2) \subset \mathrm{SO}(n-1)$ preserves a unit vector in $\mathbb{R}^{n-1}$;
- For $1 \leq p \leq (n-2)/2$, $\mathrm{SO}(p) \times \mathrm{SO}(n-1-p) \subset \mathrm{SO}(n-1)$ preserves the orthogonal projection of $\mathbb{R}^{n-1}$ onto $\mathbb{R}^p$.

Following §1.4 and applying the non-Abelian Livšic Theorem 1.5, this implies in turn that in all these cases, one can produce a smooth parallel (with respect to the dynamical connection) object over SM. In particular, this object is flow-invariant:

Theorem 3.5. *If the frame flow of M is not ergodic, there exists a non-vanishing flow-invariant section $f \in C^\infty(SM, \mathcal{E})$, where $\mathcal{E}$ is one of the following bundles:*

(1) $\mathcal{E} = \mathcal{V}$, *(2)* $\mathcal{E} = \Lambda^2\mathcal{V}$ *for* $n = 7$,
(3) $\mathcal{E} = \Lambda^3\mathcal{V}$ *for* $n = 8$ *or* $n = 134$, *(4)* $\mathcal{E} = S^2\mathcal{V}$.

For instance, if $n = 7$, $H \leqslant \mathrm{U}(3)$ and thus H fixes a an element $\Lambda^2\mathbb{R}^6$ which is an almost complex structure, namely, a skew-symmetric endomorphism squaring to $-\mathbb{1}_{\mathbb{R}^6}$. In turn, this gives rise to the existence of a flow-invariant section $f \in C^\infty(SM, \Lambda^2\mathcal{V})$ of constant algebraic type, that is, such that for every $(x, v) \in SM$, $f(x, v)$ is a skew-symmetric endomorphism on $v^\perp$ squaring to $-\mathbb{1}_{v^\perp}$. Our aim is now to show that, under some pinching condition, one can rule out the existence of such a flow-invariant geometric structure on the unit tangent bundle SM.

3.3. **Degree of flow-invariant sections.** The existence of flow invariant structures provided by Theorem 3.5 is not enough to obtain

[11] Indeed, the embedding of E_7 in SO(133) is obtained via the adjoint representation of E_7 on its Lie algebra $\mathfrak{e}_7 = \mathbb{R}^{133}$, so E_7 preserves the canonical 3-form of $\mathfrak{e}_7$.

a contradiction. Indeed, such objects do actually exist in some settings as we shall see below in §3.3.1. However, under some pinching condition and using Fourier analysis on SM, one can actually describe very accurately the analytic properties of flow-invariant objects. This is eventually what will give us the contradiction we are seeking.

3.3.1. *Examples. Link with Killing forms.* We describe some examples of flow-invariant sections of finite degree.

(1) **Tautological section.** The tautological section, defined by $s(x,v) := v$, is a flow-invariant section of the pull-back bundle π^*TM over SM of Fourier degree 1. Flow-invariance is understood as above in the sense that $\mathbf{X}s = 0$, where $\mathbf{X} = (\pi^*\nabla)_X$ and $\nabla = \nabla^{TM}$ is the Levi-Civita connection on TM. Equivalently, s corresponds to the identity endomorphism id_{TM}, viewed as a section of $S^1(T^*M)\otimes TM$, via the mapping (2.13), namely, $\pi^*(\mathrm{id}_{TM})_{(x,v)} := \mathrm{id}_{T_xM}(v) = v$. We use the notation $S^1(T^*M)$ to insist on the fact that one could consider more general objects ϕ in $S^p(T^*M) \otimes TM$ as in (2.13), and then the mapping to the unit tangent bundle would yield a section $(x,v) \mapsto \phi_x(v,...,v) \in \pi^*TM$.

(2) **Normal bundle.** The normal bundle $\mathcal{V}$ on SM is naturally identified with a subbundle of $\pi^*(TM)$ of codimension 1: as already mentioned, it is in fact the subbundle $s^\perp$ orthogonal to the tautological section s. Any section $f \in C^\infty(SM,\mathcal{V})$ which has Fourier degree 1 as a section of $\pi^*(TM)$ corresponds to an endomorphism ϕ of TM via the above identification $f_{(x,v)} = \pi^*(\phi)_{(x,v)} = \phi_x(v)$, which further satisfies $g(\phi(v),v) = 0$ for every $v \in SM$, i.e. it is skew-symmetric.

(3) **Exterior forms.** More generally, if ω is a $(p+1)$-form on M, it can be viewed as a p-form on SM taking values in the normal bundle $\mathcal{V}$ by defining $\pi^*\omega \in C^\infty(SM,\Lambda^p\mathcal{V})$ as $\pi^*\omega_{(x,v)} := v \lrcorner \omega_x$ (interior product with v). Conversely, a section of Fourier degree 1 of $\pi^*(\Lambda^p(T^*M))$ which takes values in the subbundle $\Lambda^p\mathcal{V}$ of $\pi^*(\Lambda^p(T^*M))$, corresponds to a $(p+1)$-form on M. Indeed, if $\omega \in C^\infty(M, S^1(T^*M) \otimes \Lambda^p(T^*M))$ has the property that $(\pi^*\omega)_{(x,v)} \in \Lambda^p(\mathcal{V})$ for every $v \in SM$, this just means that ω is totally skew-symmetric as a covariant $(p+1)$-tensor.

(4) **Flow-invariance and Killing forms.** It can be easily checked that the flow-invariance condition $\mathbf{X}\pi^*\omega = 0$ translates into

$$\nabla^{TM}_v \omega(v, \bullet, ..., \bullet) = 0,$$

for every $v \in TM$, that is, the covariant derivative of ω is totally skew-symmetric. Such a form is called a *Killing form* on M. A typical example where such a situation occurs with $p = 1$ is the case of a Kähler manifold (M, g, J), where $J^2 = -\mathbb{1}_{TM}$ is the almost-complex structure satisfying $\nabla^{TM} J = 0$. Since J is skew-symmetric, it defines an element of $\Lambda^2(T^*M)$. Setting $f_{(x,v)} := \pi^*(J)_{(x,v)} = J_x v$, the above discussion shows that $f \in C^\infty(SM, \mathcal{V})$ is flow-invariant. In turn, by §1.4, this shows that the transitivity group H *cannot be* equal to $\mathrm{SO}(n-1)$ since it has to fix at least one invariant vector, that is, $H \leqslant \mathrm{SO}(n-2)$, hence showing that the frame flow is not ergodic.

3.3.2. *Bounding the degree via the Pestov identity.* We will now explain the last steps in the proof of Theorem 3.2. Assuming that the frame flow is not ergodic, Theorem 1.2 (and Corollary 1.4) implies that the transitivity group H is a strict subgroup of $\mathrm{SO}(n-1)$, so by Theorem 3.5 there exists a non-vanishing section $f \in C^\infty(SM, \mathcal{E})$, where $\mathcal{E} = \Lambda^p\mathcal{V} \subset \pi^*(\Lambda^p(T^*M))$ (with $p = 1, 2, 3$) or $\mathcal{E} = S^2\mathcal{V} \subset \pi^*(S^2(T^*M))$, satisfying $\mathbf{X}f = 0$. The ultimate goal is to prove that f is of degree 1 under some pinching condition.

Corollary 2.16 shows that if the frame flow of (M, g) is not ergodic, the non-vanishing flow-invariant section $f \in C^\infty(SM, \mathcal{E})$ given by Theorem 3.5 has finite degree. The idea is that under a suitable pinching hypothesis, one can show that the section f has degree 1. By §3.3.1, it defines a Killing form on M, the existence of which is obstructed either by negative curvature, or by its algebraic properties. We now explain the remaining arguments leading to Theorem 3.2 in cases (1)–(3)[12] of Theorem 3.5.

End of the proof of Theorem 3.2. The proof is divided into three steps.

Step 1. We first show by topological arguments that the section f given by Theorem 3.5 is odd. For instance, in case (1), if non-zero, the restriction of the even part of f to a fiber of SM defines a constant length vector field on $\mathbb{S}^{n-1}$ of even degree, thus a polynomial map $\xi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ satisfying $\xi(v) = \xi(-v)$ for every $v \in \mathbb{S}^{n-1}$. In particular, the topological degree of ξ is even. On the other hand ξ is

[12]Case (4) is slightly more technical and we refer to [CLMS, Section 4] for further details.

homotopic to the identity via $\left[0, \frac{\pi}{2}\right] \ni t \mapsto \xi_t(v) := \cos(t)\xi(v) + \sin(t)v$, so its topological degree is 1, which is a contradiction.

In cases (2) and (3), since the space of K-invariant elements in $\Lambda^p\mathbb{R}^{n-1}$ is 1-dimensional for (K, p, n) in the list $(\mathrm{U}(3), 2, 7)$, $(\mathrm{G}_2, 3, 8)$, and $(\mathrm{E}_7, 3, 134)$, we deduce that f is either even or odd from the beginning. Assuming that it is even, we can again restrict the flow-invariant section f to a fiber of SM: this yields a p-form ω on $\mathbb{S}^{n-1}$ (with $p = 2$ for $n = 7$ and $p = 3$ for $n = 8$ or $n = 134$), satisfying $\omega(v) = \omega(-v)$ for every $v \in \mathbb{S}^{n-1}$. However, the tangent spaces $T_v\mathbb{S}^{n-1}$ and $T_{-v}\mathbb{S}^{n-1}$ at antipodal points are equal as vector spaces, but have opposite orientations, so for every $v \in \mathbb{S}^{n-1}$, the stabilizer of $\omega(v)$ would contain elements in $\mathrm{O}(v^\perp) \setminus \mathrm{SO}(v^\perp)$, which contradicts the last statement in Theorem 3.5.

Step 2. Under some curvature pinching assumption, we then show that the degree of f must be strictly smaller than 3, hence precisely equal to 1 by the first step. This is the key point of the proof, and is based on subtle estimates in the curvature term Q_k^E appearing in the right-hand side of the twisted Pestov identity (2.22). In order to keep the discussion simple, we will only give the main idea. Decomposing the flow-invariant section $f = f_k + f_{k-2} + ... + f_1$, where $f_k \neq 0$ and k is odd, and setting $u := f_k \neq 0$, we see by (2.14) that the flow-invariance $\mathbf{X}f = 0$ implies $\mathbf{X}_+u = 0$, that is, u is a twisted conformal Killing tensor. Applying (2.22), we thus obtain:

$$0 \leq \tfrac{(n+k-2)(n+2k-4)}{n+k-3}\|\mathbf{X}_-u\|^2 + \|Zu\|^2 = Q_k^E(u,u) \leq F(k,\delta)\|u\|^2, \quad (3.1)$$

where $F(k,\delta)$ is the maximum of the symmetric bilinear form Q_k^E on the unit sphere of $\Omega_k(\mathcal{E})$. As (2.24) indicates, it can be shown that for fixed δ, the sequence $k \mapsto F(k,\delta)$ decreases to $-\infty$. Hence, there is a $k(\delta)$ such that for all $k \geq k(\delta)$, $F(k,\delta) < 0$. In turn, this implies by (3.1) that $u \equiv 0$ which contradicts the assumption that $u \neq 0$. Now, it can be checked that the function $\delta \mapsto k(\delta)$ is a *decreasing* function, and that there exists $\delta(n) < 1$ such that for $\delta > \delta(n)$ sufficiently close to 1, $k(\delta) < 3$. Hence, we conclude that whenever $\delta > \delta(n)$, if the twisted conformal Killing tensor u in (3.1) is of degree ≥ 3, it must vanish identically. This is a contradiction and thus u is of degree 1 by the first step.

Step 3. Once we have established that the Fourier degree of the flow-invariant section $f \in C^\infty(SM, \Lambda^p\mathcal{V})$ is 1, following §3.3.1, we know that f defines a Killing $(p+1)$-form on M, with $p \in \{1, 2, 3\}$, that is a $(p+1)$-form ω such that $\nabla_v^{TM}\omega(v, \bullet, ..., \bullet) = 0$. The first two cases are ruled out by [BMS20], while we use some ad-hoc arguments in order to show that such special Killing 4-forms vanish identically (see [CLMS, Lemma 3.13]). ☐

3.4. **Unitary frame flows.** We now discuss the case of complex manifolds. Let (M^{2m}, g) be a smooth closed (compact, without boundary) Kähler manifold with *negative sectional curvature* and real dimension $2m \geq 4$. In this case, the holonomy group of M is given by $\mathrm{Hol}(M) = \mathrm{U}(m) < \mathrm{SO}(2m)$ and the transitivity group is always a subgroup $H \leqslant \mathrm{U}(m-1) < \mathrm{SO}(2m-1)$. As a consequence, the usual frame flow is *never* ergodic. However, one can consider the reduced frame flow of complex unitary frames and ask about its ergodicity. This is the content of the present paragraph.

We let $F_{\mathbb{C}}M \to M$ be the principal $\mathrm{U}(m)$-bundle of *unitary bases* over M. A point $w \in F_{\mathbb{C}}M$ over $x \in M$ is the data of an orthonormal basis $(v, \mathbf{e}_2, ..., \mathbf{e}_m)_x$ of (T_xM, h_x), where $h_x(\bullet, \bullet) = g_x(\bullet, \bullet) + ig_x(\bullet, J_x \bullet)$ is the canonical Hermitian inner product on the fibres of TM, and J is the integrable almost-complex structure on TM. Equivalently, we will see $F_{\mathbb{C}}M$ as a principal $\mathrm{U}(m-1)$-bundle over SM by the projection map $p : F_{\mathbb{C}}M \to SM$ defined as $p(v, \mathbf{e}_2, ..., \mathbf{e}_{m-1})_x = (x, v)$.

The *unitary frame flow* on $F_{\mathbb{C}}M$ is then defined as

$$\Phi_t(x, v, \mathbf{e}_2, ..., \mathbf{e}_m) := (\varphi_t(x, v), P_{\gamma_{x,v}(t)}\mathbf{e}_2, ..., P_{\gamma_{x,v}(t)}\mathbf{e}_m), \tag{3.2}$$

where $P_{\gamma_{x,v}(t)} : T_xM \to T_{\gamma_{x,v}(t)}M$ is the parallel transport along $\gamma_{x,v}$ with respect to the Levi-Civita connection.

While the geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is well-known to be ergodic [Hop36, Ano67] with respect to the Liouville measure on SM, the ergodicity of the frame flow $(\Phi_t)_{t\in\mathbb{R}}$ with respect to the natural flow-invariant smooth measure ω, that is, the Liouville measure wedged with the Haar measure on the group $\mathrm{U}(m-1)$, is still an open question (similarly to the real case). It was proved by Brin-Gromov [BG80] that this flow is ergodic whenever $\dim_{\mathbb{C}} M$ is odd but the even-dimensional case has remained open so far. We bring here a first positive answer when $\dim_{\mathbb{C}} M$ is even.

Recall that the *holomorphic curvature* of (M, g) is defined as

$$H(X) := R(X, JX, JX, X), \tag{3.3}$$

for all unitary vectors $X \in TM$, where R is the Riemann curvature tensor of (M, g). The manifold is said to be *holomorphically λ-pinched* if there exists a constant $C > 0$ such that

$$-C \leq H \leq -C\lambda. \tag{3.4}$$

The manifold is said to be *strictly* λ-pinched if the inequalities in (3.4) are strict.

In order to state our main result, we need to introduce a specific function $m \mapsto \lambda(m)$, defined for even numbers $m \geq 4$ by

$$\lambda(4) = \frac{33+40\sqrt{7}}{42+40\sqrt{7}} \simeq 0.9391...,$$

and for $m \geq 6$,

$$\lambda(m) = \frac{6\alpha_{m,k} + 32\beta_{m,k} + 6 + \gamma_{m,k}\left(6\alpha_{m,k-1} + 16\beta_{m,k-1} + \frac{29}{8} + \frac{3}{2}\delta_{m,k-1}\right) + 3\delta_{m,k}}{9\alpha_{m,k} + 32\beta_{m,k} - 6 + \gamma_{m,k}\left(9\alpha_{m,k-1} + 16\beta_{m,k-1} - \frac{29}{8} - \frac{3}{2}\delta_{m,k-1}\right) - 3\delta_{m,k}}, \tag{3.5}$$

where

$$\alpha_{m,k} := k(2m+k-2),$$
$$\beta_{m,k} := (k(2m+k-2)(2m-1))^{1/2},$$
$$\gamma_{m,k} := \frac{(2m+k-2)(m+k-2)k}{(2m+k-3)(m+k-1)(k-1)},$$
$$\delta_{m,k} := 2(m+k-2).$$

It can be checked that the function $m \mapsto \lambda(m)$ is decreasing for $m \geq 4$ and $\lambda(m) \to_{m\to\infty} 0.9166...$ (see Figure 3 below for a plot of the function $m \mapsto \lambda(m)$). The following holds:

Theorem 3.6 (Cekić-L.-Moroianu-Semmelmann, '22)**.** *Let (M, g) be a closed connected Kähler manifold with negative sectional curvature and $\dim_{\mathbb{C}} M = m \geq 2$. The unitary frame flow $(\Phi_t)_{t\in\mathbb{R}}$ is* ergodic *and* mixing *on $F_{\mathbb{C}}M$ with respect to the smooth measure ω if:*

(i) *The complex dimension m is odd or $m = 2$* [BG80],
(ii) *The complex dimension m is even, $m \neq 28$, and the manifold is strictly holomorphically $\lambda(m)$-pinched.*

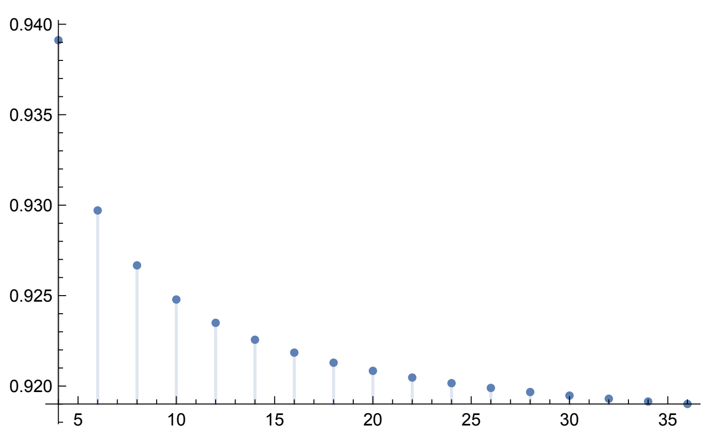


FIGURE 3. The threshold $\lambda(m)$ (vertical axis) computed with respect to the complex dimension m (horizontal axis) for $m \in \{4, 6, ..., 36\}$

We actually show that the unitary frame flow is ergodic if and only if it is mixing. We believe that ergodicity should hold *without any pinching condition* but it is clear from the proofs that our method only works with a pinching condition close to 1. Besides [BG80] in odd complex dimensions, Theorem 3.6 seems to be the first result proving ergodicity of unitary frame flows on negatively-curved Kähler manifolds of even complex dimensions, and even the case of constant holomorphic curvature $H = -1$ (that is, compact quotients $\Gamma\backslash\mathbb{CH}^m$ of the complex hyperbolic space) still seemed to be open.

As indicated in Theorem 3.6, it also seems that our technique does not apply in complex dimension $m = 28$. As in the real case, this open case is connected to an open problem in algebraic topology which is to classify reductions of the structure group of the unitary frame bundle $F_{\mathbb{C}}S^{55}$ over the sphere S^{55}. More precisely, we are unable to rule out the possible existence of a special E_6-structure on S^{55} and this eventually turns out to be problematic in order to run our argument.

The structure of the argument is similar to the real case and follows the three steps described in §3.1. The curvature term in the twisted Pestov identity are slightly more involved but it can still be understood using the complex structure of the manifold.

Eventually, let us mention that, similarly to the real case where ergodicity of the frame flow was shown to determine the high-energy behaviour of eigenfunctions of Dirac-type operators [JS07], the ergodicity of the unitary frame flow on Kähler manifolds determines the high-energy behaviour of eigenfunctions of Dolbeault Laplacians and Spin^c Dirac operators [JSZ08].

3.5. **General frame flows.** We now investigate a generalization of the geodesic frame flow to arbitrary vector bundles. We assume that (M, g) is a Riemannian manifold with Anosov geodesic flow.

Definition 3.7. Given a pair $(E, \nabla^E) \in \mathbf{A}^{\mathbb{R}}$ over M, we say that it admits a *holonomy reduction* if the full holonomy group $\mathrm{Hol}(E, \nabla^E) \leqslant \mathrm{SO}(r)$ is strictly contained in $\mathrm{SO}(r)$, where $r := \mathrm{rank}(E)$. We say that it is *irreducible* if there exists no non-trivial splitting $(E, \nabla^E) = (E_1, \nabla^{E_1}) \oplus (E_2, \nabla^{E_2})$ over M. The same definitions hold in the complex case with the obvious modifications.

Given $a = ([E], [\nabla^E])$, we consider as before the pullback pair

$$\pi^* a = \big([\pi^* E], [\pi^* \nabla^E]\big) =: ([\mathcal{E}], [\nabla^{\mathcal{E}}])$$

of a vector bundle and a connection over SM, where $\pi : SM \to M$ is the footpoint projection. Define $\mathbf{X} := \nabla^{\mathcal{E}}_X$ to be the first order differential operator of covariant differentiation along X, also the generator of parallel transport along the geodesic flowlines, and let $F\mathcal{E}$ be the principal G-bundle (with $G = \mathrm{SO}(r)$ or $\mathrm{U}(r)$) of orthonormal frames of $\mathcal{E}$. The connection $\nabla^{\mathcal{E}}$ provides a natural parallel transport of frames of $\mathcal{E}$ along geodesic flowlines and we thus obtain the *frame flow* $\Phi_t : F\mathcal{E} \to F\mathcal{E}$ of $\nabla^{\mathcal{E}}$ which *extends* the geodesic flow in the sense that $\varphi_t \circ \widehat{\pi} = \widehat{\pi} \circ \Phi_t$, where $\widehat{\pi} : F\mathcal{E} \to SM$ is the projection. Moreover, this flow commutes with right-action of the group G. The geodesic flow preserves a natural smooth measure called the Liouville measure, and $(\Phi_t)_{t\in\mathbb{R}}$ thus preserves a canonical measure ω on $F\mathcal{E}$ obtained locally as the product of the Liouville measure with the Haar measure on the group G. This fits into the framework discussed in §1.

Our aim in this paragraph is to study the ergodicity of $(\Phi_t)_{t\in\mathbb{R}}$ with respect to ω. It will be the main result of this section, Theorem 3.10, that under some low-rank assumption on E, the flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic whenever ∇^E admits no holonomy reduction. To encompass both cases (real and complex bundles), introduce the notation $q_{\mathbb{F}}(n)$, where

$$q_{\mathbb{R}}(n) := q(n) = q_{\mathrm{SO}}(n) - 1, \qquad q_{\mathbb{C}}(n) := q(n)/2 = q_{\mathrm{U}}(n) - 1, \quad (3.6)$$

where the numbers $q(n)$ were introduced in §2.3.1. By standard algebra, the representation ρ of Parry's free monoid $\mathbf{G}$ as introduced in (1.4) admits a splitting

$$\mathcal{E}_{z_\star} \simeq \mathbb{F}^r = \oplus_i^\perp \oplus_{j=1}^{n_i} V_i^{(j)}, \quad (3.7)$$

where $V_i^{(j)} \subset \mathbb{F}^r$, $n_i \geq 1$, each $V_i^{(j)} \simeq V_i$ is H-invariant and irreducible, and the representations V_i and V_k are isomorphic if and only if $i = k$, see e.g. [Lan02, Chapter XVII]. In particular, if ρ is irreducible, then $V_1^{(1)} = \mathbb{F}^r$, $n_1 = 1$, and we will say that the splitting (3.7) is *trivial.*

Lemma 3.8. *Let (M^{n+1}, g) be a closed negatively-curved Riemannian manifold. Let $a \in \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)}$ and assume that the induced representation of $\mathbf{G}$ admits the splitting (3.7). Then*

$$(E, \nabla^E) = \oplus_i^\perp \oplus_{j=1}^{n_i} \big(F_i^{(j)}, \nabla^{F_i^{(j)}}\big),$$

where $\mathcal{E} = \oplus_i^\perp \oplus_{j=1}^{n_i} \pi^* F_i^{(j)}$, $V_i^{(j)} = \big(\pi^* F_i^{(j)}\big)_{z_\star}$, and $\big([F_i^{(j)}], [\nabla^{F_i^{(j)}}]\big) = \big([F_i], [\nabla^{F_i}]\big)$ is irreducible. In particular, if $a \in \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)}$ is irreducible, then the transitivity group H acts irreducibly on $\mathbb{F}^r$ and the splitting (3.7) is trivial.

Proof. By the non-Abelian Livšic Theorem 1.5 and Remark 1.6 applied with $\mathfrak{o} = S^2$ (symmetric endomorphisms), the splitting (3.7) yields a flow-invariant smooth splitting of $\mathcal{E}$ over SM as:

$$\mathcal{E} = \oplus_i^{\perp} \oplus_{j=1}^{n_i} \mathcal{F}_i^{(j)}, \qquad \mathbb{1}_{\mathcal{E}} = \sum_i \sum_{j=1}^{n_i} \pi_{\mathcal{F}_i^{(j)}},$$

where $\pi_{\mathcal{F}_i^{(j)}}$ is the orthogonal projection onto the vector bundle $\mathcal{F}_i^{(j)}$. Let τ be the orthogonal projector onto $\mathcal{F}_i^{(j)} =: \mathcal{F}$. Note that by assumption, $\mathcal{F}$ does not split further, that is, there is no non-trivial flow-invariant subbundle of $\mathcal{F}$. By Corollary 2.16, τ has finite Fourier content. By fixing a point $x \in M$ and identifying $S_xM \simeq \mathbb{S}^n$, we get a polynomial map $\tau_x : \mathbb{S}^n \to \mathrm{Gr}_{\mathbb{F}}(k, r)$, where $k = \mathrm{rank}(\mathcal{F}) \leq q_{\mathbb{F}}(n)$. Hence, by Lemma 2.12, we deduce that τ_x is constant, that is, τ has zero Fourier degree and thus descends to a parallel orthogonal projector $\tau \in C^\infty(M, S^2E)$ on a parallel subbundle $F \subset E$ such that $\pi^*F = \mathcal{F}$. Note that $(F, \nabla^E|_F)$ is irreducible, otherwise any reduction of it would yield a reduction of $\mathcal{F}$ upstairs, which is excluded by assumption. This proves the claim. ☐

Lemma 3.9. *Let (M^{n+1}, g) be a closed negatively-curved Riemannian manifold with $n \geq 2$. Assume that $(E, \nabla^E) \in \mathbf{A}^{\mathbb{F}}_{\leq q_{\mathbb{F}}(n)}$ does not have a finite holonomy group. Then, the transitivity group H is not finite.*

Proof. If H is finite, then $Q \to SM$ is a finite covering (where the bundle $Q \subset F\mathcal{E}$ is given by Corollary 1.4). Since $n \geq 2$, $\pi_1(\mathbb{S}^n) = 0$ and thus the long exact sequence of the spherical fibration $\mathbb{S}^n \hookrightarrow SM \to M$ yields $\pi_1(M) \simeq \pi_1(SM)$. It implies that that there exists a finite cover $p : \widetilde{M} \to M$ such that $S\widetilde{M} \simeq Q$ and $S\widetilde{M} \to SM$ is the induced covering map. Lift the bundle $(E, \nabla^E) \to M$ to $(p^*E, p^*\nabla^E) \to \widetilde{M}$. Set $\widetilde{E} := p^*E$ and consider the induced frame flow on the frames of $\widetilde{\mathcal{E}} := \widetilde{\pi}^*\widetilde{E} \to S\widetilde{M}$ (where $\widetilde{\pi} : S\widetilde{M} \to \widetilde{M}$ is the projection). By construction, the transitivity group of this frame flow is trivial (see [Lef, Lemma 3.12] for more details). Hence, by Theorem 1.5 we can find a global flow-invariant orthonormal basis $\mathbf{e}_1, ..., \mathbf{e}_r \in C^\infty(S\widetilde{M}, \widetilde{\mathcal{E}})$. Now, by evaluating the frame $\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_r)$ at a point $x \in M$, we get a map $\mathbf{e}_x : \mathbb{S}^n \to \mathrm{SO}(r)$ (real case) or $\mathbf{e}_x : \mathbb{S}^n \to \mathrm{SU}(r)$ (complex case). Since $r \leq q_{\mathbb{F}}(n)$, by Lemma 2.12 this map is constant, and thus the sections $\mathbf{e}_i$ have degree 0. In turn, this implies that they define parallel sections $\mathbf{e}_i \in C^\infty(\widetilde{M}, \widetilde{E})$ over the base $\widetilde{M}$. As a consequence, $([p^*E], [p^*\nabla^E]) = ([\mathbb{F}^r], [d])$ is the trivial flat bundle equipped with the trivial flat connection. In turn, it implies that (E, ∇^E) is flat with finite holonomy group. This contradicts the assumption.

$\square$

From the discussion above, we can derive a result on ergodicity of certain isometric extensions in low rank. For that purpose, define

$$m_{\mathbb{R}}(n) := \lfloor \tfrac{1}{2}(1 + \sqrt{1 + 8q_{\mathbb{R}}(n)}) \rfloor, \qquad m_{\mathbb{C}}(n) := \lfloor \sqrt{q_{\mathbb{R}}(n)} \rfloor,$$

where $q_{\mathbb{F}}(n)$ was defined in (3.6). The following holds:

Theorem 3.10 (Cekić-L., '22)**.** *Let (M^{n+1}, g) be a closed negatively-curved Riemannian manifold with $n \geq 2$. Assume that $(E, \nabla^E) \in \mathbf{A}^{\mathbb{F}}_{\leq m_{\mathbb{F}}(n)}$ has no holonomy reduction. Then the frame flow $(\Phi_t)_{t \in \mathbb{R}}$ on $F\mathcal{E}$ is ergodic.*

The case $\dim(M) = 2$ will be discussed afterwards in Corollary 3.12. It can be easily seen that the absence of holonomy reduction is a generic property among connections. We also formulate an important remark relating Theorem 3.10 to the study of the ergodicity of the geodesic frame flow. In [Bri82, Conjecture 2.9], Brin conjectured that the frame flow over negatively-curved manifolds should be ergodic whenever the Riemannian manifold (M, g) admits no holonomy reduction (see the proof of Proposition 2.22 for the classification of possible holonomy reductions in negative curvature). Theorem 3.10 therefore proves a general version of Brin's conjecture for arbitrary flows of frames (on pullback vector bundles) but under a low-rank assumption on the bundle. Unfortunately, the usual geodesic frame flow fails to satisfy the assumptions of Theorem 3.10 because the rank of the tangent bundle is too high.

Proof. We first deal with the complex case. By construction, the transitivity group $H \leqslant \mathrm{U}(r)$ provides an H-invariant Lie subalgebra $\mathfrak{h} \leqslant \mathfrak{u}(r)$ of real rank $k \geq 0$. Note that $\mathfrak{h} \neq 0$ by Lemma 3.9, that is, $k \geq 1$. By the non-Abelian Livšic Theorem 1.5, we may consider $\tau \in C^\infty(SM, \pi^* S^2 \mathrm{End}_{\mathrm{sk}}(E))$, the flow-invariant orthogonal projection whose value at $z_\star$ is given by the orthogonal projection onto $\mathfrak{h}$ (here $\mathrm{End}_{\mathrm{sk}}$ denote the skew-Hermitian endomorphisms). By Corollary 2.16, τ has finite Fourier content, that is, it is a fiberwise polynomial section. Arguing as in the proof of Lemma 3.8, and using that the real rank of $\mathfrak{u}(r)$ is r^2, we get by restricting τ to an arbitrary sphere $S_{x_0}M \simeq \mathbb{S}^n$ (for some $x_0 \in M$) a polynomial map $\mathbb{S}^n \to \mathrm{Gr}_{\mathbb{R}}(k, r^2)$. This map needs to be constant by Lemma 2.12 as the condition $r \leq m_{\mathbb{C}}(n)$ implies $r^2 \leq q_{\mathbb{R}}(n)$. Hence, $\tau \in C^\infty(M, S^2 \mathrm{End}_{\mathrm{sk}}(E))$ defines a parallel section on M. Since the holonomy group of (E, ∇^E) is equal to

$\mathrm{U}(r)$ (by assumption) and its adjoint representation splits irreducibly as $\mathfrak{u}(r) = \mathfrak{su}(r) \oplus i\mathbb{R}I_r$, we obtain that $\tau(z_\star) \neq 0$ is either $\pi_{\mathfrak{su}(r)}, \pi_{i\mathbb{R}I_r}$ or $\mathbb{1}$. Observe that the last possibility $\tau(z_\star) = \mathbb{1}$ implies that $H = \mathrm{U}(r)$, that is, $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on $F\mathcal{E}$ by Corollary 1.4. As a consequence, it suffices to rule out the first two cases.

If $\tau(z_\star) = \pi_{\mathfrak{su}(r)}$, then H is a finite disjoint union of copies of $SU(r)$, and the effective action of H on $\det \mathbb{C}^r = \Lambda^r\mathbb{C}^r$ is that of a finite Abelian group $\mathbb{Z}_p$ (for some $p \geq 1$) and thus H acts trivially on $(\det \mathbb{C}^r)^{\otimes p}$. In turn, by the non-Abelian Livšic Theorem 1.5 and Remark 1.6 (applied with $\mathfrak{o}(E) := (\det E)^{\otimes p}$), the line bundle $\pi^*(\det E)^{\otimes p}$ is *transparent* over SM (in the terminology of Paternain [Pat09]), that is, there exists a flow-invariant section $s \in C^\infty(SM, \pi^*(\det E)^{\otimes p})$ and this has finite Fourier content by Corollary 2.16. Restricting to a point $x_0 \in M$, we thus a get a polynomial map $\mathbb{S}^n \to \mathbb{S}^1$ (where $\mathbb{S}^1$ is the unit circle of $(\det E_{x_0})^{\otimes p}$) and this map needs to be constant since $n \geq 2$. Hence $s \in C^\infty(M, (\det E)^{\otimes p})$ defines a parallel section on the base M, that is, $(\det E)^{\otimes p}$ is the trivial flat line bundle and (E, ∇^E) thus admits a holonomy reduction, which contradicts the assumptions.

If $\tau(z_\star) = \pi_{i\mathbb{R}I_r}$, we can consider a finite cover $p : \widetilde{M} \to M$ as in the proof of Lemma 3.9, equipped with the pullback bundle $(p^*E, p^*\nabla^E)$, so that the associated transitivity group $\widetilde{H}$ is connected (see also [Lef, Lemma 3.12]), and thus $\widetilde{H} = \mathrm{U}(1)$. The induced representation of $\widetilde{H}$ on $\mathbb{C}^r$ splits as a sum of irreducible representations $\mathbb{C}^r = \mathbb{C} \oplus ... \oplus \mathbb{C}$ and thus by the non-Abelian Livšic Theorem 1.5, there exists r smooth flow-invariant complex line bundles $\mathcal{L}_1, ..., \mathcal{L}_r$ such that $\widetilde{\pi}^*(p^*E) = \mathcal{L}_1 \oplus ... \oplus \mathcal{L}_r$ (recall $\widetilde{\pi} : S\widetilde{M} \to \widetilde{M}$ is the projection). The orthogonal projection $\pi_{\mathcal{L}_i} \in C^\infty(S\widetilde{M}, \widetilde{\pi}^*(S^2p^*E))$ onto each factor is flow-invariant so it is fiberwise polynomial by Corollary 2.16 and thus we obtain a polynomial map $\mathbb{S}^n \to \mathrm{Gr}_{\mathbb{C}}(1, r)$. The assumption $r \leq m_{\mathbb{C}}(n)$ implies $r^2 \leq q_{\mathbb{R}}(n)$ and using that $q_{\mathbb{R}}(n)$ is even, this also implies that $r \leq q_{\mathbb{R}}(n)/2$ (arguing separately for the cases $n = 2, 3$ and $n \geq 4$). By Lemma 2.12, the inequality $r \leq q(n)/2$ then implies that these algebraic maps are constant, that is, $\pi_{\mathcal{L}_i} \in C^\infty(\widetilde{M}, S^2p^*E)$ is a parallel section on the base $\widetilde{M}$ and $p^*E = \mathcal{L}_1 \oplus ... \oplus \mathcal{L}_r$ splits as a sum a parallel complex line bundles. Hence, $(p^*E, p^*\nabla^E)$ admits a holonomy reduction to $\mathrm{U}(1) \times ... \times \mathrm{U}(1) \subset \mathrm{U}(r)$. In turn, this implies that (E, ∇^E) admits a holonomy reduction, which contradicts the assumption if $r \geq 2$ (if $r = 1$ there is nothing to prove since $\mathfrak{su}(1) = \{0\}$).

Let us now deal with the real case. Further assume that $r \neq 1, 2, 4$; we will deal with the remaining cases afterwards. As in the complex case, the transitivity subgroup $H \leqslant \mathrm{SO}(r)$ provides an H-invariant Lie algebra $0 \neq \mathfrak{h} \leqslant \mathfrak{so}(r)$. By the non-Abelian Livšic Theorem 1.5 and Remark 1.6 applied with $\mathfrak{o} = S^2\Lambda^2$, we get a flow-invariant projector $\tau \in C^\infty(SM, \pi^* S^2\Lambda^2 E)$ onto a smooth flow-invariant subbundle of $\pi^*\Lambda^2 E$ of rank $k \geq 1$ (whose restriction to $z_\star$ is identified with $\mathfrak{h}$). As before, using that the rank of $\Lambda^2 E$ is $\frac{1}{2}r(r-1)$, we get by restriction of τ to a sphere $S_{x_0}M \simeq \mathbb{S}^n$ an algebraic map $\tau : \mathbb{S}^n \to \mathrm{Gr}_{\mathbb{R}}(k, \frac{1}{2}r(r-1))$. By assumption $r \leq m_{\mathbb{R}}(n)$ implies that $\frac{1}{2}r(r-1) \leq q_{\mathbb{R}}(n) = q(n)$ and thus this map is constant by definition of $q(n)$ and Lemma 2.12. As a consequence, $\tau \neq 0$ is of degree zero and descends on the base to an orthogonal parallel projector $\tau \in C^\infty(M, S^2\Lambda^2 E)$ onto a rank k subbundle. But (E, ∇^E) has no holonomy reduction by assumption, so its holonomy group is $\mathrm{SO}(r)$ and since $\mathrm{SO}(r)$ acts irreducibly on $\Lambda^2\mathbb{R}^r$ (since $\mathrm{SO}(r)$ is simple if $r \neq 1, 2, 4$ and $\Lambda^2\mathbb{R}^r$ is isomorphic to the adjoint representation), we obtain that $\tau = \mathbb{1}$ is the identity, that is, $\mathfrak{h} = \mathfrak{so}(r)$ and $H = \mathrm{SO}(r)$. By Corollary 1.4, we conclude that the frame flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on $F\mathcal{E}$.

For $r = 4$, we need to slightly adapt the argument. First of all, observe that the assumption $r = 4 \leq m_{\mathbb{R}}(n)$ yields $q(n) \geq 6 \geq q(4) = 4$, that is, $n \geq 4$ as $n \mapsto q(n)$ is non-decreasing by Lemma 2.10. As before, up to taking a finite cover, we can also directly assume that H is connected. Now, $\Lambda^2\mathbb{R}^4$ splits as $\Lambda^2\mathbb{R}^4 = \Lambda^+\mathbb{R}^4 \oplus \Lambda^-\mathbb{R}^4$, the space of self-dual and anti self-dual 2-forms, which give non-isomorphic irreducible representations of $\mathrm{SO}(4)$ (see Proposition 2.22 below where this is further discussed). Hence, the section $\tau \neq 0$ is either $\mathbb{1}$ (in which case, the frame flow is ergodic) or $\pi_{\Lambda^\pm\mathbb{R}^4}$, one of the two orthogonal projections onto $\Lambda^\pm\mathbb{R}^4$. In both cases, since H is connected, we obtain that H is equal to one of the two $\mathrm{SU}(2)$ factors of $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$. Hence, H acts trivially on either $\Lambda^+\mathbb{R}^4$ or $\Lambda^-\mathbb{R}^4$ and, without loss of generality, we can assume that it acts trivially on $\Lambda^-\mathbb{R}^4$. Applying once again Theorem 1.5, we deduce that, in particular, $\pi^*\Lambda^- E$ admits a flow-invariant section $s \in C^\infty(SM, \pi^*\Lambda^- E)$. This yields a polynomial mapping $\mathbb{S}^n \to \mathbb{S}^2$ which is necessarily constant since $n \geq 4$ (by the preliminary remark) so $s \in C^\infty(M, \Lambda^- E)$ is a parallel section. But then, this contradicts the assumption that (E, ∇^E) has no holonomy reduction.

For $r = 2$, either $H = \mathrm{SO}(2)$ (ergodicity) or H is a finite Abelian group, in which case we can directly assume (up to taking a finite cover of M) that $H = \{0\}$. But then, there exists by Theorem 1.5 a flow-invariant section $s \in C^\infty(SM, \pi^* E)$ which must be of degree 0 by the same arguments as before, that is, $s \in C^\infty(M, E)$ is parallel. In turn, this implies that $(E, \nabla^E) = (\mathbb{R}^2, d)$ is the trivial flat bundle, which contradicts the holonomy assumption. Eventually, the case $r = 1$ is empty. This concludes the proof. ☐

Note that, in the complex case, further assuming that the connection $(E, \nabla^E) \in \mathbf{A}^{\mathbb{C}}$ preserves the determinant, one can replace $\mathrm{U}(r)$ by $\mathrm{SU}(r)$ in the argument and $m_{\mathbb{C}}(n)$ is then replaced by $m'_{\mathbb{C}}(n) := \lfloor \sqrt{1 + q_{\mathbb{R}}(n)} \rfloor$.

If $\mathcal{E}$ is a real vector bundle over SM, one can also consider the lift $(\Phi_t)_{t\in\mathbb{R}}$ of the geodesic flow to $S\mathcal{E}$, the unit sphere bundle of $\mathcal{E}$ over SM, given by parallel transport of sections of $\mathcal{E}$ along geodesic flowlines.

Corollary 3.11. *Let (M^{n+1}, g) be a closed negatively-curved Riemannian manifold and $n \geq 4$. Let $(E, \nabla^E) \in \mathbf{A}^{\mathbb{R}}_{\leq 4}$ be an orthogonal connection on a real vector bundle $E \to M$ such that $r := \mathrm{rank}(E) \leq 4$. Further assume that it is irreducible and that it does not have a finite holonomy group. Then $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on $S\mathcal{E}$.*

Proof. We deal with the case $r = 4$, the other cases being similar. Observe that $q_{\mathbb{R}}(n) \geq q_{\mathbb{R}}(4) = 4$ by Lemma 2.10 and (2.23). Using Lemmas 3.9 and 3.8, the transitivity group $H \leqslant \mathrm{SO}(4)$ acts irreducibly on $\mathbb{R}^4$ and is not finite. The only possibility is that H contains $\mathrm{SU}(2)$. But then, since $\mathrm{SU}(2)$ acts transitively on $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, we can conclude by Theorem 1.2 that the flow $(\Phi_t)_{t\in\mathbb{R}}$ is ergodic on $S\mathcal{E}$. ☐

For surfaces, the situation is slightly different but it involves studying the limiting case where the rank has the dimension of the base. If (M^2, g) is a (closed oriented) Riemannian surface, let $(T^*_{\mathbb{C}}M)^{1,0} := \kappa \to M$ be the canonical line bundle. Using the Gysin exact sequence, it can be checked that $\mathcal{E} := \pi^*\kappa \to SM$ is trivial. Moreover, letting ∇^{LC} be the Levi-Civita connection on κ, the induced frame flow $(\Phi_t)_{t\in\mathbb{R}}$ on $F\mathcal{E} \to SM$ is trivial in the sense that it has trivial transitivity group and is thus conjugate to the flow $(\varphi_t, \mathbf{1}_{\mathrm{U}(1)})_{t\in\mathbb{R}}$ on $SM \times \mathrm{U}(1) \simeq F\mathcal{E} \simeq S\mathcal{E}$. In the terminology of Paternain [Pat09], the pair

$$k := (\kappa, \nabla^{\mathrm{LC}}) \in \mathbf{A}^{\mathbb{C}}_1$$

defines a *transparent connection*, that is, its holonomy along every closed geodesic is trivial. The following is actually just a reformulation of Proposition 2.5.

Corollary 3.12. *Let (M^2, g), be a closed Anosov Riemannian surface. Let $a := (E, \nabla^E) \in \mathbf{A}^{\mathbb{C}}_1$ be a unitary connection on a complex line bundle $E \to M$. Then the frame flow $(\Phi_t)_{t\in\mathbb{R}}$ on $F\mathcal{E} \to SM$ is ergodic, unless there exists $p \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$ such that $a^{\otimes p} = k^{\otimes m}$.*

Equivalently, the last equality can be written

$$a = ((\kappa^{\otimes m})^{\otimes 1/p}, (\nabla_{\mathrm{LC}}^{\otimes m})^{\otimes 1/p}),$$

where the index $\otimes 1/p$ stands for the p-th unit root. Topologically, the p-th unit root of the line bundle $\kappa^{\otimes m} \to M$, when it exists, is always unique. Actually, it exists if and only if the first Chern class of $\kappa^{\otimes m}$ is divisible by p, that is, $mc_1(\kappa) = m(2g-2)$ is divisible by p (where g is the genus of M), and it is then given by the unique complex line bundle over M whose first Chern class is $m(2g-2)/p$. However, $(\nabla_{\mathrm{LC}}^{\otimes m})^{\otimes 1/p}$ is not unique and the choice of such an p-th root of the connection is parametrized by $H^1(M, \mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^{2g}$ (see [For81, Section 21]).

Proof. The proof follows from the above description and earlier observations due to Paternain, see [Pat09, Theorems 3.1 and 3.2]. Indeed, assume that the frame flow $(\Phi_t)_{t\in\mathbb{R}}$ is not ergodic. Then the transitivity group $H \leqslant \mathrm{SO}(2)$ is finite, that is, $H = \mathbb{Z}/p\mathbb{Z}$ for some $p \in \mathbb{Z}_{\geq 1}$. But then the frame flow $(\Phi_t^{\otimes p})_{t\in\mathbb{R}}$ induced by $a^{\otimes p}$ has trivial transitivity group. This is equivalent to saying that $a^{\otimes p}$ is transparent (the holonomy is trivial along every closed geodesic loop). Now, Paternain [Pat09] classified all transparent connections on complex line bundles over Anosov surfaces and found that they are precisely given by $\{k^{\otimes m} \mid m \in \mathbb{Z}\}$. $\square$

4. Lens rigidity of Anosov manifolds

4.1. **Statement of the problem. Main results.** The result of this paragraph can be found in [CGL22].

4.1.1. *Lens data of Riemannian manifolds.* Let (M, g) be a smooth connected Riemannian manifold with strictly convex boundary (i.e. the second fundamental form is positive on ∂M). Let $\mathcal{M} := SM$ be the unit tangent bundle of (M, g) and define the incoming (-) and outgoing (+) boundary of $\mathcal{M}$ as:

$$\partial_\pm \mathcal{M} := \{(x, v) \in \mathcal{M} \mid x \in \partial M, \pm g_x(v, \nu(x)) > 0\},$$

where ν is the unit outward pointing normal vector to the boundary. For any $(x, v) \in \partial_- \mathcal{M}$, the maximally extended geodesic $\gamma_{(x,v)}$, with initial condition $\gamma_{(x,v)}(0) = x$, $\dot{\gamma}_{(x,v)} = v$ is defined on a time interval $[0, \ell_g(x, v)]$ where $\ell_g(x, v) \in \mathbb{R}_+ \cup \{\infty\}$. When $\ell_g(x, v) < \infty$, we define

$$S_g(x, v) := \big(\gamma_{(x,v)}(\ell_g(x, v)), \dot{\gamma}_{(x,v)}(\ell_g(x, v))\big)$$

to be the outgoing tangent vector at $\partial_+ \mathcal{M}$, see Figure 4.

Definition 4.1 (Lens data)**.** The map $S_g : \partial_- \mathcal{M} \setminus \{\ell_g = \infty\} \to \partial_+ \mathcal{M}$ is called the *scattering map* and the function $\ell_g : \partial_- \mathcal{M} \setminus \{\ell_g = \infty\} \to \mathbb{R}_+$ the *length map*. The pair (ℓ_g, S_g) is the *lens data* of the Riemannian manifold (M, g).

The lens data encodes the boundary data one can measure on the geodesic flow from "outside of the manifold". A natural inverse problem that arises from tomography consists in determining the geometry, namely, the Riemannian metric g inside M, from the measurement of the lens data (ℓ_g, S_g). In geophysics, this is related to recovering the speed of propagation of waves inside a domain such as the Earth, for instance, see [PSU14b]. When two metrics g and g' agree on ∂M, it makes sense to say that they have the same lens data as there is a natural identification between the boundary of their respective unit tangent bundles via the unit disk bundle of the boundary. The *lens rigidity problem* is concerned with the following question:

Question 4.2. *Assume that (M, g) and (M', g') are two Riemannian metrics with strictly convex boundary such that there exists an isometry $I \in \mathrm{Diff}(\partial M, \partial M')$ with $I^*(g'|_{T\partial M'}) = g|_{T\partial M}$. Does the following implication*

$$(\ell_g, S_g) = I^*(\ell_{g'}, S_{g'}) \Longrightarrow \exists \psi \in \mathrm{Diffeo}(M, M'),\ \psi|_{\partial M} = I, \quad \psi^* g' = g$$

hold true?

We say that a manifold (M, g) is *lens rigid* if there is no other Riemannian manifold (up to isometry) having the same lens data as (ℓ_g, S_g). In the following, in order to simplify the notation, we will assume that $M = M', I = \mathrm{id}$.

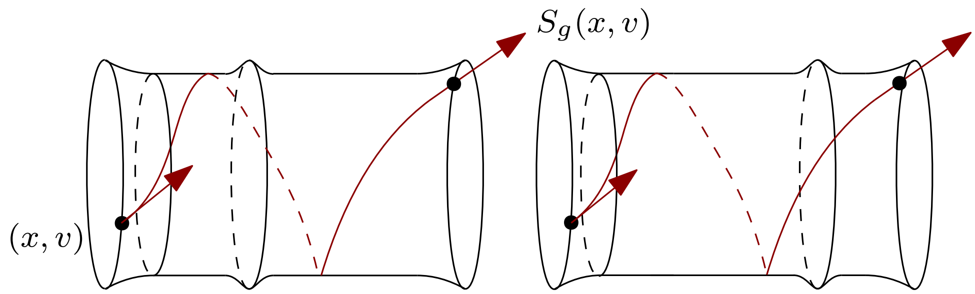


FIGURE 4. A surface with strictly convex boundary which is not lens rigid. Example taken from [CH16].

There are simple counter-examples of manifolds for which lens rigidity does not hold: considering certain perturbations of the flat cylinder $\mathbb{S}^1 \times [0, 1]$ (see Figure 4 and [CH16] where this is further discussed), one can easily obtain non-isometric metrics with same lens data. Such cases have *trapped geodesics*, that is some maximally extended geodesics with infinite length, or equivalently $\ell_g(x, v) = \infty$ for some $(x, v) \in \partial_-\mathcal{M}$. It turns out that all existing counter-examples to lens rigidity have trapped geodesics.

4.1.2. *Lens rigidity for non-trapping manifolds.* Even among manifolds without a trapped set, the lens rigidity problem is still widely open. The recent breakthrough of Stefanov-Uhlmann-Vasy [SUV21] is the closest result in this direction, showing lens rigidity in dimension $n \geq 3$ under the additional assumption that the manifold (M, g) is foliated by strictly convex hypersurfaces. This includes all simply connected non-positively curved manifolds with strictly convex boundary. In the class of real analytic metrics such that from each $x \in \partial M$ there is a maximal geodesic free of conjugate points, the lens rigidity was proved by Vargo [Var09]. A local lens rigidity result was also proved near analytic metrics by Stefanov-Uhlmann [SU09] under certain assumptions on the conjugate points.

There is also a subclass of metrics that have attracted a lot of attention since the work of Michel [Mic82], namely the class of *simple manifolds*, which are manifolds with strictly convex boundary that have no trapped geodesics and no conjugate points. These manifolds are diffeomorphic to the unit ball in $\mathbb{R}^n$. In this case, knowing the lens data is equivalent to knowing the restriction $d_g|_{\partial M\times\partial M}$ of the Riemannian distance function $d_g \in C^0(M \times M)$ to the boundary, also called the *boundary distance*. The lens rigidity problem for this subclass of metrics is also called the *boundary rigidity problem*. In dimension $n = 2$, it was proved by Otal [Ota90b] (in negative curvature), Croke [Cro91] (in non-positive curvature), and Pestov-Uhlmann [PU05] (in general) that simple surfaces are boundary rigid, and thus lens rigid. We also mention the results by Croke-Dairbekov-Sharafutdinov [CDS00] and Stefanov–Uhlmann [SU04] for local boundary rigidity results, the work by Gromov [Gro83] and Burago–Ivanov [BI10] for rigidity results of flat and close to flat simple manifolds, and we finally refer more generally to the review article by Croke [Cro04] and the recent book of Paternain–Salo–Uhlmann [PSU22] for an overview of the boundary rigidity problem.

4.1.3. *Lens rigidity for manifolds with non-empty trapped set.* In most situations, trapped geodesics appear since all Riemannian manifolds (M, g) with strictly convex boundary and non-trivial topology, i.e. nontrivial fundamental group, always have trapped geodesics (and they even have closed geodesics in the interior M°). As far as manifolds with trapped geodesics are concerned, very little is known on the lens rigidity problem. It is not even clear what would be the most general class of manifolds for which lens rigidity could hold and the example above in Figure 4 shows that it seems hopeless to consider general manifolds with both trapped geodesics and conjugate points.

The only result considering cases with both trapped geodesics and conjugate points seems to be the local rigidity result of Stefanov-Uhlmann [SU09]. In dimension $n \geq 3$, under a certain topological assumption, it is proved that if (M, g_0) is real analytic[13], with strictly convex boundary, and for each $(x, v) \in SM$ there is $w \in v^\perp$ such that the maximally extended geodesic tangent to w at x has finite length (it is not trapped) and is free of conjugate points, then the following holds: if g is another metric with $\|g - g_0\|_{C^N}$ small enough for some $N \gg 1$ and $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then g and g_0 are isometric via a boundary-preserving diffeomorphism. On the other hand, it is not clear (geometrically speaking) what type of manifolds are contained in this class and there are many interesting geometric cases not contained in it. For example, there exist convex co-compact hyperbolic 3-manifolds $M := \Gamma\backslash\mathbb{H}^3$ (with constant sectional curvature -1) whose convex core

[13] Or more generally if a certain localized X-ray transform is injective.

$\mathcal{C}$ has positive measure and totally geodesic boundary. Thus, cutting the ends of such examples at a finite positive distance of $\mathcal{C}$, one obtains a metric not satisfying the assumptions of [SU09] due to the totally geodesic surfaces bounding $\mathcal{C}$.

From our point of view, there is a very natural class of metrics with non-trivial trapped set where the lens ridigity problem seems well-posed and interesting from a geometrical point of view. We call elements of this class manifolds of *Anosov type*; it contains as a strict subclass the set of negatively-curved metrics with strictly convex boundary.

Definition 4.3. A Riemannian manifold (M, g) with boundary is of *Anosov type* if:

(i) it has strictly convex boundary,
(ii) no conjugate points,
(iii) the trapped set for the geodesic flow $(\varphi_t^g)_{t\in\mathbb{R}}$ on $\mathcal{M} := SM$, defined by

$$K^g := \bigcap_{t\in\mathbb{R}} \varphi_t^g(\mathcal{M}^\circ) \subset \mathcal{M}^\circ,$$

is *hyperbolic* in the following sense. There exists a continuous flow-invariant splitting

$$\forall y \in K^g, \quad T_y\mathcal{M} = \mathbb{R}X_g(y) \oplus E_-(y) \oplus E_+(y),$$

where X_g is the geodesic vector field, and constants $\nu, C > 0$ such that

$$\forall \pm t \geq 0, \quad \forall y \in K^g, \quad \forall v \in E_\mp(y), \quad \|d\varphi_t^g(y)v\| \leq Ce^{-\nu|t|}\|v\|, \tag{4.1}$$

for an arbitrary choice of metric $\|\cdot\|$ on $\mathcal{M}$.

Example 4.4. The main two examples of manifolds of Anosov type are:

(i) Riemannian manifolds with negative sectional curvature and strictly convex boundary (see [Kli95, Theorem 3.2.17 and Section 3.9]),
(ii) strictly convex subdomains of closed Riemannian manifolds with Anosov geodesic flows.

Manifolds of Anosov type have a trapped set with fractal structure and zero Lebesgue measure. It implies that almost-every point in $\mathcal{M}$ is reachable from geodesics with endpoints on $\partial\mathcal{M}$. This case can be interpreted as an intermediate rigidity problem between the *length spectrum rigidity* of manifolds with Anosov geodesic flows, where one asks if the lengths of closed geodesics determine the metric up to isometry, and the boundary rigidity problem of simple manifolds. In the closed case, Vignéras [Vig80] exhibited counter-examples to the length spectrum rigidity: in constant negative curvature, there are non-isometric metrics on surfaces with the same length spectrum. The well-posed rigidity problem is rather that of the *marked length spectrum* problem, also known as the Burns-Katok conjecture [BK85]. Similarly, for manifolds with boundary and non-trivial topology, the same problem of “marking" of geodesics is a serious difficulty. The first natural question one may consider is the following, known as *marked lens rigidity* or *marked boundary rigidity* problem for Riemannian manifolds of Anosov type.

Definition 4.5 (Marked lens data)**.** Let g_1, g_2 be two metrics of Anosov type on M. We say that g_1 and g_2 have the same *marked lens data* if for each $(x, v) \in \partial_-\mathcal{M} \setminus \{\ell_g = \infty\}$ one has $(\ell_{g_1}(x, v), S_{g_1}(x, v)) = (\ell_{g_2}(x, v), S_{g_2}(x, v))$ and the g_1- and g_2-geodesics with initial conditions (x, v) are homotopic via a homotopy fixing the endpoints.

Technically, having same marked lens data is the same as having same boundary distance function on the universal cover $\widetilde{M}$ (which is now a non-compact space). It is important to observe that the absence of conjugate points is necessary in order to even define the notion of marked lens rigidity as one needs uniqueness of geodesics with fixed endpoints x, x' on ∂M in each homotopy class of curves with fixed extremities x, x'. The following conjecture is somehow similar to the Burns-Katok conjecture in the closed case and to the boundary rigidity problem of negatively curved simple metrics:

Conjecture 4.6 (Marked lens rigidity of manifolds of Anosov type)**.** *Let M be a smooth manifold with boundary and assume that g_1, g_2 are two smooth metrics of Anosov type on M in the sense of Definition 4.3, such that $g_1|_{T(\partial M)} = g_2|_{T(\partial M)}$. If g_1 and g_2 have same marked lens data, then there exists a smooth diffeomorphism ψ, homotopic to the identity and equal to the identity on the boundary ∂M, such that $\psi^* g_2 = g_1$.*

In dimension 2, Conjecture 4.6 was proved by Guillarmou-Mazzuchelli in [GM18] using the method of Otal [Ota90a], and in higher dimensio, we proved it in [Lef18] for pairs of metrics g_1, g_2 that are close enough in C^k norm for $k \gg 1$ large enough (local marked lens rigidity). However, it is still open in general. The fact that there is no smooth 1-parameter family $(g_s)_{s \in (-1,1)}$ of non-isometric negatively curved metrics with the same marked lens data[14] is called *infinitesimal rigidity* and was proved by Guillarmou [Gui17b].

Here, we consider the more difficult problem of lens rigidity in the class of manifolds of Anosov type. Since, contrary to the closed case, there are still no counter-examples to lens rigidity, we make the following conjecture of lens rigidity in the class of metrics of Anosov type:

Conjecture 4.7 (Lens rigidity of manifolds of Anosov type)**.** *We let (M_1, g_1), (M_2, g_2) be two smooth Riemannian manifolds of Anosov type such that $(\partial M_1, g_1|_{\partial M_1}) = (\partial M_2, g_2|_{\partial M_1})$. If $(\ell_{g_1}, S_{g_1}) = (\ell_{g_2}, S_{g_2})$, then there exists a smooth diffeomorphism ψ, equal to the identity on the boundary, such that $\psi^* g_2 = g_1$.*

There are already partial answers to Conjecture 4.7:

(i) In dimension 2, Croke and Herreros [CH16] proved that cylinders with negative curvature and strictly convex boundary are lens rigid,

(ii) In dimension 2, Guillarmou shows in [Gui17b] that the scattering map S_g determines (M, g) up to conformal diffeomorphism fixing the boundary. Recovering the conformal factor of the metric is still an open question.

(iii) In dimension $n \geq 3$, Stefanov-Uhlmann-Vasy [SUV21] proved that for general metrics with strictly convex boundary, the lens data determines the metric in a neighborhood of ∂M; applying this result in the setting of negatively curved manifold, one can recover the metric outside the convex core of the manifold (which contains the projection of the trapped set).

(iv) In [GGJ22], Bonthonneau-Guillarmou-Jézéquel recently proved Conjecture 4.7 under the extra assumption that $(M_1, g_1), (M_2, g_2)$ are real analytic, but only using the equality $S_{g_1} = S_{g_2}$ of the scattering maps.

Our first result is the following local rigidity result answering Conjecture 4.7 for metrics close to each other.

Theorem 4.8 (Cekić-Guillarmou-L., '22)**.** *Let (M, g_0) be a Riemannian manifold of Anosov type. Assume that either $\dim M = 2$ or that the curvature of g_0 is non-positive. Then there exists $N \gg 1, \delta > 0$ such that the following holds: for any smooth metric g on M such that $\|g - g_0\|_{C^N} < \delta$, if $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then there exists a smooth diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and $\psi^* g = g_0$.*

[14]In this case, having the same marked lens data is equivalent to having the same lens data.

More generally, Theorem 4.8 holds under the general assumption that g_0 is of Anosov type and that its *X-ray transform operator* $I_2^{g_0}$ on divergence-free symmetric 2-tensors is injective, see (4.2) for a definition of $I_2^{g_0}$. The fact that $I_2^{g_0}$ is injective on divergence-free tensors was proved in [Gui17b] in non-positive curvature and in general on Anosov surfaces by [Lef19a] (without any assumption on the curvature). It was also proved in [GGJ22] that $I_2^{g_0}$ is injective for real-analytic metrics g_0 which implies that generic smooth metrics of Anosov type have an injective X-ray transform operator $I_2^{g_0}$; generic injectivity of $I_2^{g_0}$ follows from [CL21b] as well, admitting also Theorem 4.10 below. As a corollary of Theorem 4.8, we obtain:

Corollary 4.9. *Let (M, g_0) be a negatively-curved Riemannian manifold with strictly convex boundary. Then, there exists $N \gg 1, \delta > 0$ such that the following holds: for any smooth metric g on M such that $\|g - g_0\|_{C^N} < \delta$, if $(\ell_g, S_g) = (\ell_{g_0}, S_{g_0})$, then there exists a smooth diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathrm{id}$ and $\psi^* g = g_0$.*

We observe that Corollary 4.9 and Theorem 4.8 are not a consequence of [SU09] (nor of [SUV21]) mentioned above since: 1) our result contains the case of surfaces (dimension $n = 2$); 2) the assumption on the trapped set in [SU09] does not cover all hyperbolic trapped sets (typically, the example $M = \Gamma \setminus \mathbb{H}^3$ mentioned above is not covered when the boundary of the convex core $\mathcal{C}$ is totally geodesic), whereas we do not make any specific assumption on the topology, and neither do we assume that g_0 is analytic or that it has an injective localized X-ray transform. Theorem 4.8 is also clearly stronger than the marked local rigidity result we obtained in [Lef18], since we are now able to remove the *marking* assumption on the lens data.

4.2. **Proof ideas.** The removal of the marking assumption is not simply a technical artefact but a crucial aspect of Theorem 4.8. Indeed, without the marking assumption, one can no longer use the fact that the geodesic flows of g and g_0 are conjugate with a conjugacy preserving the Liouville measure. This conjugacy was a fundamental aspect of both proofs of [GM18, Lef18]. In the proof of Theorem 4.8, one has to rely on a completely different argument, which is the linearisation of the pair (ℓ_g, S_g). Nevertheless, since g has a big set of trapped geodesics (typically a fractal set), this creates many singularities for (ℓ_g, S_g) and its linearization. The analysis one has to perform is then quite involved. One needs to combine several different key tools, in particular:

(i) the proof of the C^2-regularity with respect to g of the operator $\mathcal{S}_g : C^\infty(\partial_+\mathcal{M}) \to \mathcal{D}'(\partial_-\mathcal{M})$ defined by $\mathcal{S}_g f := f \circ S_g$, see §4.2.1;
(ii) the exponential decay in $t \to \infty$ of the volume of points $(x, v) \in \mathcal{M} = SM$ that remain trapped for time t;
(iii) an elliptic estimate for the normal operator $\Pi_2 := I_2^* I_2$, see §4.2.2.

4.2.1. *Smoothness of the resolvent.* The first item is obtained by re-proving certain results of [DG16] on the resolvent of an Axiom A vector field X, but now with an explicit control of the dependence with respect to the vector field X. In particular, as a byproduct of this analysis we show the following result that could prove useful for other applications such as Fried's conjecture for manifolds with boundary, in the spirit of [DGRS20]:

Theorem 4.10. *Let $\mathcal{M}$ be a smooth manifold with boundary and let X_0 be a smooth vector field so that $\partial\mathcal{M}$ is strictly convex for the flow of X_0. Assume that the trapped set $K^{X_0} := \cap_{t\in\mathbb{R}}\varphi_t^{X_0}(\mathcal{M}^\circ)$ of the flow $(\varphi_t^{X_0})_{t\in\mathbb{R}}$ of X_0 is hyperbolic. Then, there exist $\delta > 0$, $N \gg 1$, such that for all $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ with $\|X - X_0\|_{C^N} < \delta$, the following holds:*

(i) *the resolvent $R^X(z) := (-X+z)^{-1} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$, initially defined in the half-plane $\{z \in \mathbb{C} \mid \Re(z) \gg 1\}$, extends meromorphically to $\mathbb{C}$ as a bounded operator $R^X(z) : C^\infty_{\mathrm{comp}}(\mathcal{M}^\circ) \to \mathcal{D}'(\mathcal{M}^\circ)$,*
(ii) *if $z_0 \in \mathbb{C}$ is not a pole of $R^{X_0}(z)$, then the map*

$$C^\infty(\mathcal{M}, T\mathcal{M}) \ni X \mapsto R^X(z_0) \in \mathcal{L}(C^\infty_{\mathrm{comp}}(\mathcal{M}^\circ), \mathcal{D}'(\mathcal{M}^\circ)),$$

is C^2-regular[15] *with respect to X.*

Here, we denote by $\mathcal{L}(A, B)$ the space of continuous linear maps between functional spaces A and B. In fact, we prove the result in anisotropic Sobolev spaces, but only need it in the distributional sense. We show that the scattering operator $\mathcal{S}_g$ has a Schwartz kernel that can be written as a restriction of the Schwartz kernel of $R^{X_g}(0)$ on $\partial_-\mathcal{M} \times \partial_+\mathcal{M}$, implying that the map $g \mapsto \mathcal{S}_g$ is C^2-regular as operators acting on some appropriate Sobolev spaces.

[15] Even though we only need C^2, our proof actually shows it is C^k for all $k \in \mathbb{N}$.

4.2.2. *Elliptic estimate.* We let

$$I_2^{g_0} : C^\infty(M, S^2T^*M) \to L^\infty_{\mathrm{loc}}(\partial_-\mathcal{M} \setminus \{\ell_{g_0} = \infty\})$$

be the X-ray transform on symmetric 2-tensors with respect to g_0, defined as

$$I_2^{g_0} h(x, v) := \int_0^{\ell_{g_0}(x,v)} h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt, \qquad (4.2)$$

if $\varphi_t^{g_0}(x, v) = (\gamma(t), \dot\gamma(t)) \in \mathcal{M}$. This operator appears as the differential of the length map $g \mapsto \ell_g(x, v)$ with respect to the metric g (at g_0) and therefore plays a fundamental role in the study of the lens rigidity problem.

Obviously, the operator $I_2^{g_0}$ has a big kernel which contains at least the tangent space to the orbit of the metric g_0 under the action of the group of diffeomorphisms equal to the identity on the boundary ∂M. One can actually show that the space of symmetric 2-tensors on M splits as

$$C^\infty(M, S^2T^*M) = \ker D_{g_0}^*|_{C^\infty(M,S^2T^*M)} \oplus D_{g_0}(C_0^\infty(M, T^*M)), \quad (4.3)$$

where D_{g_0} is the symmetric derivative and $D_{g_0}^*$ its adjoint, see §2.3.3 where these operators were introduced, and $C_0^\infty(M, T^*M)$ denotes 1-forms vanishing to first order on the boundary ∂M. The tensors $D_{g_0}p$ in (4.3) are called *potential tensors* and are equal to $D_{g_0}p = \mathcal{L}_V g_0$, where $V := p^\sharp$ is the vector field identified to p via the metric. In other words, potential tensors give the tangent space at g_0 to the orbit $\mathcal{O}(g_0) := \{\phi^* g_0 \mid \phi \in \mathrm{Diffeo}^0(M)\}$. The tensors in $\ker D_{g_0}^*$ are called *divergence-free* or *solenoidal* tensors and correspond to a genuine variation of the Riemannian structure. It is immediate to check that

$$D_{g_0}(C_0^\infty(M, T^*M)) \subset \ker I_2^{g_0}, \qquad (4.4)$$

and the question is: how bigger is this kernel? It is conjectured that generically (4.4) should be an equality, and that it should always be an equality for Anosov manifolds. This is known for instance on all negatively-curved manifolds with strictly convex boundary, see [Gui17b].

An important operator to consider is the *normal operator* $\Pi_2^{g_0} := (I_2^{g_0})^* I_2^{g_0}$ which enjoys good analytic properties. Roughly speaking, this is a pseudodifferential operator of order -1, elliptic on solenoidal tensors , but the presence of the boundary ∂M creates some difficulties to understand its precise behaviour. As a consequence, it is usually easier to embed artificially (M, g_0) in $(M_e, g_{0e}) \supset (M, g_0)$, a Riemannian extension of the manifold (M, g_0) which is also of Anosov type in the sense of Definition 4.3. We will denote by

$$E_0 : L^2(M, S^mT^*M) \to L^2(M_e, S^mT^*M_e)$$

the operator of extension by 0.

Proposition 4.11. *Let (M, g_0) be a manifold of Anosov type, and further assume that $I_2^{g_0}$ is solenoidal injective. Let (M_e, g_{0e}) be an extension of Anosov type of (M, g). Then, there exists $C > 0$ such that for all $f \in L^2(M, S^2T^*M) \cap \ker D_{g_0}^*$:*

$$\|f\|_{L^2(M)} \leq C\|\Pi_2^{g_{0e}} E_0 f\|_{H^1(M_e)}.$$

Proposition 4.11 is based on the study of the geodesic flow resolvent *à la* Dyatlov-Guillarmou [DG16] on anisotropic Sobolev spaces and also relies on the injectivity of $I_2^{g_0}$.

4.2.3. *Ending the proof.* The strategy of the proof then goes as follows. First of all, we put the metric g in solenoidal gauge (with respect to g_0), namely we find a first diffeomorphism $\psi \in \text{Diff}(M)$ such that $\psi|_{\partial M} = \text{id}$ and $g' = \psi^* g$ is divergence-free with respect to g_0.

We then show the following key estimate: there are $C, \mu > 0$ such that, if $(\ell_{g_0}, S_{g_0}) = (\ell_g, S_g)$ and $\|g' - g_0\|_{C^N} < \delta$ for some small $\delta > 0$, then

$$\|I_2^{g_0}(g' - g_0)\|_{H^{-6}(\partial_-\mathcal{M})} \leq C\|g' - g_0\|^{1+\mu}_{C^N(M, \otimes_S^2 T^*M)}. \tag{4.5}$$

The proof of this estimate is involved. It is based on some complex interpolation argument using the holomorphic map $\mathbb{C} \ni z \mapsto e^{-z\ell_{g_0}} I_2^{g_0}(g' - g_0)$ and the C^2-smoothness of the scattering map $g \mapsto \mathcal{S}_g$ as a continuous map from $C^\infty(\partial_+\mathcal{M})$ to $H^{-6}(\partial_-\mathcal{M})$. It also relies on some volume estimates on the set of geodesics trapped for time $t \to \infty$ that follow from [Gui17b].

Finally, slightly extending (M, g_0) to some (M_e, g_{0e}), using the mapping properties of the adjoint $(I_2^{g_{0e}})^*$, interpolation arguments, and (4.5), one obtains for $h := g' - g_0$:

$$\|h\|_{L^2} \leq C\|\Pi_2^{g_{0e}} E_0 h\|_{H^1} \leq C\|h\|^{1+\mu}_{C^N}, \tag{4.6}$$

where E_0 is the zero extension operator to M_e, $\Pi_2^{g_{0e}} = (I_2^{g_{0e}})^* I_2^{g_{0e}}$ is the normal operator, and the estimate on the left is an elliptic estimate of Proposition 4.11. It is left to interpolate C^N between L^2 and $C^{N'}$ in (4.6), where $N' \gg N$, to get for some $0 < \mu' < \mu$:

$$\|h\|_{L^2} \leq C\|h\|_{L^2}\|h\|^{\mu'}_{C^{N'}} \leq C\|h\|_{L^2}\|g - g_0\|^{\mu'}_{C^{N'}}.$$

For $\|g - g_0\|_{C^{N'}}$ small enough, this readily implies that $g' = \phi^* g = g_0$, concluding the proof.

5. Geodesic Lévy flights and expected stopping time for random searches

5.1. **Lévy flight foraging hypothesis.** The *Lévy flight foraging hypothesis* is a well-known hypothesis in the field of biology asserting that animals foraging behaviours should be modelled by Lévy flights insofar as they may optimize search efficiencies. While this hypothesis has been around for more than twenty years, it is still controversial and subject to many research articles investigating whether Brownian motion or Lévy flights are optimal search strategies [PCM14, SK86, VDLRS11, BN13, BLMV11, DGV22]. The purpose of this chapter is to shed a new theoretical light on this question by means of a precise mathematical study.

More precisely, we will investigate the *narrow capture problem* which consists in finding a small target in space for a motion whose law is that of a Lévy flight. The interesting quantity to understand then is the *expected capture time*, namely, the expected time that a process starting at a given point p will eventually find the target. This small target typically models a prey hunted by a predator whose foraging behaviour is modelled by the Lévy process. The Lévy flight foraging hypothesis can then be phrased as follows: *is the expected capture time significantly lower if one uses a search based on Lévy flights rather than on Brownian motion?*

For bounded domains in the Euclidian space, there are various search strategies based on Brownian motion and in this case an important set of literatures already investigated the expected time of finding small targets [SSH08, SSH06, GC15, CWS10, CF11, AKKL12]. However, while the background geometry of many animal foraging behaviours and constraint optimization searches are naturally curved, we have only recently started addressing the question of expected stopping time for Brownian motions on Riemannian manifolds [NTTT22, NTT21b, NTT21a]. Thus far, nothing has been done for stopping time for Lévy flight based searches even in flat geometry. We address this question here for a class of isotropic pure jump Lévy processes introduced by Applebaum–Estrade [AE00]. In particular, we investigate the asymptotics of the expected stopping time for a Lévy flight based random search to find a target the size of a small geodesic ball whose radius converges to zero.

5.2. **Statement of the problem. Main results.** The results of this paragraph can be found in [CGLT22]. We assume throughout that (M, g) is a smooth closed (that is, compact without boundary) connected n-dimensional Riemannian manifold with $n \geqslant 2$. We let

$(X_t)_{t\geqslant 0}$ be a cadlag semi-martingale on M which is an isotropic Lévy process in the sense of [AE00], induced by the isostropic Lévy measure

$$\nu_p(A) = C(n,\alpha)\int_A \frac{\mathrm{d}T_p(v)}{|v|_g^{n+2\alpha}}, \quad A\subset T_pM, \quad \alpha\in(0,1) \tag{5.1}$$

on each tanget space. Here T_p is the volume form on T_pM induced by the metric $g|_{T_pM}$ and $C(n,\alpha)$ is the constant[16]

$$C(n,\alpha) := \frac{4^\alpha\Gamma(n/2+\alpha)}{\pi^{n/2}|\Gamma(-\alpha)|}. \tag{5.2}$$

Fix $p_0\in M$ and let $B_\varepsilon(p_0)$ be the open geodesic ball of radius $\varepsilon>0$ centered at p_0. We define the *expected stopping time* as:

$$\tau_\varepsilon = \inf\left\{t\geqslant 0 \mid X_t\in\overline{B_\varepsilon(p_0)}\right\} \quad \text{and} \quad u_\varepsilon(q) = \mathbb{E}\left(\tau_\varepsilon \mid X_0 = q\right), \tag{5.3}$$

for each $q\in M$.

Below, we will denote by $\mathbb{S}^n$ the Riemannian n-dimensional sphere equipped with the round metric and by $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus with the flat metric. We say that a manifold is *Anosov* if its geodesic flow is Anosov on its unit tangent bundle. In particular, this includes all negatively-curved manifolds.

We will prove the following result.

Theorem 5.1 (Chaubet-Guedes Bonthonneau-L.-Tzou, '22)**.** *Assume that $M=\mathbb{S}^n,\mathbb{T}^n$ or is Anosov. Then the following holds.*

(i) *There is $c(n,\alpha)>0$ such that the average of u_ε has the expansion*

$$\frac{1}{|M|}\int_M u_\varepsilon \mathrm{dvol}_g \sim \frac{|M|c(n,\alpha)}{\varepsilon^{n-2\alpha}}, \quad \varepsilon\to 0.$$

(ii) *For each $p\neq p_0\in M$ (and $p\neq -p_0$ if $M=\mathbb{S}^n$),*

$$u_\varepsilon(p) - \frac{1}{|M|}\int_M u_\varepsilon \mathrm{dvol}_g \to |M|\mathrm{G}_{\mathscr{A}}(p,p_0), \quad \varepsilon\to 0,$$

where $\mathrm{G}_{\mathscr{A}}$ is the Green's function of the generator of $(X_t)_{t\geqslant 0}$ (see Theorem 5.5 and Corollary 5.6).

(iii) *If $M=\mathbb{S}^n$, $n>1+4\alpha$ and $1>(n-4)\alpha$ then for some $\tilde{c}(n,\alpha)\neq 0$,*

$$\left|u_\varepsilon(-p_0) - \frac{1}{M}\int_M u_\varepsilon \mathrm{dvol}_g\right| \sim \frac{|M|\tilde{c}(n,\alpha)}{\varepsilon^{n-1-4\alpha}}, \quad \varepsilon\to 0.$$

[16]This constant is chosen to be consistent with the definition of the fractional Laplacian on $\mathbb{R}^n$, which is the infinitesmial generator of 2α-stable isotropic Lévy processes in Euclidean space.

We will state below more precise results (Theorems 5.7 and 5.9) giving an explicit expression of the constants and the size of the remainders. While such results exist for Brownian motions in Euclidean domains [SSH08, SSH06, GC15, CWS10, CF11, AKKL12] and on general manifolds [NTTT22, NTT21b, NTT21a], this is the first such detailed analytical calculation for Lévy flights for such a broad class of geometries.

We emphasize that Theorem 5.1 shows that the asymptotics of the deviation of the expected stopping time from its average heavily depends on the underlying geometry. In particular on the sphere, antipodal points are conjugate[17], and this leads to a singular behavior of the expected stopping time at those points. Nevertheless, we expect that point (i) of Theorem 5.1 should remain valid for general Riemannian manifolds, regardless of the geometry. Theorem 5.1 follows from a detailed study of the analytic properties of the generator $\mathscr{A}$ of the Lévy process, see Theorems 5.5 and 5.4 below.

We finally observe that in the physical dimensions $n = 2$, the expected stopping time for the Brownian motion was shown to be of size $\mathcal{O}(|\log \varepsilon|)$ in [NTTT22] whereas it is here of size $\mathcal{O}(\varepsilon^{-(2-2\alpha)})$ by Theorem 5.1.

5.2.1. *Results on the generator.* While it is well understood that the infinitesimal generator for 2α-stable jump processes on Euclidean spaces are precisely the fractional powers of the Laplacian, the same may not hold for Lévy processes on closed compact Riemannian manifolds. In fact it was shown in [AE00] that if $(X_t)_{t\geqslant 0}$ is a cadlag semi-martingale valued in a Riemannian manifold (M, g), then it is an isotropic Lévy process iff it is a Feller process with infinitesimal generator $a\Delta_g + \mathscr{A}$ for some constant $a \geqslant 0$ and for $u \in C^\infty(M)$,

$$\mathscr{A} u(p) := \text{p.v.} \int_{v \in T_pM\setminus 0} \big(u(\exp_p(v)) - u(p)\big)\, \nu_p(\mathrm{d}v). \tag{5.4}$$

Here p.v. means that we take the principal value of the integral, $\{\nu_p\}_{p\in M}$ is a field of measures on T_pM induced from an isotropic Lévy measure ν on $\mathbb{R}^n$ by $\nu_p(A) = \nu(r^{-1}(A))$ whenever $\pi(r) = p$ and $r \in FM$ is an element of the orthonormal frame bundle over M. Alternatively, one can re-write the principal value of the integral (5.4) as

$$\frac{1}{2} \int_{v \in T_pM\setminus 0} \big(u(\exp_p(v)) + u(\exp_p(-v)) - 2u(p)\big)\, \nu_p(\mathrm{d}v).$$

[17]On the sphere, conjugate points correspond to pair of points that may be connected by a non-trivial continuous one-parameter family of geodesic paths.

Note that thanks to the isotropic assumption on ν, this definition is independent of the choice of frame $r \in FM$.

When the leading term in the generator is $a\Delta_g$ (i.e. $a > 0$), some mapping properties were analyzed in [AB21]. However, not much is known about the case when $a = 0$ (i.e. the process is "pure jump"). This is due to the fact that (5.4) is now the dominant driver of the process and integrating the exponential map is difficult to control beyond the injectivity radius on a general Riemannian manifold. We address this challenge for a broad class of Riemannian manifolds.

Throughout the chapter, we make the choice

$$\nu(A) = C(n,\alpha) \int_A \frac{1}{|v|^{n+2\alpha}} \mathrm{d}v, \quad \alpha \in (0,1), \tag{5.5}$$

for the Lévy measure, which is motivated by the fact that such processes on $\mathbb{R}^n$ are generated by the fractional Laplacian on the Euclidean space. Note that after pulling back by an element of the fiber of the orthonormal frame bundle FM over $p \in M$, this measure becomes the Lévy measure on T_pM described earlier in (5.1).

We will prove:

Theorem 5.2 (Chaubet-Guedes Bonthonneau-L.-Tzou, '22). *We let $(X_t)_{t\geqslant 0}$ be a cadlag semi-martingale valued on a Riemannian manifold (M,g) which is either $\mathbb{S}^n, \mathbb{T}^n$ or Anosov. If it is an isotropic Lévy process with pure jump induced by the Lévy measure (5.1), then its infinitesimal generator $\mathscr{A}$ is a non-positive Fredholm operator*

$$\mathscr{A} : W^{s,m}(M) \to W^{s-2\alpha,m}(M),$$

for all $s \in \mathbb{R}, m \in (1,\infty)$, that has discrete spectrum with one dimensional null-space and co-kernel.

We now give more details on our results on the generator on this Lévy process. The explicit presence of the exponential map in (5.4) suggests that the behaviour of $\mathscr{A}$ depends more on the geometry and the dynamics of geodesics than the fractional Laplacian. If $M = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the flat torus, the operator $\mathscr{A}$ happens (not surprisingly) to be the fractional Laplacian:

Theorem 5.3. *If (M,g) is the torus $\mathbb{T}^n$, the infinitesimal generator given by (5.4) is*

$$-\mathscr{A} = (-\Delta)^\alpha,$$

where Δ is the non-positive Laplace operator on $\mathbb{T}^n$. In particular, $\mathscr{A}$ is an elliptic, classical, pseudo-differential operator of order 2α.

This result is a byproduct of [AH14, Example 1] but can also be obtained by a simple explicit computation. Obviously, for general Riemannian manifolds, such an explicit computation will not be available. It turns out that in the case of the round unit sphere, $\mathscr{A}$ *does not* in fact resemble the fractional powers of the Laplacian and is actually a Fourier Integral Operator. On the unit sphere with round metric, we denote by $\mathscr{J} : \mathcal{D}'(\mathbb{S}^n) \to \mathcal{D}'(\mathbb{S}^n)$ the pullback by the antipodal map. We will prove that the following result holds.

Theorem 5.4. *If (M, g) is the sphere $\mathbb{S}^n$, the infinitesimal generator given by (5.4) can be written*

$$\mathscr{A} = \mathscr{A}_{2\alpha} + \mathscr{A}_0 + \mathscr{A}_{-1}\mathscr{J}$$

where for each $\ell = 2\alpha, 0, -1$, $\mathscr{A}_\ell \in \Psi^\ell_{\mathrm{cl}}(M)$ is a classical formally selfadjoint pseudodifferential operator of order ℓ. The operators $\mathscr{A}_{2\alpha}$ and $\mathscr{A}_{-1}$ have principal symbols $\sigma_{\mathscr{A}_{2\alpha}}(x, \eta) = -|\eta|_g^{2\alpha}$ and $\sigma_{\mathscr{A}_{-1}}(x, \eta) = c(n)|\eta|_g^{-1}$, for some constant $c(n) > 0$. All operators commute with the operator $\mathscr{J}$.

We shall see that since the integral kernel of $\mathscr{A}$ has singularities at both $p = q$ and $p = -q$ (antipodal point), it cannot be the fractional Laplacian. It is natural to deduce that the complications arising on the sphere are due to geodesics focusing at a single point (i.e. conjugate points). If we make assumptions about the manifold (M, g) as to rule out such behaviour, we should expect $\mathscr{A}$ to have a simpler expression. This is indeed the case if we assume that (M, g) is *Anosov*. We will prove the following result.

Theorem 5.5. *If (M, g) is a closed connected Anosov Riemannian manifold, the infinitesimal generator given by (5.4) can be written*

$$\mathscr{A} = \mathscr{A}_{2\alpha} + \mathscr{A}_0$$

where for each $\ell = 2\alpha, 0$, $\mathscr{A}_\ell \in \Psi^\ell_{\mathrm{cl}}(M)$ is a classical formally selfadjoint pseudodifferential operator of order ℓ.

More precisely, for each $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\chi(t) = 1$ for t near 0 and $\mathrm{supp}(\chi) \subset [0, r_{\mathrm{inj}}^2/2]$, where r_{inj} is the injectivity radius of (M, g), the operator (5.4) writes

$$\begin{aligned}\mathscr{A}u(p) = C(n,\alpha)\ \text{p.v.} \int_M \chi(\mathrm{dist}_g(p,q)^2) \frac{u(q) - u(p)}{\mathrm{dist}_g(p,q)^{n+2\alpha}} J(p,q) \mathrm{dvol}_g(q) \\ + w(p)u(p) + \int_M K(p,q)u(q)\mathrm{dvol}_g(q)\end{aligned} \tag{5.6}$$

for some smooth functions $w \in C^\infty(M)$ *and* $K \in C^\infty(M \times M)$*. Here we set* $J(p,q) = \det \mathrm{d}_q \exp_p^{-1}$.

An immediate observation is that when (M,g) is Anosov, the result of Theorem 5.5 implies that the operator $\mathscr{A}$ is an elliptic pseudodifferential operator with principal symbol $\sigma_{\mathscr{A}}(x,\xi) = -|\eta|_g^{2\alpha}$ if $\alpha \geqslant 1/2$ and $\sigma_{\mathscr{A}}(x,\xi) = -|\eta|_g^{2\alpha} + \sigma_{\mathscr{A}_0}(x,\xi)$ if $\alpha < 1/2$. Also remark that, when (M,g) is Anosov, the trace formula of Duistermaat-Guillemin [DG75b] implies that the spectrum of $\mathscr{A}$ determines uniquely the lengths of periodic geodesics.

Theorems 5.3, 5.4 and 5.5 imply the following:

Corollary 5.6. *If* (M,g) *is* $\mathbb{S}^n, \mathbb{T}^n$ *or Anosov, the following holds.*

(i) $-\mathscr{A}$ *extends to a formally selfadjoint Fredholm operator*

$$-\mathscr{A} : W^{s,m}(M) \to W^{s-2\alpha,m}(M),$$

for all $s \in \mathbb{R}, m \in (1,\infty)$*, with non-negative discrete spectrum and smooth eigenfunctions for all* $s \in \mathbb{R}$*. The null-space consists of only constant functions.*

(ii) *There exists* $\mathscr{A}^+ : W^{s,m}(M) \to W^{s+2\alpha,m}(M)$ *with* $\mathrm{Ker}(\mathscr{A}^+) = \mathbb{C}\cdot\mathbf{1}$ *and* $\mathrm{Ran}(\mathscr{A}^+) \perp \mathbb{C}\cdot\mathbf{1}$ *such that*

$$\mathscr{A}^+\mathscr{A} = \mathscr{A}\mathscr{A}^+ = I - P$$

where P *is the* L^2 *orthonormal projection to the space of constant functions. The Schwartz kernel of* $\mathscr{A}^+$*,* $\mathrm{G}_{\mathscr{A}}(p,q)$*, which we will call the Green's function, satisfies, for each* $p \in M$ *and* $u \in C^\infty(M)$*,*

$$\int_M \mathrm{G}_{\mathscr{A}}(p,\cdot)\mathscr{A}u \,\mathrm{dvol}_g = u(p) - |M|^{-1}\int_M u\mathrm{dvol}_g$$

The heat kernel $e^{t\mathscr{A}}$ has bounded integral kernel for all $t > 0$ and is therefore trace class. Since the spectrum is discrete and the operator is semidefinite, the solutions of the heat equation converge exponentially in $L^2(M)$ to the constant function because $\mathrm{Ker}(\mathscr{A}) = \mathbb{C}\cdot\mathbf{1}$. This is an important remark because the heat kernel is precisely what governs the probability of finding X_t at a point q if $X_0 = p$. At last, we have the Poincaré inequality on the sphere, torus, and Anosov case:

$$-\int_M u\,\mathscr{A}u\,\mathrm{dvol}_g \geqslant c\|u\|_{L^2}^2. \tag{5.7}$$

for all $u \perp \mathbb{C}\cdot\mathbf{1}$.

5.2.2. *Applications to random searches.* As in §5.2, let $(X_t)_{t\geq 0}$ be a cadlag semi-martingale that is an isotropic Lévy process with infinitesimal generator $\mathscr{A}$ defined by (5.4). Let $B_\varepsilon(p_0)$ be the open geodesic ball of radius $\varepsilon > 0$ centred at p_0. We define τ_ε and u_ε by (5.3). Let $c(n,\alpha)$ be the constant given by

$$c(n,\alpha) := \begin{cases} \dfrac{2^{-2\alpha}(1-\alpha)\Gamma(1-\alpha)^2}{\pi^2}, & \text{if } n = 2, \\ \dfrac{2^{1-2\alpha}\Gamma(n/2-\alpha)\Gamma(n/2-\alpha+1)}{\pi^{n/2}(n-2)\Gamma(n/2-1)}, & \text{if } n \geqslant 3. \end{cases} \tag{5.8}$$

Then we have the following result, which is a more precise version of Theorem 5.1, involving remainder terms.

Theorem 5.7. *If (M,g) is a closed connected Anosov Riemannian manifold, then:*

(i) *As $\varepsilon \to 0$, the average of u_ε over M has expansion*

$$\frac{1}{|M|}\int_M u_\varepsilon \mathrm{dvol}_g = \varepsilon^{2\alpha-n}|M|c(n,\alpha)(1+\mathcal{O}(E(\alpha,\varepsilon))),$$

where the error term $E(\alpha,\varepsilon)$ is given by

$$E(\alpha,\varepsilon) = \begin{cases} \varepsilon^{2\alpha}, & \text{if } \alpha < 1/2, \\ \varepsilon|\log\varepsilon|, & \text{if } \alpha = 1/2, \\ \max(\varepsilon, \varepsilon^{n-2\alpha}), & \text{if } \alpha > 1/2. \end{cases} \tag{5.9}$$

(ii) *For all $\varepsilon > 0$, $u_\varepsilon \in C^\infty(M \setminus \partial B_\varepsilon(p_0)) \cap L^\infty(M)$. Moreover, for all $p \neq p_0$, we have as $\varepsilon \to 0$*

$$u_\varepsilon(p) - \frac{1}{|M|}\int_M u_\varepsilon \mathrm{dvol}_g = |M|\mathrm{G}_{\mathscr{A}}(p,p_0) + \mathcal{O}(E(\alpha,\varepsilon)) \tag{5.10}$$

where $\mathrm{G}_{\mathscr{A}}(p,q)$ is the Green's function of $\mathscr{A}$ given by (iii) of Theorem 5.5.

For the torus, the same result holds, up to changing the error term:

Theorem 5.8. *If (M,g) is $\mathbb{T}^n$, then the conclusions of Theorem 5.7 hold if we replace the error term (5.9) by*

$$E(\alpha,\varepsilon) = \begin{cases} \max(\varepsilon, \varepsilon^{n-2\alpha}), & \text{if } \alpha \neq 1/2, \\ \varepsilon|\log\varepsilon|, & \text{if } \alpha = 1/2. \end{cases} \tag{5.11}$$

These asymptotics are similar to the ones computed in [NTT21b, NTTT22] for the Brownian motion. When $\alpha > 0$ is small the situation on the sphere is quite different from that of Anosov manifolds. Due to the singularity structure of $\mathscr{A}$ when $M = \mathbb{S}^n$, a propagation phenomena occurs from p_0 to $-p_0$ to create, as $\varepsilon \to 0$, a blowup of the quantity

$$\left| u_\varepsilon(-p_0) - |M|^{-1} \int_M u_\varepsilon \mathrm{dvol}_g \right|.$$

We will prove that the following holds:

Theorem 5.9. *If (M, g) is $\mathbb{S}^n$, then:*

(i) *The average value of u_ε over M is the same as in Theorem 5.7.*
(ii) *For all $\varepsilon > 0$, we have*

$$u_\varepsilon \in C^\infty\left(M \setminus (\partial B_\varepsilon(p_0) \cup \partial B_\varepsilon(-p_0))\right) \cap L^\infty(M)$$

and (5.10) holds whenever $p \notin \{p_0, -p_0\}$ where $\mathrm{G}_{\mathscr{A}}$ is given by Corollary 5.6.
(iii) *If $n > 1 + 4\alpha$ and $1 > (n-4)\alpha$, then at $p = -p_0$ we have*

$$\left| u_\varepsilon(-p_0) - \frac{1}{|M|} \int_M u_\varepsilon \mathrm{dvol}_g \right| = \frac{\tilde{c}(\alpha, n)|M|}{\varepsilon^{n-1-4\alpha}} + o(\varepsilon^{-n+1+4\alpha}) \tag{5.12}$$

for some $\tilde{c}(\alpha, n) > 0$, which we do not make explicit.

Following Theorem 5.9, it would be interesting to understand the generator $\mathscr{A}$ and the narrow capture problem in other settings than the sphere where conjugate points appear, like Zoll manifolds for instance. This is left for future investigation.

5.3. **Proof ideas.** We now briefly explain the ideas behind the proof of Theorems 5.7, 5.8 and 5.9.

5.3.1. *Pseudodifferential behaviour.* The first step is to prove Theorems 5.3, 5.4 and 5.5, namely that the generator of the Lévy process is an elliptic pseudodifferential operator (or FIO in the sphere case) of order 2α. For the sphere and the torus, the proof relies on an explicit computation. However, when the manifold is Anosov, the proof relies on the hyperbolic behaviour of the geodesic flow: the generator is shown to be a kind of generalized X-ray transform, quite similar to the one introduced in §2.4 (with the flat trivial line bundle), and the formalism of anisotropic Sobolev spaces can be applied. The key lemma in the proof is the propagation of singularities for pseudodifferential operators of real principal type, see §A.3 where this is further discussed. Then, Corollary 5.6 is an immediate consequence of the theory of pseudodifferential operators, see (A.4).

5.3.2. *Key lemmas.* Recall that, if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $(X_t)_{t\geqslant 0}$ denotes the Brownian motion and τ denotes the exit time of Ω for this process, then the expected exit time

$$u(p) := \mathbb{E}(\tau \mid X_0 = p)$$

solves the equation $\Delta u = -1$ in Ω with Dirichlet boundary condition $u|_{\partial\Omega} = 0$, see [Tay13, Chapter 11] for instance. We claim that the expected stopping time for our Lévy flight satisfies similar properties as the ones connecting the exit time of the Brownian motion to the Dirichlet Laplacian.

Proposition 5.10. *If (M, g) is the round sphere, the flat torus, or Anosv, the expected stopping time satisfies the following properties ($1 < m < 1/\alpha$):*

(i) *In the Anosov and flat torus case,*

$$u_\varepsilon \in L^\infty(M) \cap C^\infty(M \setminus \partial B_\varepsilon(p_0)) \cap W^{2\alpha,m}(\overline{\Omega_\varepsilon}),$$

while in the sphere case

$$u_\varepsilon \in L^\infty(M) \cap C^\infty(M \setminus \partial B_\varepsilon(\pm p_0)) \cap W^{2\alpha,m}(\overline{\Omega_\varepsilon}),$$

for all $\varepsilon > 0$.

(ii) *One has the fundamental relation:*

$$\mathscr{A} u_\varepsilon = -1, \text{ on } \Omega_\varepsilon := M \setminus \overline{B_\varepsilon(p_0)}, \qquad u_\varepsilon = 0, \text{ on } \overline{B_\varepsilon(p_0)}. \tag{5.13}$$

This is very similar to the equation satisfied by the expected exit time for the Brownian motion. Note, however, that due to the non-locality of the generator $\mathscr{A}$, the boundary condition $u = 0$ on $\partial\Omega$ in the Laplacian case has to be replaced here by $u_\varepsilon = 0$ on $\overline{B_\varepsilon(p_0)}$. This will actually create a lot of troubles in the proofs and showing (5.13) actually requires some effort.

Using (5.13), we can introduce $F_\varepsilon \in \mathcal{D}'(M)$ such that

$$\mathscr{A} u_\varepsilon = F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}. \tag{5.14}$$

By construction, F_ε is a distribution supported in $\overline{B_\varepsilon(p_0)}$. An important idea will be to study the properties of F_ε (and not that of u_ε directly), and then to deduce properties from (5.14) properties for the expected stopping time u_ε.

Before stating the result, we need to introduce some notation. First, we introduce rescaled geodesic coordinates centred at p_0: let $(\mathbf{e}_1, ..., \mathbf{e}_n) \in T_{p_0}M$ be a orthonormal basis and define $\psi_\varepsilon : \mathbb{B}^n \to B_\varepsilon(p_0)$ by

$$\psi_\varepsilon(x) := \exp_{p_0}(\varepsilon x_1 \mathbf{e}_1 + \cdots + \varepsilon x_n \mathbf{e}_n). \tag{5.15}$$

The following holds:

Proposition 5.11. *The distribution $F_\varepsilon \in \mathcal{D}'(M)$ satisfies:*

(i) $\operatorname{supp}(F_\varepsilon) \subset \overline{B_\varepsilon(p_0)}$ *and* $F_\varepsilon \in C^\infty(M \setminus \partial B_\varepsilon(p_0))$,

(ii) $u_\varepsilon = \mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) + C_\varepsilon$ *where*

$$\begin{aligned} C_\varepsilon &:= |M|^{-1} \int_M u_\varepsilon(p) \mathrm{dvol}_g(p) \\ &= \varepsilon^{2\alpha-n}|M| c(n,\alpha)(1 + \mathcal{O}(E(\alpha,\varepsilon))), \end{aligned} \tag{5.16}$$

and $c(n, \alpha)$ is given by (5.8) and the error term $E(\alpha, \varepsilon)$ is

$$E(\alpha,\varepsilon) = \begin{cases} \varepsilon^{2\alpha}, & \text{if } \alpha < 1/2, \\ \varepsilon|\log\varepsilon|, & \text{if } \alpha = 1/2, \\ \max(\varepsilon, \varepsilon^{n-2\alpha}), & \text{if } \alpha > 1/2. \end{cases} \tag{5.17}$$

(iii) $F_\varepsilon \in L^m(\overline{B_\varepsilon(p_0)}) \cap C^\infty(B_\varepsilon(p_0))$ *for all $m \in (1, 1/\alpha)$ and in the coordinate system (5.15), F_ε has expansion*

$$F_\varepsilon(\psi_\varepsilon(x)) = -\frac{|M|}{\varepsilon^n} \left(\int_{\mathbb{B}^n} \frac{\mathrm{d}x}{(1-|x|^2)^\alpha} \right)^{-1} \left(\frac{1}{(1-|x|^2)^\alpha} + \mathcal{O}_{L^m(\overline{\mathbb{B}^n})}(E(\alpha,\varepsilon)) \right). \tag{5.18}$$

Remark 5.12. If (M, g) is $\mathbb{T}^n$, then the error term (5.17) can be replaced by (5.11).

Although the statements may sound natural, the proof of Propositions 5.10 and 5.11 is actually involved, due to the nonlocality of the generator $\mathscr{A}$ which causes trouble understanding the analytic properties of $u_\varepsilon, F_\varepsilon$ on $\partial B_\varepsilon(p_0)$. We mainly follow the strategy of [Get61] which deals with a similar problem in a simpler setting where $\mathscr{A}$ is the fractional Laplacian in $\mathbb{R}^n$.

5.3.3. *Structure of the argument.* Propositions 5.10 and 5.11 do not reflect the structure of the argument: basically, the main issue is that we cannot show *directly* that $\mathscr{A} u_\varepsilon = -1$ on Ω_ε. The idea, somehow, is to revert the logic of the argument. First, let us make some quick observations. Assuming that the fundamental equation $\mathscr{A} u_\varepsilon = -1$ holds on $\Omega_\varepsilon = M \setminus \overline{B_\varepsilon(p_0)}$ and $u_\varepsilon = 0$ on $\overline{B_\varepsilon(p_0)}$, that is, (5.14) holds, we get by applying $\mathscr{A}^+$ to both sides of (5.14) that

$$\mathscr{A}^+ \mathscr{A} u_\varepsilon = u_\varepsilon - |M|^{-1} \int_M u_\varepsilon(p) \mathrm{dvol}_g(p) = \mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}),$$

that is $\mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) = u_\varepsilon - C_\varepsilon$. Since u_ε vanishes on $B_\varepsilon(p_0)$, we thus get:

$$\mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) = -C_\varepsilon, \qquad \text{on } B_\varepsilon(p_0). \tag{5.19}$$

Moreover, integrating (5.14) with respect to dvol_g, we get

$$\langle F_\varepsilon, \mathrm{dvol}_g\rangle = |\Omega_\varepsilon|. \tag{5.20}$$

The pair of equations (5.19) - (5.20) with unknowns $(F_\varepsilon, C_\varepsilon)$ will be called the *integral equation*. The argument then goes as follows:

Step 1: Existence and uniqueness to the integral equation. First, we *construct* a pair of solution $(\widetilde{F}_\varepsilon, \widetilde{C}_\varepsilon)$ to the integral equation (5.19) - (5.20) such that $\widetilde{F}_\varepsilon$ has support in $\overline{B_\varepsilon(p_0)}$ and control precisely its analytic properties, that is, show that it satisfies the content of Proposition 5.11. More precisely, we show:

Proposition 5.13 (Existence and uniqueness of regular solutions to the integral equation)**.** *Let (M, g) be either Anosov, the flat torus, or the round sphere. For $\varepsilon > 0$ small enough, there* exists *a* unique $\widetilde{F}_\varepsilon \in L^m(\overline{B_\varepsilon(p_0)}) \cap C^\infty(B_\varepsilon(p_0))$ *with $m \in (1, 1/\alpha)$ and constant $\widetilde{C}_\varepsilon$ solving* (5.19) - (5.20)*. Moreover, $\widetilde{C}_\varepsilon$ satisfies the expansion* (5.16) *and $\widetilde{F}_\varepsilon$ satisfies the expansion* (5.18) *in Proposition 5.11.*

Step 2: Uniqueness of solutions to the fundamental equation. We then set

$$\widetilde{u}_\varepsilon := \mathscr{A}^+(\widetilde{F}_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) + \widetilde{C}_\varepsilon. \tag{5.21}$$

By construction, the distribution $\widetilde{u}_\varepsilon$ satisfies $\mathscr{A}\widetilde{u}_\varepsilon = -1$ on Ω_ε, $\widetilde{u}_\varepsilon = 0$ on $B_\varepsilon(p_0)$. Moreover, as a consequence of Proposition 5.13, $\widetilde{u}_\varepsilon$ lies in some Sobolev space of positive regularity, that is, $\widetilde{u}_\varepsilon \in \dot{W}^{2\alpha,m}(\overline{\Omega_\varepsilon})$, where:

$$\dot{W}^{2\alpha,m}(\overline{\Omega_\varepsilon}) := \left\{u \in W^{2\alpha,m}(M) \mid \mathrm{supp}(u) \subset \overline{\Omega_\varepsilon}\right\}.$$

We then show that the following uniqueness result holds:

Proposition 5.14 (Uniqueness of regular solutions to the fundamental equation)**.** *Let (M, g) be the round sphere, the torus, or Anosov. Let $w \in \dot{W}^{2\alpha,m}(\overline{\Omega_\varepsilon})$, $1 < m < \infty$. Assume that $\mathscr{A}w = -1$ on Ω_ε. Then $w = u_\varepsilon$.*

This uniqueness result should be compared with [Get61, Corollary 5.1]. Proposition 5.14 will therefore imply that $u_\varepsilon = \widetilde{u}_\varepsilon \in \dot{W}^{2\alpha,m}(\overline{\Omega_\varepsilon})$ satisfies the fundamental equation $\mathscr{A}u_\varepsilon = -1$ on Ω_ε. The idea behind Proposition 5.14 is to use an integral representation formula relating u_ε to the generator of a bounded semi-group on Ω_ε.

The proofs of Propositions 5.10 and 5.11 are then straightforward, combining both Propositions 5.13 and 5.14.

Proof of Proposition 5.10. By the previous Propositions, we have that $u_\varepsilon = \widetilde{u}_\varepsilon \in \dot{W}^{2\alpha,m}(\overline{\Omega_\varepsilon})$ and u_ε satisfies the fundamental relation (5.13). Moreover, in the case of Anosov manifolds or the torus, $u_\varepsilon \in C^\infty(M \setminus \partial B_\varepsilon(p_0))$ follows from standard elliptic regularity since $u_\varepsilon = 0$ in $B_\varepsilon(p_0)$, $\mathscr{A} u_\varepsilon = -1$ on Ω_ε and $\mathscr{A}$ is pseudodifferential elliptic. In the sphere case, we get similarly that $u_\varepsilon \in C^\infty(M \setminus \partial B_\varepsilon(\pm p_0))$ by elliptic regularity of $\mathscr{A}$ (up to antipodal points). ☐

Proof of Proposition 5.11. It suffices to prove that $\widetilde{F}_\varepsilon = F_\varepsilon$, where F_ε is defined by (5.14) and $\widetilde{F}_\varepsilon$ solves the integral equation (5.19) - (5.20), and similarly that $\widetilde{C}_\varepsilon = C_\varepsilon$. But by definition of $\widetilde{u}_\varepsilon$ in (5.21), and by (5.14), one has using Proposition 5.14:

$$\widetilde{u}_\varepsilon = \mathscr{A}^+(\widetilde{F}_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) + \widetilde{C}_\varepsilon = u_\varepsilon = \mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon}) + C_\varepsilon.$$

Integrating the previous equation over M yields $C_\varepsilon = \widetilde{C}_\varepsilon$. Hitting the previous equation with $\mathscr{A}$ then yields $F_\varepsilon = \widetilde{F}_\varepsilon$. This concludes the proof. ☐

5.3.4. *Proof of the stopping time asymptotics.* We show how Theorems 5.7, 5.8 and 5.9 can be deduced from Propositions 5.10 and 5.11. We only deal with the Anosov case in order to simplify the exposition.

Proof of Theorem 5.7. First of all, observe that, using Proposition 5.11, the following holds: let $G \in \mathcal{D}'(M)$ be smooth in a neighbourhood of p_0, then for all $\varepsilon > 0$ sufficiently small,

$$\int_{B_\varepsilon(p_0)} F_\varepsilon(q) G(q) \mathrm{dvol}_g(q) = |M| G(p_0) + \mathcal{O}(E(\alpha, \varepsilon)), \tag{5.22}$$

where $E(\alpha, \varepsilon)$ is given by (5.17).

We now fix $p \neq p_0$ assume that $\varepsilon > 0$ is sufficiently small such that $p \notin B_\varepsilon(p_0)$. Since $u_\varepsilon - C_\varepsilon = \mathscr{A}^+(F_\varepsilon - \mathbf{1}_{\Omega_\varepsilon})$, we have:

$$\begin{aligned} u_\varepsilon(p) - C_\varepsilon &= \left(\mathscr{A}^+ 1_{B_\varepsilon(p_0)}\right)(p) + \int_{B_\varepsilon(p_0)} \mathrm{G}_{\mathscr{A}}(p,q) F_\varepsilon(q) \mathrm{dvol}_g(q). \\ &= O(\varepsilon^n) + \int_{B_\varepsilon(p_0)} \mathrm{G}_{\mathscr{A}}(p,q) F_\varepsilon(q) \mathrm{dvol}_g(q). \end{aligned}$$

Since $\mathscr{A}^+$ is a pseudodifferential operator, $\mathrm{G}_{\mathscr{A}}(p,q)$ is smooth away from the the set $\{p = q\}$. Since we have taken $\varepsilon > 0$ so that $p \notin B_\varepsilon(p_0)$, $F_\varepsilon(q)$ is integrated against a smooth function of q. We now use the expansion produced in (5.22) to get

$$u_\varepsilon(p) - C_\varepsilon = |M| \mathrm{G}_{\mathscr{A}}(p, p_0) + \mathcal{O}(E(\alpha, \varepsilon)).$$

Recalling now that $\Omega_\varepsilon = M \setminus B_\varepsilon(p_0)$ and $C_\varepsilon = |M|^{-1} \int_M u_\varepsilon(q) \mathrm{dvol}_g(q)$ has expansion given in Proposition 5.11 concludes the proof of Theorem 5.7. □

Appendix A. A Hitchhiker guide to microlocal analysis

We refer to [Hö3, Hö7, Shu01, GS94] for an extensive treatment of pseudodifferential operators on closed manifolds.

A.1. Definitions. First properties. Ellipticity.

Let $\mathcal{M}$ be a closed n-dimensional manifold. For $k \in \mathbb{R}$, we define $S^k(T^*\mathcal{M}) \subset C^\infty(T^*\mathcal{M})$, the space of symbols of order k, as the set of smooth functions a satisfying the following bounds, in any coordinate chart $U \subset \mathbb{R}^n$: for all $\alpha, \beta \in \mathbb{N}^n$, there exists $C := C(U, \alpha, \beta) > 0$ such that

$$\forall (x,\xi) \in T^*U \simeq \mathbb{R}^n \times \mathbb{R}^n, \qquad |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C\langle \xi \rangle^{k-|\alpha|}. \tag{A.1}$$

It can be checked that (A.1) is invariant by diffeomorphism, which implies that $S^k(T^*\mathcal{M})$ is intrinsically defined on $\mathcal{M}$.

We define $\Psi^{-\infty}(\mathcal{M})$, the set of *smoothing operators*, as the space of linear operators on $\mathcal{M}$ with smooth Schwartz Kernel. Denote by Op a quantization procedure on $\mathcal{M}$, given in a local coordinate patch $U \subset \mathbb{R}^n$ by:

$$\mathrm{Op}(a)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i\xi\cdot(x-y)} a(x,\xi) f(y) |dy||d\xi|,$$

where $a \in S^k(T^*U)$ and $f \in C^\infty_{\mathrm{comp}}(U)$. The set of *pseudodifferential operators* of order $k \in \mathbb{R}$ is then defined as

$$\Psi^k(\mathcal{M}) := \left\{ \mathrm{Op}(a) + R \mid a \in S^k(T^*\mathcal{M}), R \in \Psi^{-\infty}(\mathcal{M}) \right\}.$$

It can be checked that $\Psi^k(\mathcal{M})$ is intrinsically defined and independent on the choice of quantization Op.

There exists a well-defined *principal symbol map* $\sigma : \Psi^k(\mathcal{M}) \to S^k(T^*\mathcal{M})/S^{k-1}(T^*\mathcal{M})$ such that we have the following exact sequence:

$$0 \longrightarrow \Psi^{k-1}(\mathcal{M}) \longrightarrow \Psi^k(\mathcal{M}) \longrightarrow S^k(T^*\mathcal{M})/S^{k-1}(T^*\mathcal{M}) \longrightarrow 0.$$

The elliptic set $\mathrm{ell}(A) \subset T^*\mathcal{M} \setminus \{0\}$ of an operator $A \in \Psi^k(T^*\mathcal{M})$ is defined as the (open) conic set of points $(x_0, \xi_0) \in T^*\mathcal{M} \setminus \{0\}$ such that there exists a constant $C > 0$ such that the following holds:

$$\begin{aligned} |\xi| \geq C \text{ and } d_{S^*\mathcal{M}}\left((x, \xi/|\xi|), (x_0, \xi_0/|\xi_0|)\right) < 1/C \\ \Longrightarrow |\sigma_A(x,\xi)| \geq \langle \xi \rangle^k / C. \end{aligned} \tag{A.2}$$

Here $d_{S^*\mathcal{M}}$ is any metric on the cosphere bundle $S^*\mathcal{M} := T^*\mathcal{M}/\mathbb{R}_+$, where the $\mathbb{R}_+$-action is given by radial dilation in the fibers of $T^*\mathcal{M}$. An operator is said to be *elliptic* if $\mathrm{ell}(A) = T^*\mathcal{M} \setminus \{0\}$. The *characteristic set* $\Sigma(A)$ of an operator is the closed conic subset defined as the complement of the elliptic set in $T^*\mathcal{M}$.

Example A.1. Let $X \in C^\infty(\mathcal{M}, T\mathcal{M})$ be a vector field on $\mathcal{M}$, seen as a differential operator of order 1. Then, its principal symbol is $\sigma_X(x,\xi) = i\langle \xi, X(x)\rangle$ and thus, using (A.2), it is immediate to check that $\Sigma(X) = \{\langle \xi, X(x)\rangle = 0\}$ and $\mathrm{ell}(A) = (T^*\mathcal{M} \setminus \{0\}) \setminus \Sigma(X)$.

The important property of elliptic operators is that they are invertible modulo smoothing remainders:

Lemma A.2. *Let $A \in \Psi^k(\mathcal{M})$ and further assume that $(x_0, \xi_0) \in \mathrm{ell}(A)$. Then, there exists $\chi \in S^0(T^*\mathcal{M})$, supported on a small conic neighborhood of (x_0,ξ_0) and equal to 1 on a smaller conic neighborhood of (x_0,ξ_0), and $B \in \Psi^{-k}(T^*\mathcal{M}), R_L, R_R \in \Psi^{-\infty}(\mathcal{M})$ such that*

$$BA = \mathrm{Op}(\chi) + R_L, \qquad AB = \mathrm{Op}(\chi) + R_R.$$

In particular, if $\mathrm{ell}(A) = T^*\mathcal{M} \setminus \{0\}$, one can take $\chi = \mathbf{1}$ and $\mathrm{Op}(\mathbf{1}) = \mathbb{1}$.

The *wavefront set* $\mathrm{WF}(A)$ (or the *microsupport*) of an operator $A \in \Psi^k(\mathcal{M})$ is the (closed) conic subset of $T^*\mathcal{M}\setminus\{0\}$ satisfying the following property: $(x_0,\xi_0) \notin \mathrm{WF}(A)$ if and only if for all $m \in \mathbb{R}$, for all $b \in S^m(T^*\mathcal{M})$ supported in a small conic neighborhood of (x_0,ξ_0), one has $A\,\mathrm{Op}(b) \in \Psi^{-\infty}(\mathcal{M})$. In other words, the complement of the wavefront set of A is the set of codirections where A behaves as a smoothing operator.

The wavefront set $\mathrm{WF}(u)$ of a distribution $u \in \mathcal{D}'(SM)$ is the (closed) conic subset of $T^*\mathcal{M} \setminus \{0\}$ satisfying the following property: $(x_0,\xi_0) \notin \mathrm{WF}(u)$ if and only if there exists a small open conic neighborhood V of (x_0,ξ_0) such that for all $k \in \mathbb{R}$, for all $A \in \Psi^k(\mathcal{M})$ with wavefront set contained in V, one has $Au \in C^\infty(\mathcal{M})$. The wavefront set captures the set of (co)directions in which the distribution is irregular.

A.2. **Sobolev spaces. Elliptic estimate.** Let g be an arbitrary metric on $\mathcal{M}$ and denote by $\Delta_g \geq 0$ the nonnegative Hodge Laplacian acting on functions. For all $s \in \mathbb{R}$, the operator $(\mathbb{1} + \Delta)^s$ defined using the spectral theorem (applied to the selfadjoint operator Δ_g on $L^2(\mathcal{M}, \mathrm{vol}_g)$) is an invertible pseudodifferential operator of order $2s$.

For $s \in \mathbb{R}$, $u \in C^\infty(\mathcal{M})$, we set

$$\|u\|_{H^s} := \|(\mathbb{1} + \Delta)^{s/2} u\|_{L^2}, \tag{A.3}$$

and define $H^s(\mathcal{M})$ to be the completion of $C^\infty(\mathcal{M})$ with respect to the norm (A.3). Note that $H^s(\mathcal{M})$ is intrinsically defined, that is, it is independent of the choice of metric g, and changing the metric only changes the norm (A.3) by an equivalent norm. A distribution $u \in \mathcal{D}'(SM)$ is said to *microlocally H^s near* $(x_0,\xi_0) \in T^*\mathcal{M} \setminus \{0\}$ if the following holds: there exists a small open conic neighborhood V of (x_0, ξ_0) such that for all $A \in \Psi^0(\mathcal{M})$ with $\mathrm{WF}(A) \subset V$, one has $Au \in H^s(\mathcal{M})$.

The following boundedness result for pseudodifferential operators holds: for all $k \in \mathbb{R}, A \in \Psi^k(\mathcal{M})$ and $s \in \mathbb{R}$,

$$A : H^{s+k}(\mathcal{M}) \to H^s(\mathcal{M}) \tag{A.4}$$

is bounded. In particular, combining the parametrix construction of Lemma A.2 for elliptic operators and the boundedness (A.4), one obtains the following:

Lemma A.3. *Let $A \in \Psi^k(\mathcal{M})$ be an elliptic pseudodifferential operator. Then, for all $s \in \mathbb{R}, N > 0$, there exists a constant $C > 0$ such that: for all $u \in C^\infty(\mathcal{M})$,*

$$\|u\|_{H^{s+k}} \leq C \left(\|Au\|_{H^s} + \|u\|_{H^{-N}}\right). \tag{A.5}$$

Moreover, if $u \in \mathcal{D}'(\mathcal{M})$ is merely a distribution, $u \in H^{-N}(\mathcal{M})$ and $Au \in H^s(\mathcal{M})$, then $u \in H^{s+k}(\mathcal{M})$ and (A.5) holds.

We now describe a similar bound to (A.5) in the case where A is not elliptic but of *real principal type*. This is known as the *propagation of singularities* for pseudodifferential operators.

A.3. **Propagation of singularities.** Let $P \in \Psi^m(\mathcal{M})$ be a pseudodifferential operator. We will say that P is of *real principal type* if its principal symbol is real-valued and homogeneous (of order m). For such an operator, we denote by $H_P \in C^\infty(T^*\mathcal{M}, T(T^*\mathcal{M}))$ the Hamiltonian vector field on $T^*\mathcal{M}$ (equipped with the standard Liouville 2-form) generated by the principal symbol σ_P, and by $(\Phi_t)_{t\in\mathbb{R}}$ the Hamiltonian flow it generates.

Lemma A.4. *Let $P \in \Psi^m(\mathcal{M})$ be a pseudodifferential operator of real principal type. Let $A, B, B_0 \in \Psi^0(\mathcal{M})$ such that the following holds: for every $(x, \xi) \in \mathrm{WF}(A)$, there exists a time $T > 0$ such that $\Phi_{-T}(x, \xi) \in \mathrm{ell}(B)$ and for all $t \in [0, T]$, $\Phi_{-t}(x, \xi) \in \mathrm{ell}(B_0)$. Then, for all $s \in \mathbb{R}, N > 0$, there exists a constant $C > 0$ such that: for all $u \in C^\infty(\mathcal{M})$,*

$$\|Au\|_{H^{s+m-1}} \leq C \left(\|Bu\|_{H^{s+m-1}} + \|B_0 Pu\|_{H^s} + \|u\|_{H^{-N}}\right). \tag{A.6}$$

Moreover, if $u \in \mathcal{D}'(\mathcal{M})$ is merely a distribution, $u \in H^{-N}(\mathcal{M})$, $Bu \in H^{s+m-1}(\mathcal{M})$ and $B_0 Pu \in H^s(\mathcal{M})$, then $u \in H^{s+m-1}(\mathcal{M})$ and (A.6) holds.

An important case where this propagation result is used is with $P := -iX \in \Psi^1(\mathcal{M})$, where X is a smooth vector field on $\mathcal{M}$: this appears in

§2.4 for the study of the (generalized) X-ray transform and in §5.3.1 in order to show the pseudodifferential behaviour of the generator of Lévy flights on Anosov manifolds. By Example A.1, the principal symbol of P is $\sigma_P(x,\xi) = \langle \xi, X(x) \rangle$ and the Hamiltonian flow generated by H_P is simply given by $\Phi_t(x,\xi) = (\varphi_t(x), d\varphi_t^{-\top}(x)\xi)$, the symplectic lift of the flow $(\varphi_t)_{t\in\mathbb{R}}$ generated by X. (Here, ${}^{-\top}$ stands for the inverse transpose.)

References

[AB21] David Applebaum and Rosemary Shewell Brockway. L2 properties of Lévy generators on compact riemannian manifolds. *Journal of Theoretical Probability*, 34(2):1029–1042, 2021.

[Abr70] R. Abraham. Bumpy metrics. In *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, pages 1–3. Amer. Math. Soc., Providence, R.I., 1970.

[Ada62] John Frank Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[AE00] D. Applebaum and A. Estrade. Isotropic Lévy processes on Riemannian manifolds. *Ann. Probab.*, 28(1):166–184, 2000.

[AH14] David Applebaum and Herbert Heyer. *Probability on compact Lie groups*, volume 70. Springer, 2014.

[AKKL12] Habib Ammari, Kostis Kalimeris, Hyeonbae Kang, and Hyundae Lee. Layer potential techniques for the narrow escape problem. *Journal de mathématiques pures et appliquées*, 97(1):66–84, 2012.

[Ami86] Arif Amirov. Existence and uniqueness theorems for the solution of an inverse problem for the transfer equation. *Sibirsk. Mat. Zh.*, 27(6):3–20, 1986.

[Ano67] Dmitri Viktorovitch Anosov. Geodesic flows on closed Riemannian manifolds of negative curvature. *Trudy Mat. Inst. Steklov.*, 90:209, 1967.

[Ano82] D. V. Anosov. Generic properties of closed geodesics. *Izv. Akad. Nauk SSSR Ser. Mat.*, 46(4):675–709, 896, 1982.

[Bar91] J. W. Barrett. Holonomy and path structures in general relativity and Yang-Mills theory. *Internat. J. Theoret. Phys.*, 30(9):1171–1215, 1991.

[Bea13] Chris Beasley. Localization for Wilson loops in Chern-Simons theory. *Adv. Theor. Math. Phys.*, 17(1):1–240, 2013.

[Ber53] Marcel Berger. Sur les groupes d'holonomie des variétés riemanniennes. *C. R. Acad. Sci. Paris*, 237:472–473, 1953.

[Ber60] Marcel Berger. Pincement riemannien et pincement holomorphe. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 14:151–159, 1960.

[BG80] Michael Brin and Mikhael Gromov. On the ergodicity of frame flows. *Invent. Math.*, 60(1):1–7, 1980.

[BI10] Dmitri Burago and Sergei Ivanov. Boundary rigidity and filling volume minimality of metrics close to a flat one. *Ann. of Math. (2)*, 171(2):1183–1211, 2010.

[BK84] Michael Brin and Hermann Karcher. Frame flows on manifolds with pinched negative curvature. *Compositio Math.*, 52(3):275–297, 1984.

[BK85] K. Burns and A. Katok. Manifolds with nonpositive curvature. *Ergodic Theory Dynam. Systems*, 5(2):307–317, 1985.

[BL07] Oliver Butterley and Carlangelo Liverani. Smooth Anosov flows: correlation spectra and stability. *J. Mod. Dyn.*, 1(2):301–322, 2007.

[BLMV11] Olivier Bénichou, Claude Loverdo, Michel Moreau, and Raphael Voituriez. Intermittent search strategies. *Reviews of Modern Physics*, 83(1):81, 2011.

[BMS20] María Laura Barberis, Andrei Moroianu, and Uwe Semmelmann. Generalized vector cross products and Killing forms on negatively curved manifolds. *Geom. Dedicata*, 205:113–127, 2020.

[BN13] Paul C Bressloff and Jay M Newby. Stochastic models of intracellular transport. *Reviews of Modern Physics*, 85(1):135, 2013.

[Bon20] Yannick Guedes Bonthonneau. Perturbation of Ruelle resonances and Faure-Sjöstrand anisotropic space. *Rev. Un. Mat. Argentina*, 61(1):63–72, 2020.

[BP74] Michael I. Brin and Yakov B. Pesin. Partially hyperbolic dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:170–212, 1974.

[BP03] Keith Burns and Mark Pollicott. Stable ergodicity and frame flows. *Geom. Dedicata*, 98:189–210, 2003.

[Bri75a] Michael Brin. Topological transitivity of a certain class of dynamical systems, and flows of frames on manifolds of negative curvature. *Funkcional. Anal. i Priložen.*, 9(1):9–19, 1975.

[Bri75b] Michael Brin. The topology of group extensions of C-systems. *Mat. Zametki*, 18(3):453–465, 1975.

[Bri82] Michael Brin. Ergodic theory of frame flows. In *Ergodic theory and dynamical systems, II (College Park, Md., 1979/1980)*, volume 21 of *Progr. Math.*, pages 163–183. Birkhäuser, Boston, Mass., 1982.

[Bry93] Jean-Luc Brylinski. *Loop spaces, characteristic classes and geometric quantization*, volume 107 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[BtD85] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.

[BW10] Keith Burns and Amie Wilkinson. On the ergodicity of partially hyperbolic systems. *Ann. of Math. (2)*, 171(1):451–489, 2010.

[ČC06] Martin Čadek and Michael Crabb. G-structures on spheres. *Proc. London Math. Soc. (3)*, 93(3):791–816, 2006.

[CDS00] Christopher B. Croke, Nurlan S. Dairbekov, and Vladimir A. Sharafutdinov. Local boundary rigidity of a compact Riemannian manifold with curvature bounded above. *Trans. Amer. Math. Soc.*, 352(9):3937–3956, 2000.

[CF11] Xinfu Chen and Avner Friedman. Asymptotic analysis for the narrow escape problem. *SIAM journal on mathematical analysis*, 43(6):2542–2563, 2011.

[CGL22] Mihajlo Cekić, Colin Guillarmou, and Thibault Lefeuvre. Local lens rigidity for manifolds of Anosov type. *arXiv e-prints*, page arXiv:2204.02476, April 2022.

[CGLT22] Yann Chaubet, Yannick Guedes Bonthonneau, Thibault Lefeuvre, and Leo Tzou. Geodesic Lévy flights and expected stopping time for random searches. *arXiv e-prints*, page arXiv:2211.13973, November 2022.

[CH16] Christopher B. Croke and Pilar Herreros. Lens rigidity with trapped geodesics in two dimensions. *Asian J. Math.*, 20(1):47–57, 2016.

[Che55] Shiing-shen Chern. On curvature and characteristic classes of a Riemann manifold. *Abh. Math. Sem. Univ. Hamburg*, 20:117–126, 1955.

[CLa] Mihajlo Cekić and Thibault Lefeuvre. On polynomial structures over spheres. *in preparation*.

[CLb] Mihajlo Cekić and Thibault Lefeuvre. The Holonomy Inverse Problem. *arXiv:2105.06376*.

[CL21a] Mihajlo Cekić and Thibault Lefeuvre. Generic Dynamical Properties of Connections on Vector Bundles. *International Mathematics Research Notices*, 04 2021. rnab069.

[CL21b] Mihajlo Cekić and Thibault Lefeuvre. Generic injectivity of the X-ray transform. *arXiv e-prints*, July 2021.

[CLMS] Mihajlo Cekić, Thibault Lefeuvre, Andrei Moroianu, and Uwe Semmelmann. On the ergodicity of the frame flow on even-dimensional manifolds. *arXiv:2111.14811*.

[CLMS22] Mihajlo Cekić, Thibault Lefeuvre, Andrei Moroianu, and Uwe Semmelmann. On the ergodicity of the unitary frame flow on kähler manifolds. *in preparation*, 2022.

[Cro90] Christopher B. Croke. Rigidity for surfaces of nonpositive curvature. *Comment. Math. Helv.*, 65(1):150–169, 1990.

[Cro91] Christopher B. Croke. Rigidity and the distance between boundary points. *J. Differential Geom.*, 33(2):445–464, 1991.

[Cro04] Christopher B. Croke. Rigidity theorems in Riemannian geometry. In *Geometric methods in inverse problems and PDE control*, volume 137 of *IMA Vol. Math. Appl.*, pages 47–72. Springer, New York, 2004.

[CS98] Christopher B. Croke and Vladimir A. Sharafutdinov. Spectral rigidity of a compact negatively curved manifold. *Topology*, 37(6):1265–1273, 1998.

[CWS10] Alexei F Cheviakov, Michael J Ward, and Ronny Straube. An asymptotic analysis of the mean first passage time for narrow escape problems: Part ii: The sphere. *Multiscale Modeling & Simulation*, 8(3):836–870, 2010.

[DG75a] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29(1):39–79, 1975.

[DG75b] J. J. Duistermaat and V. W. Guillemin. The spectrum of positive elliptic operators and periodic bicharacteristics. *Invent. Math.*, 29:39–79, 1975.

[DG16] Semyon Dyatlov and Colin Guillarmou. Pollicott-Ruelle resonances for open systems. *Ann. Henri Poincaré*, 17(11):3089–3146, 2016.

[DGRS20] Nguyen Viet Dang, Colin Guillarmou, Gabriel Rivière, and Shu Shen. The Fried conjecture in small dimensions. *Invent. Math.*, 220(2):525–579, 2020.

[DGV22] Serena Dipierro, Giovanni Giacomin, and Enrico Valdinoci. Efficiency functionals for the lévy flight foraging hypothesis. *Journal of Mathematical Biology*, 85(4):1–50, 2022.

[DK00] J. J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext. Springer-Verlag, Berlin, 2000.

[Dol02] Dmitry Dolgopyat. On mixing properties of compact group extensions of hyperbolic systems. *Israel J. Math.*, 130:157–205, 2002.

[DZ16] Semyon Dyatlov and Maciej Zworski. Dynamical zeta functions for Anosov flows via microlocal analysis. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):543–577, 2016.

[FH19] Todd Fisher and Boris Hasselblatt. *Hyperbolic flows*. Berlin: European Mathematical Society (EMS), 2019.

[For81] Otto Forster. *Lectures on Riemann surfaces*, volume 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981. Translated from the German by Bruce Gilligan.

[Fri01] Thomas Friedrich. Weak Spin(9)-structures on 16-dimensional Riemannian manifolds. *Asian J. Math.*, 5(1):129–160, 2001.

[FRS08] Frédéric Faure, Nicolas Roy, and Johannes Sjöstrand. Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances. *Open Math. J.*, 1:35–81, 2008.

[FS11] Frédéric Faure and Johannes Sjöstrand. Upper bound on the density of Ruelle resonances for Anosov flows. *Comm. Math. Phys.*, 308(2):325–364, 2011.

[FT13] Frédéric Faure and Masato Tsujii. Band structure of the Ruelle spectrum of contact Anosov flows. *C. R. Math. Acad. Sci. Paris*, 351(9-10):385–391, 2013.

[GC15] Daniel Gomez and Alexei F Cheviakov. Asymptotic analysis of narrow escape problems in nonspherical three-dimensional domains. *Physical Review E*, 91(1):012137, 2015.

[Get61] RK Getoor. First passage times for symmetric stable processes in space. *Transactions of the American Mathematical Society*, 101(1):75–90, 1961.

[GGJ22] Yannick Guedes Bonthonneau, Colin Guillarmou, and Malo Jézéquel. Scattering rigidity for analytic metrics. *arXiv e-prints*, page arXiv:2201.02100, January 2022.

[Gil81] R. Giles. Reconstruction of gauge potentials from Wilson loops. *Phys. Rev. D (3)*, 24(8):2160–2168, 1981.

[GK80] Victor Guillemin and David Kazhdan. Some inverse spectral results for negatively curved 2-manifolds. *Topology*, 19(3):301–312, 1980.

[GK21] Colin Guillarmou and Benjamin Küster. Spectral theory of the frame flow on hyperbolic 3-manifolds. *Ann. Henri Poincaré*, 22(11):3565–3617, 2021.

[GKL19] Colin Guillarmou, Gerhard Knieper, and Thibault Lefeuvre. Geodesic stretch, pressure metric and marked length spectrum rigidity. *arXiv e-prints*, page arXiv:1909.08666, Sep 2019.

[GL06] Sébastien Gouëzel and Carlangelo Liverani. Banach spaces adapted to Anosov systems. *Ergodic Theory Dynam. Systems*, 26(1):189–217, 2006.

[GL19] Colin Guillarmou and Thibault Lefeuvre. The marked length spectrum of Anosov manifolds. *Ann. of Math. (2)*, 190(1):321–344, 2019.

[GL20] Yannick Guedes Bonthonneau and Thibault Lefeuvre. Radial source estimates in Hölder-Zygmund spaces for hyperbolic dynamics. *arXiv:2011.06403*, November 2020.

[GM18] Colin Guillarmou and Marco Mazzucchelli. Marked boundary rigidity for surfaces. *Ergodic Theory Dynam. Systems*, 38(4):1459–1478, 2018.

[GPSU16] Colin Guillarmou, Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. The X-ray transform for connections in negative curvature. *Comm. Math. Phys.*, 343(1):83–127, 2016.

[Gro83] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[GS94] Alain Grigis and Johannes Sjöstrand. *Microlocal analysis for differential operators*, volume 196 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994. An introduction.

[Gui73] Victor Guillemin. Fourier integral operators for systems. *Unpublished Report*, 1973. Available at https://msp.org/extras/Guillemin/V.Guillemin-Fourier_Integral_Operators-1974.pdf.

[Gui17a] Colin Guillarmou. Invariant distributions and X-ray transform for Anosov flows. *J. Differential Geom.*, 105(2):177–208, 2017.

[Gui17b] Colin Guillarmou. Lens rigidity for manifolds with hyperbolic trapped sets. *J. Amer. Math. Soc.*, 30(2):561–599, 2017.

[GWW92] C. Gordon, D. Webb, and S. Wolpert. Isospectral plane domains and surfaces via Riemannian orbifolds. *Invent. Math.*, 110(1):1–22, 1992.

[H03] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[H07] Lars Hörmander. *The analysis of linear partial differential operators. III.* Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[Ham99] U. Hamenstädt. Cocycles, symplectic structures and intersection. *Geom. Funct. Anal.*, 9(1):90–140, 1999.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[HM79] Roger E. Howe and Calvin C. Moore. Asymptotic properties of unitary representations. *J. Functional Analysis*, 32(1):72–96, 1979.

[Hop36] Eberhard Hopf. Fuchsian groups and ergodic theory. *Trans. Amer. Math. Soc.*, 39(2):299–314, 1936.

[HP06] Boris Hasselblatt and Yakov Pesin. Partially hyperbolic dynamical systems. In *Handbook of dynamical systems. Vol. 1B*, pages 1–55. Elsevier B. V., Amsterdam, 2006.

[JS07] Dmitry Jakobson and Alexander Strohmaier. High energy limits of Laplace-type and Dirac-type eigenfunctions and frame flows. *Comm. Math. Phys.*, 270(3):813–833, 2007.

[JSZ08] Dmitry Jakobson, Alexander Strohmaier, and Steve Zelditch. On the spectrum of geometric operators on Kähler manifolds. *J. Mod. Dyn.*, 2(4):701–718, 2008.

[Kac66] Mark Kac. Can one hear the shape of a drum? *Amer. Math. Monthly*, 73(4, part II):1–23, 1966.

[Kli95] Wilhelm P. A. Klingenberg. *Riemannian geometry*, volume 1 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 1995.

[Kob54] Shôshichi Kobayashi. La connexion des variétés fibrées. II. *C. R. Acad. Sci. Paris*, 238:443–444, 1954.

[Kuw90] Ruishi Kuwabara. Isospectral connections on line bundles. *Math. Z.*, 204(4):465–473, 1990.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Lef] Thibault Lefeuvre. Isometric extensions of Anosov flows via microlocal analysis. *arXiv:2112.05979*.

[Lef18] Thibault Lefeuvre. Local marked boundary rigidity under hyperbolic trapping assumptions. *arXiv e-prints*, page arXiv:1804.02143, Apr 2018.

[Lef19a] Thibault Lefeuvre. On the s-injectivity of the x-ray transform on manifolds with hyperbolic trapped set. *Nonlinearity*, 32(4):1275–1295, 2019.

[Lef19b] Thibault Lefeuvre. *Sur la rigidité des variétés riemanniennes*. Theses, Université Paris-Saclay, December 2019.

[Leo71] Peter Leonard. G-structures on spheres. *Trans. Amer. Math. Soc.*, 157:311–327, 1971.

[Liv04] Carlangelo Liverani. On contact Anosov flows. *Ann. of Math. (2)*, 159(3):1275–1312, 2004.

[Lol94] Renate Loll. Gauge theory and gravity in the loop formulation. In *Canonical gravity: from classical to quantum (Bad Honnef, 1993)*, volume 434 of *Lecture Notes in Phys.*, pages 254–288. Springer, Berlin, 1994.

[Mic82] René Michel. Sur la rigidité imposée par la longueur des géodésiques. *Invent. Math.*, 65(1):71–83, 1981/82.

[Mil64] J. Milnor. Eigenvalues of the Laplace operator on certain manifolds. *Proc. Nat. Acad. Sci. U.S.A.*, 51:542, 1964.

[Moo87] Calvin C. Moore. Exponential decay of correlation coefficients for geodesic flows. In *Group representations, ergodic theory, operator algebras, and mathematical physics (Berkeley, Calif., 1984)*, volume 6 of *Math. Sci. Res. Inst. Publ.*, pages 163–181. Springer, New York, 1987.

[MS39] S. B. Myers and N. E. Steenrod. The group of isometries of a Riemannian manifold. *Ann. of Math. (2)*, 40(2):400–416, 1939.

[Muk75] Ravil Galatdinovich Mukhometov. Inverse kinematic problem of seismic on the plane. *Akad. Nauk. SSSR*, 6:243–252, 1975.

[Muk81] Ravil Galatdinovich Mukhometov. On a problem of reconstructing Riemannian metrics. *Sibirsk. Mat. Zh.*, 22(3):119–135, 237, 1981.

[NTT21a] Medet Nursultanov, William Trad, and Leo Tzou. Narrow escape problem in the presence of the force field. *Mathematical Methods in the Applied Sciences*, 2021.

[NTT21b] Medet Nursultanov, Justin C Tzou, and Leo Tzou. On the mean first arrival time of brownian particles on riemannian manifolds. *Journal de Mathématiques Pures et Appliquées*, 150:202–240, 2021.

[NTTT22] Medet Nursultanov, William Trad, Justin C Tzou, and Leo Tzou. The narrow capture problem on general riemannian surfaces. *arXiv preprint arXiv:2209.12425*, 2022.

[Ota90a] Jean-Pierre Otal. Le spectre marqué des longueurs des surfaces à courbure négative. *Ann. of Math. (2)*, 131(1):151–162, 1990.

[Ota90b] Jean-Pierre Otal. Sur les longueurs des géodésiques d'une métrique à courbure négative dans le disque. *Comment. Math. Helv.*, 65(2):334–347, 1990.

[Par99] William Parry. The Livšic periodic point theorem for non-abelian cocycles. *Ergodic Theory Dynam. Systems*, 19(3):687–701, 1999.

[Pat09] Gabriel P. Paternain. Transparent connections over negatively curved surfaces. *J. Mod. Dyn.*, 3(2):311–333, 2009.

[PCM14] Vladimir V Palyulin, Aleksei V Chechkin, and Ralf Metzler. Lévy flights do not always optimize random blind search for sparse targets. *Proceedings of the National Academy of Sciences*, 111(8):2931–2936, 2014.

[Pes04] Yakov B. Pesin. *Lectures on partial hyperbolicity and stable ergodicity*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[PP90] William Parry and Mark Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188):268, 1990.

[PS88] Leonid N. Pestov and Vladimir A. Sharafutdinov. Integral geometry of tensor fields on a manifold of negative curvature. *Sibirsk. Mat. Zh.*, 29(3):114–130, 221, 1988.

[PS00] Charles Pugh and Michael Shub. Stable ergodicity and julienne quasiconformality. *J. Eur. Math. Soc. (JEMS)*, 2(1):1–52, 2000.

[PSU13] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography on simple surfaces. *Invent. Math.*, 193(1):229–247, 2013.

[PSU14a] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Spectral rigidity and invariant distributions on Anosov surfaces. *J. Differential Geom.*, 98(1):147–181, 2014.

[PSU14b] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. Tensor tomography: progress and challenges. *Chin. Ann. Math. Ser. B*, 35(3):399–428, 2014.

[PSU22] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann. *Geometric Inverse Problems: With Emphasis on Two Dimensions*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[PU05] Leonid Pestov and Gunther Uhlmann. Two dimensional compact simple Riemannian manifolds are boundary distance rigid. *Ann. of Math. (2)*, 161(2):1093–1110, 2005.

[Sch99] Klaus Schmidt. Remarks on Livšic' theory for nonabelian cocycles. *Ergodic Theory Dynam. Systems*, 19(3):703–721, 1999.

[Sha94] V. A. Sharafutdinov. *Integral geometry of tensor fields*. Inverse and Ill-posed Problems Series. VSP, Utrecht, 1994.

[Sha09] Vladimir A. Sharafutdinov. Local audibility of a hyperbolic metric. *Siberian Mathematical Journal*, 50(5):929, 2009.

[Shu01] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer-Verlag, Berlin, second edition, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[SK86] MF Shlesinger and J Klafter. On growth and form. *Levy Walks vs. Levy Flights*, pages 279–283, 1986.

[SSH06] Amit Singer, Zeev Schuss, and David Holcman. Narrow escape, part ii: The circular disk. *Journal of statistical physics*, 122(3):465–489, 2006.

[SSH08] Amit Singer, Z Schuss, and David Holcman. Narrow escape and leakage of brownian particles. *Physical Review E*, 78(5):051111, 2008.

[SU04] Plamen Stefanov and Gunther Uhlmann. Stability estimates for the X-ray transform of tensor fields and boundary rigidity. *Duke Math. J.*, 123(3):445–467, 2004.

[SU09] Plamen Stefanov and Gunther Uhlmann. Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds. *J. Differential Geom.*, 82(2):383–409, 2009.

[Sun85] Toshikazu Sunada. Riemannian coverings and isospectral manifolds. *Ann. of Math. (2)*, 121(1):169–186, 1985.

[SUV21] Plamen Stefanov, Gunther Uhlmann, and András Vasy. Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge. *Ann. of Math. (2)*, 194(1):1–95, 2021.

[Tay13] Michael Taylor. *Partial differential equations II: Qualitative studies of linear equations*, volume 116. Springer Science & Business Media, 2013.

[Tot07] Burt Totaro. The automorphism group of an affine quadric. *Math. Proc. Cambridge Philos. Soc.*, 143(1):1–8, 2007.

[TZ] Masato Tsujii and Zhiyuan Zhang. Smooth mixing Anosov flows in dimension three are exponential mixing. *arXiv:2006.04293*.

[Var09] James Vargo. A proof of lens rigidity in the category of analytic metrics. *Math. Res. Lett.*, 16(6):1057–1069, 2009.

[VDLRS11] Gandhimohan M Viswanathan, Marcos GE Da Luz, Ernesto P Raposo, and H Eugene Stanley. *The physics of foraging: an introduction to random searches and biological encounters*. Cambridge University Press, 2011.

[Vig80] Marie-France Vignéras. Variétés riemanniennes isospectrales et non isométriques. *Ann. of Math. (2)*, 112(1):21–32, 1980.

[Wal04] Gerard Walschap. *Metric structures in differential geometry*, volume 224 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2004.

[Wil74] Kenneth G. Wilson. Confinement of quarks. *Phys. Rev. D*, 10:2445–2459, Oct 1974.

[Wil10] Amie Wilkinson. Conservative partially hyperbolic dynamics. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1816–1836. Hindustan Book Agency, New Delhi, 2010.

[Woo68] Reginald Wood. Polynomial maps from spheres to spheres. *Invent. Math.*, 5:163–168, 1968.

[Yiu86] Paul Y. H. Yiu. Quadratic forms between spheres and the nonexistence of sums of squares formulae. *Math. Proc. Cambridge Philos. Soc.*, 100(3):493–504, 1986.

[Yiu94] Paul Yiu. Quadratic forms between Euclidean spheres. *Manuscripta Math.*, 83(2):171–181, 1994.

[Zel04] Steve Zelditch. The inverse spectral problem. In *Surveys in differential geometry. Vol. IX*, volume 9 of *Surv. Differ. Geom.*, pages 401–467. Int. Press, Somerville, MA, 2004. With an appendix by Johannes Sjöstrand and Maciej Zworski.

[Zel14] Steve Zelditch. Survey on the inverse spectral problem. *ICCM Not.*, 2(2):1–20, 2014.

Université de Paris and Sorbonne Université, CNRS, IMJ-PRG, F-75006 Paris, France.

E-mail address: tlefeuvre@imj-prg.fr