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Sur la rigidité des variétés riemanniennes

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A mes parents,

« Je ne sais pas si beaucoup d'hommes ont comme moi depuis l'enfance pressenti toute leur vie. Rien ne m'est arrivé que je n'aie obscurément prévu dès mes premières années. Les ruines d'Angkor, je me souviens si bien de certain soir d'avril, un peu voilé, où en vision elles m'apparurent ! »

Un pèlerin d'Angkor, Pierre Loti

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tamment durant cette semaine passée ensemble à l'Institut d'Oberwolfach dans la Forêt Noire. Je salue enfin le groupe de travail qui s'est constitué à Orsay autour du livre de Melrose [Mel93] sur le b-calcul : Yann Chaubet, Hugo Federico, Benjamin Küster et Nir Schwartz.

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Introduction

« *On affirme, en Orient, que le meilleur moyen pour traverser un carré est d'en parcourir trois côtés.* »

Les Sept Piliers de la sagesse,
Thomas Edward Lawrence

Cette introduction reprend en partie l'article *Le chant de la Terre*, publié dans **Images des Mathématiques**.

Sommaire

1.1 La théorie des problèmes inverses

Avant d'aborder pleinement les problèmes mathématiques qui nous intéresseront dans cette thèse, nous les motivons par quelques discussions informelles tirées de considérations venant de la physique.

1.1.1 La Physique au XXème siècle

La théorie des *problèmes inverses* est une branche des mathématiques aussi vaste que ramifiée, et désormais motivée par un nombre croissant d'applications à la vie quotidienne. S'il fallait en donner une définition quelque peu raisonnable, nous pourrions dire qu'*un problème inverse consiste à déterminer les caractéristiques physiques d'un objet inaccessible à la mesure par l'étude de sa réponse à une stimulation ondulatoire.* Qu'un objet ne soit pas observable *immédiatement*, c'est-à-dire par le simple recours à un instrument d'observation tel qu'un télescope, est — pourrait-on dire — le propre de la physique moderne et, en ce sens, tout problème physique traitant de l'infiniment petit ou de l'infiniment grand pourrait être qualifié de *problème inverse*. Si Rutherford découvre en 1909 le modèle planétaire de l'atome [1], battant ainsi en brèche le modèle antérieur de Thomson qui voulait qu'un atome soit constitué d'un seul noyau renfermant les deux charges opposées, ce n'est pas en observant la structure de l'atome par le truchement d'un microscope surpuissant : c'est en étudiant la faible déviation de particules α bombardant une fine feuille d'or que Rutherford met au jour la structure *lacunaire* de la matière, ainsi que l'existence d'un noyau chargé positivement.

Partant, la majorité des avancées de la Physique du XXème se sont constituées sur l'observation d'évènements indirectement liés à l'existence même des objets. Autrement dit : la confirmation des modèles théoriques s'est faite en observant les conséquences de ce qu'ils prédisaient, non pas les objets qu'ils manipulaient en tant que tels. L'exemple le plus significatif qui puisse être mentionné est certainement celui des trous noirs. Par nature, un trou noir ne peut pas se *voir* puisqu'aucune lumière ne peut en échapper. Aussi paradoxal que cela puisse être, les astronomes sont désormais tout à fait capables de prédire l'existence d'un trou noir — de le localiser et même désormais de le « photographier » [2] ! — grâce à diverses techniques, telles que l'*observation* de lentilles gravitationnelles, c'est-à-dire la forte déviation de la lumière (une onde !) qui nous parviendrait d'une étoile située directement derrière le trou noir [3]. On voit bien que les mots ordinaires peinent ici à donner du sens à ce paradoxe de la physique contemporaine : *rien ne se voit mais tout s'observe.*

Il y aurait donc foule de problèmes que l'on pourrait qualifier d'*inverses* et parmi ceux-ci, certains revêtiraient des natures tantôt analytiques, tantôt géométriques : ten-

1. Rutherford pensera qu'un atome est constitué d'un noyau de petit volume qui porte la charge positive, ainsi que d'électrons portant la charge négative et gravitant autour du noyau à la manière de planètes autour de leur étoile. Cette répartition de la charge sera conservée dans les modèles postérieurs, mais son analogie avec le système solaire sera mise à mal par la théorie de l'électromagnétisme : les électrons de Rutherford, s'ils gravitaient autour du noyau de l'atome, devraient rayonner et perdre de l'énergie jusqu'à s'effondrer sur le noyau, rendant ainsi toute matière instable.

2. C'est la photographie du disque d'accrétion d'un trou noir, et non l'objet en tant que tel, par nature invisible, qui a été rendue publique le 10 avril 2019.

3. La Mécanique quantique, c'est-à-dire la Physique de l'infiniment petit, repose sur la paradigme déjà séculaire — il remonte au débat entre Huygens et Newton qui agita le XVIIème siècle — de la dualité onde-corpuscule : toute particule peut à la fois être considérée comme un *corps* physique et comme une onde. L'idée qu'une stimulation ondulatoire se propage à travers un objet inconnu n'est donc jamais bien loin.

ter de tous les énumérer ne mèneraient pas à grand chose. Notre étude se bornera à quelques problèmes de nature avant tout *géométrique*. Plus précisément, nous nous intéresserons à des milieux dont les caractéristiques physiques peuvent être *a priori* décrites au moyen de la théorie de la *géométrie riemannienne*[4]. Les ondes se propagent alors selon le principe de moindre action de Maupertuis en minimisant globalement *l'action* — c'est-à-dire la différence entre l'énergie cinétique et l'énergie potentielle — ou encore en minimisant localement leur *temps de trajet*[5]. Avant de donner une définition mathématique moins équivoque des questions qui nous intéresserons, nous présentons trois exemples concrets qui les illustrent.

La propagation des ondes sismiques. [6]Au début du XXème siècle, grâce à l'amélioration technique des sismographes, les géophysiciens ont mis en évidence l'existence de deux types d'onde qui se propageaient dans la croûte terrestre à la suite d'un séisme : les *ondes P et S* (voir Figure 1.1). Les premières sont des ondes dites de *compression* : ce sont les plus rapides à se propager, se déplaçant à la vitesse de 6 km/s au voisinage de la surface, et sont donc les premières à être enregistrées par les sismographes. Puis viennent les ondes S, dites ondes de *cisaillement*, plus lentes mais aussi bien plus dévastatrices car elles tendent à déplacer la matière perpendiculairement au sens de propagation de l'onde. L'étude des temps de propagation de ces ondes tout au long du XXème siècle a conduit à des modèles de plus en plus précis de la structure interne de la Terre avec une croûte terrestre (ou continentale) de faible épaisseur — de l'ordre de quelques dizaines de kilomètres —, un manteau allant jusqu'à 3000 km de profondeur, puis un noyau dont une partie (appelée la graine) est liquide et empêche la propagation des ondes S.

Suite aux premières découvertes quant à l'existence de ces ondes, Herglotz [Her05], en 1905, puis Wiechert et Zoeppritz [EW07], en 1907, ont suggéré un modèle mathématique pour décrire la structure interne de la Terre : cette dernière est modélisée par une boule fermée $\overline{B}(0,R)$ centrée en l'origine et de rayon $R \sim 6300$ km, sa structure est à symétrie sphérique et isotrope. Cela revient à supposer que la métrique du milieu considéré est de la forme $g = c^{-2}(r)g_{\text{eucl}}$, c décrivant la vitesse de propagation des ondes. En outre, afin que le modèle soit fidèle à l'observation, Herglotz et Wichert-Zoeppritz

4. Notons que cela écarte d'emblée les géométries dites *lorentziennes* qui décrivent la structure de l'espace-temps en Relativité générale.

5. Au lecteur qui serait peu familier de ces notions, cet exemple élémentaire peut éclairer. L'été, sur une route rectiligne exposée en plein soleil, il nous est fréquent d'observer des *mirages* : ce que nous distinguons alors au loin n'est plus le bitume, ce sont des taches de ciel qui semblent s'être noyées tout au bout de la route (et que nous n'atteindrons jamais !). L'explication est simple : la chaleur dégagée par le bitume dévie les rayons lumineux et les courbe au voisinage du sol, ce qui nous fait voir le ciel à la place de la route. Une façon physique de formuler ce problème est de dire que l'*indice de réfraction* de la lumière a été modifié par la température. Par les lois de Snell-Descartes, cette modification inhomogène (mais isotrope) de l'indice entraîne une modification de la trajectoire de la lumière. Une façon mathématique de la formuler est de dire qu'une modification de l'indice correspond à une modification de la *métrique* de l'espace, ce qui a tendance à le *courber*. Autrement dit, un rayon lumineux paraît se déplacer dans un milieu dont la géométrie serait courbé, tout comme un avion entre Paris et Sydney se déplace à la surface de la Terre selon un arc de cercle (il ne va pas tout droit, sinon il devrait passer *à travers* le globe !). La courbure des rayons lumineux est ainsi interprétée comme une *courbure intrinsèque* de l'espace dans lequel ils vivent : c'est sur ce principe général que repose la géométrie riemannienne.

6. Voir l'article de vulgarisation que j'ai consacré à cette question sur le site *Images des Mathématiques* : `https://images.math.cnrs.fr/Le-chant-de-la-Terre.html`.

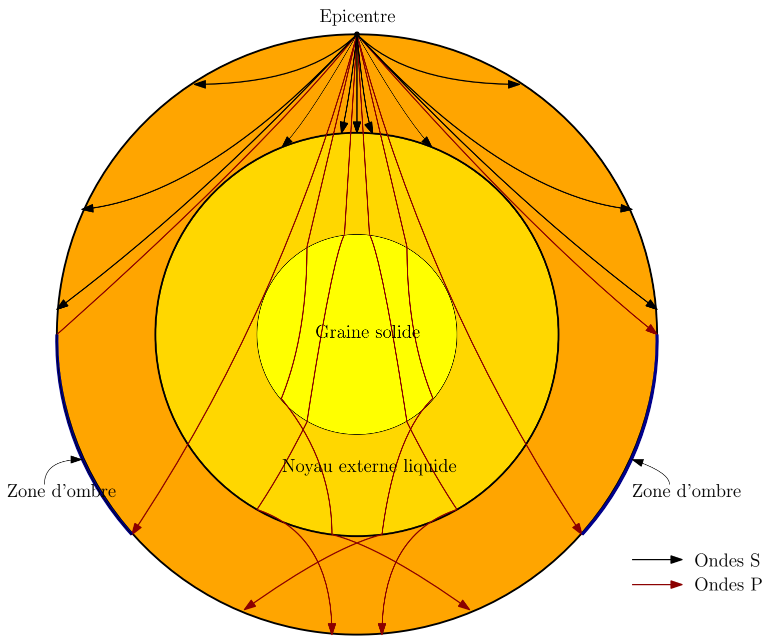

FIGURE 1.1 – Propagation des ondes P et S dans la Terre

supposaient que c vérifie la condition supplémentaire

$$\frac{d}{dr}(r/c(r)) > 0, \tag{1.1.1}$$

autrement dit que la vitesse des ondes augmente avec la profondeur. Cette hypothèse est même plus précise : elle traduit le fait que la trajectoire des ondes est de plus en plus courbée à mesure que celles-ci se rapprochent du centre de la Terre. D'un point de vue mathématique, si $\mathbb{S}_r = \{|x| = r\}$ est la sphère de rayon $0 \leq r \leq R$, alors (1.1.1) est équivalent à la stricte convexité des sphères $\mathbb{S}_r$ pour la métrique g, au sens où la seconde forme fondamentale y est définie positive en tant que forme quadratique. Selon le principe de moindre action, les ondes sismiques de type P sont supposées se propager en minimisant l'action, c'est-à-dire selon les géodésiques de la métrique g. On suppose également que suffisamment de données sismiques ont été collectées pour que, étant donnée une paire de points quelconque $(x, y) \in \mathbb{S}_R^2$ à la surface de la Terre, le temps de parcours d'une onde de x à y soit connu. Autrement dit, on suppose connue la fonction dite de *distance au bord*

$$d_g : \mathbb{S}_R \times \mathbb{S}_R \to \mathbb{R}_+, \qquad (x, y) \mapsto d_g(x, y), \tag{1.1.2}$$

où $d_g(x, y)$ désigne la distance riemannienne entre x et y calculée par rapport à la métrique g. La question est alors la suivante :

Etant connue la fonction d_g, est-il possible d'en déduire la fonction c, c'est-à-dire de reconstruire la métrique g ?

Ce problème difficile soulève en réalité deux questions qui lui sous-jacentes. La première est d'ordre *théorique* : est-il théoriquement possible de reconstruire la fonction c ? Autrement dit, étant donné deux métriques $g = c^{-2}g_{\text{eucl}}$ et $g' = c'^{-2}g_{\text{eucl}}$, si l'on suppose que les fonctions de distance au bord des métriques coïncident, i.e. $d_g = d_{g'}$, est-il vrai que $c = c'$? On voit là se dessiner un problème d'*injectivité*. Si on peut répondre positivement à cette question, on dira que *la fonction de distance au bord détermine la métrique* ou encore que la variété $(\overline{B}, g)$ — ici la Terre, munie de sa métrique supposée — est *rigide au bord*. L'autre problème est de nature plus *pratique* : supposons qu'il soit théoriquement possible de reconstruire c (autrement dit que la variété est rigide au bord), peut-on alors explicitement le faire ? Existe t-il un algorithme permettant de calculer c à partir de la fonction d_g ? C'est le problème pratique de la *reconstruction de la métrique*, problème que nous n'étudierons pas dans cette thèse. Précisons que le problème théorique est celui qui a d'abord historiquement intéressé les mathématiciens : la première formulation mathématique précise est due à Michel [Mic82] en 1982, et nous aurons l'occasion d'y revenir plus en détails. Ce n'est que très récemment — dans les cinq ou six dernières — que les premiers progrès significatifs ont été faits quant au problème de la reconstruction grâce aux travaux de Uhlmann-Vasy [UV16] et Stefanov-Uhlmann-Vasy [SUV17], ce dernier ayant même été couvert par la publication d'un billet dans la revue *Nature* (ce qui est suffisamment rare concernant un article de Mathématique pour que cela soit souligné !).

1.1.2 Transformée de Radon, transformée en rayons X.

En 1917, dans un article depuis resté célèbre, Radon [Rad17] introduit une transformée sur les fonctions f à support compact dans le plan en leur associant une fonction $\mathcal{R}f$, définie sur l'ensemble des droites L du plan par intégration de f le long de L, c'est-à-dire que $\mathcal{R}f(L) = \int_L f dL$. Il montre que cette application $\mathcal{R}$ est *inversible* et qu'il est possible de *reconstruire* la fonction f à partir de la connaissance de sa transformée $\mathcal{R}f$. La formule qu'il établit est la suivante :

$$f = \frac{1}{2}\,\Delta^{1/2}\mathcal{R}^*\mathcal{R}f, \tag{1.1.3}$$

où $\Delta^{1/2}$ est le multiplicateur de Fourier par $|\xi|$, et au point $x \in \mathbb{R}^2$, si $L_\theta(x)$ désigne la droite passant par x avec un angle θ par rapport à l'axe des abscisses,

$$\mathcal{R}^*\mathcal{R}f(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}f(L_\theta(x))d\theta. \tag{1.1.4}$$

est la moyenne calculée sur toute les droites passant par le point x.

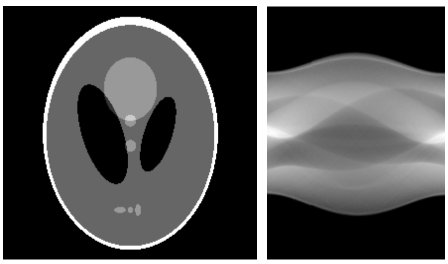

Figure 1.2 – A gauche : la fonction f initiale, dont la valeur est représentée en intensité de gris. A droite : sa transformée de Radon, où l'ensemble des droites est paramétré en utilisant un paramétrage normal.

C'est cette idée qui est reprise dans les dispositifs à imagerie médicale par rayons X, encore appelée tomographie par rayons X. Le corps dont on veut connaître la structure est bombardée par des rayons X — des ondes électromagnétiques à très haute fréquence, de l'ordre de 10^{16} à 10^{20} Hz/s — qui le traversent et viennent frapper un écran situé derrière lui. La présence d'une grille anti-diffusante permet de ne conserver que les photons qui se sont déplacés de façon rectiligne : se forme alors sur l'écran une image par contraste radiographique. Dûe à l'absorption d'une partie des photons par le corps au cours de leur trajet, l'intensité $I(x)$ de l'onde mesurée sur l'écran au point x, c'est-à-dire le nombre de photons par unité de surface et de temps, est donnée par la loi de Beer-Lambert :

$$I(x) = I_0 \exp\left(-\int_{x_0}^{x_1} \mu(E, Z(s))ds\right), \tag{1.1.5}$$

où I_0 est l'intensité initiale (supposée uniforme), x_0 et x_1 désignent respectivement le point d'entrée et de sortie du corps pour le rayon arrivant en x et μ est le coefficient d'atténuation, variant en fonction de l'énergie E des photons et du numéro atomique $Z(s)$ de la structure rencontrée au point s. La mesure de l'intensité permet donc de connaître la *transformée en rayons X* de la fonction d'atténuation μ que la formule de Radon rend ensuite possible d'inverser pour retrouver le coefficient d'atténuation μ.

Dans l'exemple précédent, les photons se déplacent en majorité en ligne droite (une faible partie est déviée par un processus de diffusion élastique que la grille d'anti-diffusion se charge d'atténuer), c'est-à-dire selon les lois de la géométrie euclidienne. Il est tout à fait possible de généraliser la discussion précédente à des géométries qui seraient courbées, telles que celles mentionnées au paragraphe §1.1.1. La transformée en rayons X d'une fonction évaluée sur une géodésique est alors l'intégrale de la fonction le long de cette même géodésique. De façon générale, la question qui nous intéressera est la suivante :

Etant connue la transformée en rayons X d'une fonction, est-il possible de reconstruire cette fonction ?

Tout comme au paragraphe §1.1.1, deux problèmes se posent en réalité : est-il *théoriquement* possible de reconstruire la fonction, autrement dit, la transformée en rayons X est-elle injective ? Et si oui, est-il possible de donner un algorithme de reconstruction ? Mentionnons au passage le fait que des transformées en rayons X plus générales peuvent être définies sur des tenseurs de rang quelconque, et non seulement des fonctions, chose que nous étudierons par la suite.

La géométrie spectrale. En 1966, dans un article au *American Mathematical Monthly*, Kac [Kac66] jette les bases de la géométrie spectrale dans une formulation depuis restée célèbre :

Peut-on entendre la forme d'un tambour ?

Il considère un tambour, modélisé par une membrane élastique Ω dont le bord $\partial\Omega$ est fixé dans le plan (Oxy). Le soulèvement vertical $u(t, x, y)$ du tambour selon l'axe (Oz) est régi par l'équation des ondes

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0,$$

avec pour conditions initiales $u(t = 0) = u_0, \partial_t u(t = 0) = u_0'$. Ici, Δ désigne le laplacien de Dirichlet sur la surface Ω. Il est bien connu que les solutions de l'équation des ondes se décomposent sous forme harmonique en

$$u(t,x,y) = \sum_{n=0}^{+\infty} (u_n^{(+)} e^{+i\sqrt{\lambda_n}t} + u_n^{(-)} e^{-i\sqrt{\lambda_n}t}) \psi_n(x,y),$$

où les ψ_n sont les fonctions propres normalisées du laplacien, associées aux valeurs propres λ_n/c^2. Ces fréquences propres sont appelées les *tons purs* de la membrane, physiquement mesurables. La question que pose Kac peut alors se reformuler en ces termes : étant connues les fréquences propres de vibration de la membrane, est-il possible d'en déduire sa géométrie ? Autrement dit : les valeurs propres du laplacien déterminent-elles la forme de la membrane Ω ?

Une première réponse qui peut-être apportée au problème de Kac est que certaines quantités géométriques peuvent être directement lues sur les valeurs propres du laplacien. Si $N(R)$ désigne le nombre de valeurs propres du laplacien $\leq R$, alors la loi de Weyl stipule que ce nombre croît en :

$$N(R) \sim_{r\to+\infty} \mathrm{vol}(\Omega)\frac{R}{2\pi}. \tag{1.1.6}$$

Le volume de la membrane est donc déterminé par les fréquences propres, mais ce n'est bien sûr qu'une information très partielle.

Il a fallu attendre 1992 pour que le problème de Kac soit résolu : Gordon-Webb-Wolpert [GWW92] ont démontré l'existence de domaines planaires isospectraux (ayant même spectre du laplacien) non isométriques. Mais, de façon plus générale, le problème a rapidement été formulé pour des variétés riemannienne compactes (à bord ou fermées) : il consiste à savoir si l'isospectralité des variétés implique leur isométrie. Peu avant, Milnor [Mil64] avait déjà remarqué qu'il existe des paires de tores de dimension 16 isospectraux mais non isométriques et c'est en 1980, suite aux travaux de Vigneras [Vig80], qu'on a su qu'il existait des paires de surfaces hyperboliques isospectrales mais non isométriques. Sans contraintes supplémentaires, le spectre du laplacien est donc un invariant géométrique trop peu robuste pour contraindre entièrement la métrique de la variété. La question se pose alors naturellement de chercher une quantité géométrique qui coderait entièrement la géométrie de la variété. En courbure négative, qui est un contexte où le flot géodésique est « chaotique », le candidat qui pourrait sembler convenir est le *spectre des longueurs*, c'est-à-dire la suite des longueurs des géodésiques périodiques. Or il se trouve que, au moins de façon générique c'est-à-dire pour « presque toutes les métriques », le spectre des longueurs est déterminé par le spectre du laplacien qui, lui-même, n'est pas suffisant pour déterminer la métrique de la variété comme nous l'avons évoqué. Une donnée plus riche est fournie par le *spectre marqué des longueurs*, c'est-à-dire la suite des longueurs des géodésiques périodiques, repérées par leur classe libre d'homotopie. Et c'est une célèbre conjecture de 1985, dûe à Burns et Katok [BK85], que le spectre marqué des longueurs des variétés à courbure négative devrait déterminer la métrique de ces variétés. Elle a été démontrée indépendamment en dimension deux en 1990 par Croke [Cro90] et Otal [Ota90] mais, depuis, le problème est resté largement ouvert. Dans cette thèse, nous apportons un résultat qui étaie significativement la validité de cette conjecture (voir le Théorème VI) en toute dimension.

1.2 Organisation de cette thèse

1.2.1 Quelques mots sur la littérature

Dans les paragraphes qui vont suivre, nous introduisons plus précisément les sujets qui vont nous intéresser ici et détaillons les résultats antérieurs à cette thèse. C'est probablement à partir de cette page qu'un lecteur non initié à la géométrie se verrait contraint d'abandonner la lecture.

A de rares exceptions près (dont le célèbre papier de Michel [Mic82] et les travaux de Guillemin-Kazhdan [GK80a]), la littérature antérieure aux années 1990 concernant les problèmes inverses géométriques est principalement issue de l'école sibérienne, centrée autour de Dairbekov, Pestov, Mukhometov, Romanov et Sharafutdinov. Le *Integral geometry of tensor fields* de Sharafutdinov [Sha94], publié en 1994, résume à peu près toutes les connaissances de l'époque concernant la géométrie intégrale. On y trouve déjà l'étude systématique des tenseurs symétriques ainsi que le recours à l'analyse microlocale. Mais la caractéristique de l'école russe est de travailler uniquement en coordonnées, ce qui a pour principal défaut de compliquer des calculs qui, faits de manière intrinsèque, peuvent devenir triviaux. Aussi renvoyons-nous vers le cours de Paternain [Pat] pour une introduction géométrique accessible à ces problèmes. Mon mémoire de M2 peut également servir d'entrée en matière, de nombreuses preuves y étant détaillées. Nous renvoyons également à l'Appendice B, où les principaux résultats concernant les tenseurs symétriques sont rappelés.

A partir du début des années 2000, le recours à l'analyse microlocale devient plus systématique sous l'impulsion d'un certain nombre de travaux dont Dairbekov-Uhlmann [DU10], Pestov-Uhlmann [PU05], Stefanov-Uhlmann [SU04, SU05, SU09]. Encore plus récemment, à partir des années 2010, son utilisation est devenue cruciale dans un large nombre de résultats, notamment pour l'étude de l'injectivité locale de la transformée en rayons X : le travail fondateur de Uhlmann-Vasy [UV16] a ensuite conduit à de nombreux théorèmes, tous exploitant le même principal général. En parallèle, l'étude analytique des flots uniformément hyperboliques par des outils d'analyse microlocale (voir par exemple les travaux de Faure-Sjöstrand [FS11] ou Dyatlov-Zworski [DZ16]) a trouvé une application particulièrement efficace dans les problèmes inverses géométriques présentant un caractère hyperbolique, comme c'est le cas des variétés à courbure négative. Les deux papiers de Guillarmou [Gui17a, Gui17b] sont, en ce sens, fondateurs de cette approche. C'est principalement à partir des nouvelles idées introduites dans [Gui17a, Gui17b] que se sont constitués les principaux résultats de cette thèse. Nous renvoyons à l'Appendice A pour une brève introduction à l'analyse microlocale.

1.2.2 Plan de la thèse

Cette thèse a donné lieu à la publication de huit articles scientifiques :

1. [Lef19] *On the s-injectivity of the X-ray transform for manifolds with hyperbolic trapped set*, (https://arxiv.org/abs/1807.03680), **Nonlinearity**, vol. 32, n°4 (2019), 1275–1295,
2. [Lef18b] *Local marked boundary rigidity under hyperbolic trapping assumptions*, (https://arxiv.org/abs/1804.02143), à paraître dans **Journal of Geometric Analysis** (2019),

3. [Lef18a] *Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces*, (https://arxiv.org/abs/1805.05155), à paraître dans **Commentarii Mathematici Helvetici** (2019),
4. [GL19d] *The marked length spectrum of Anosov manifolds*, (https://arxiv.org/abs/1806.04218), avec Colin Guillarmou, **Annals of Mathematics**, vol. 190, n°1 (2019),
5. [GL19a] *Classical and microlocal analysis of the X-ray transform on Anosov manifolds*, (https://arxiv.org/abs/1904.12290), avec Sébastien Gouëzel, à paraître dans **Analysis & PDE** (2019),
6. [GKL19] *Geodesic stretch and marked length spectrum rigidity*, (https://arxiv.org/abs/1909.08666), avec Colin Guillarmou et Gerhard Knieper,
7. [GL19b] *Local rigidity of manifolds with hyperbolic cusps I. Linear theory and pseudodifferential calculus*, (https://arxiv.org/abs/1907.01809), avec Yannick Guedes Bonthonneau,
8. [GL19c] *Local rigidity of manifolds with hyperbolic cusps II. Nonlinear theory*, (https://arxiv.org/abs/1910.02154), avec Yannick Guedes Bonthonneau.

Les trois premiers articles sont rassemblés dans la dernière partie de cette thèse et traitent de problèmes de rigidité sur les variétés ouvertes. Les trois suivants sont rassemblés dans la première partie et traitent de problèmes de rigidité sur des variétés fermées, c'est-à-dire sans bord et compactes. Enfin, les deux derniers articles traitent de problèmes de rigidité sur des variétés non-compactes mais sans bord (variétés à pointes hyperboliques réelles) et constituent la seconde partie de cette thèse. Mentionnons également au passage l'article de vulgarisation *Le chant de la Terre*, paru dans **Images des mathématiques** (https://images.math.cnrs.fr/Le-chant-de-la-Terre.html).

Nous avons jugé plus pertinent de substituer à l'organisation chronologique de ce manuscrit (au sens de la parution des articles) une organisation thématique qui va probablement du moins technique au plus technique. Par là, nous espérons faciliter la lecture en ne présentant dans un premier temps (cas des variétés fermées) que les idées principales et les outils techniques qui leur sont sous-jacentes. Le passage aux variétés non-compactes ou à bord ne modifie la donne que dans la mesure où les techniques employées sont plus complexes, d'où la raison de ne les traiter que dans un second temps. Paradoxalement, la formulation de la théorie des problèmes inverses sur les variétés à bord est aussi bien naturelle : du reste, c'est celle qui revêt le sens physique le plus immédiat.

1.3 Principaux résultats sur les variétés ouvertes

Soit (M, g) une variété lisse compacte à bord. On notera SM son fibré unitaire tangent et

$$\partial_\pm SM = \{(x, v) \in TM, x \in \partial M, |v|_x = 1, \mp g_x(v, \nu) < 0\},$$

où ν est le vecteur unitaire normal au bord ∂M pointant vers l'extérieur. Pour $(x, v) \in SM$, les temps de sortie dans le passé (-) et le futur (+) sont définis par :

$$\begin{aligned}\ell_+(x, v) &:= \sup\{t \geq 0, \varphi_t(x, v) \in SM\} \in [0, +\infty] \\ \ell_-(x, v) &:= \inf\{t \leq 0, \varphi_t(x, v) \in SM\} \in [-\infty, 0]\end{aligned}$$

On dit qu'un point (x, v) est *capté dans le futur* (resp. *dans le passé*) si $\ell_+(x, v) = +\infty$ (resp. $\ell_-(x, v) = -\infty$). Les queues entrantes (-) et sortantes (+) de SM sont définies par :

$$\Gamma_\mp := \{(x, v) \in SM, \ell_\pm(x, v) = \pm\infty\}$$

L'ensemble capté K pour le flot géodésique sur SM est l'ensemble des points qui ne s'échappent jamais de la variété, ni dans le passé, ni dans le futur :

$$K := \Gamma_+ \cap \Gamma_- = \cap_{t\in\mathbb{R}}\varphi_t(SM)$$

1.3.1 La transformée en rayons X

Nous pouvons à présent introduire la *transformée en rayons X* des fonctions sur SM.

Definition 1.3.1. La transformée en rayons X, notée $I : C_c^\infty(SM \setminus \Gamma_-) \to C_c^\infty(\partial_- SM \setminus \Gamma_-)$, est définie par :

$$If : (x, v) \mapsto \int_0^{+\infty} f(\varphi_t(x, v))dt$$

Puisque f est à support compact dans $SM \setminus \Gamma_-$, l'intégrale est bien finie puisqu'elle est calculée en tout point $(x, v) \in \partial_- SM$ sur un temps uniformément fini. Si $f \in C^\infty(M, \otimes_S^m T^*M)$ est un m-tenseur symétrique, on peut voir f comme une fonction lisse sur SM par l'identification $\pi_m^* f : (x, v) \mapsto f_x(\otimes^m v)$ (voir l'Appendice B). On définit alors la transformée en rayons des m-tenseurs symétriques par $I_m := I \circ \pi_m^*$.

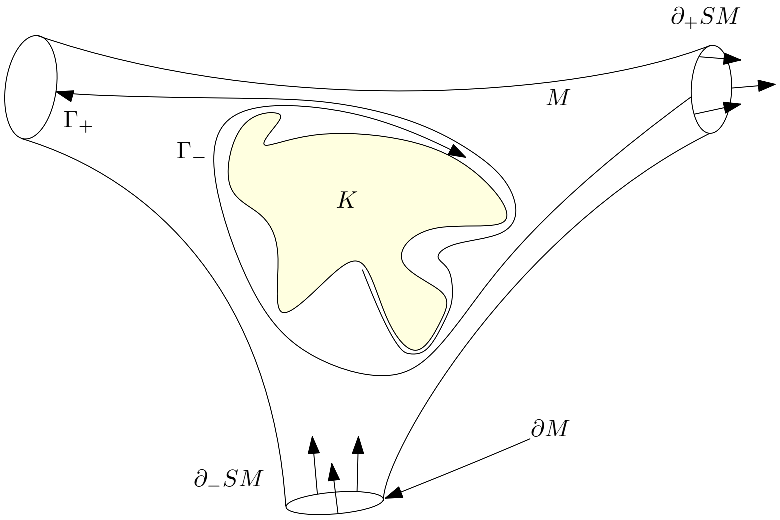

FIGURE 1.3 – La variété M.

Le noyau de I contient des élements évidents qui sont les cobords Xu, où $u \in C^\infty(SM)$ s'annule sur ∂SM. Si $\sigma : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \otimes_S^m T^*M)$ désigne l'opérateur de symétrisation des tenseurs, on note $D := \sigma \circ \nabla$ la dérivée symétrisée. On a alors la formule $X\pi_{m-1}^* = \pi_m^* D$. Le noyau de I_m contient donc tous les tenseurs de la forme Dp, où $p \in C^\infty(M, \otimes_S^{m-1} T^*M)$ s'annule sur ∂M : on les appelle les *tenseurs potentiels*. En fait, tout tenseur symétrique se décompose de façon unique en $f = h + Dp$, où $p|_{\partial M} = 0$ et $D^*h = 0$ — on dit que h est *solénoïdal* —, D^* étant l'adjoint formel de D pour le produit scalaire sur $L^2(M, \otimes_S^m T^*M)$. On dira que I_m est injective sur les tenseurs solénoïdaux — ou *s-injective* — si I_m restreinte à $C_{\mathrm{sol}}^\infty(M, \otimes_S^m T^*M)$ (les tenseurs symétriques solénoïdaux lisses) est injective.

De manière générale, il est conjecturé que la transformée I_m est s-injective sur les variétés sans ensemble capté, pour tout $m \geq 0$. En revanche, dès que la variété admet un ensemble capté, il est nécessaire de faire des hypothèses supplémentaires. Par exemple, en dimension 2, s'il est possible de plonger une partie cylindrique plate dans la surface, alors il est facile de construire des fonctions non triviales, supportées dans cette partie cylindrique, dans le noyau de la transformée I_0. Mais il est vraisemblable que sous l'hypothèse que l'ensemble capté est hyperbolique, la transformée I_m soit encore injective pour tout $m \geq 0$.

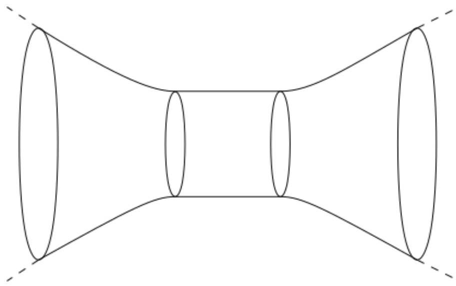

FIGURE 1.4 – Le plongement d'un cylindre euclidien prévient la s-injectivité de I_m

Pour des raisons techniques, la majorité des cas traités dans la littérature sont ceux d'une variété à bord strictement convexe [7] sans ensemble capté et sans points conjugués. En ajoutant la condition que la variété est simplement connexe, on obtient une variété dite *simple*, une définition équivalente étant de dire que la fonction exponentielle est en tout point un difféomorphisme sur son image (en particulier, de telles variétés sont des boules topologiques). Dans ce cas, Mukhometov [Muk77] a établi la s-injectivité de I_0, Anikonov-Romanov [AR97] celle de I_1, Paternain-Salo-Uhlmann [PSU13] celle de I_m pour tout $m \geq 0$ sur les surfaces, Pestov-Sharafutdinov [PS87] celle de I_m pour tout $m \geq 0$ en dimension $n \geq 2$ et courbure strictement négative.

Dans le cas où la variété est de dimension supérieure ou égale à 3 et admet un feuilletage par des hypersurfaces strictement convexes, Uhlmann-Vasy [UV16] ont montré grâce à des techniques fines d'analyse microlocale la s-injectivité de I_0, puis Stefanov-Uhlmann-Vasy [SUV17] celle de I_1 et I_2, et enfin De Hoop-Uhlmann-Zhai [dUZ18] ont généralisé le résultat à tout ordre $m \geq 0$. Cette condition de feuilletage est en particulier vérifiée pour les variétés à bord strictement convexe

- simplement connexe et de courbure sectionnelle négative,
- ou de courbure sectionnelle positive.

Ces variétés n'admettent pas d'ensemble capté mais, dans le second cas, il peut exister des points conjugués. Mentionnons au passage qu'*on ne sait toujours pas si les variétés simples vérifient la condition de feuilletage convexe.*

Seulement récemment, grâce à des techniques d'analyse microlocale que nous détaillerons par la suite, la condition sur l'ensemble capté a pu être étudiée. En particulier, Guillarmou [Gui17b] a montré que I_m est s-injective pour $m = 0, 1$ sur les variétés à bord strictement convexe, sans points conjugués et ensemble capté hyperbolique — variétés que nous appellerons par la suite *simples avec topologie* — et à tout ordre $m \geq 0$ sous l'hypothèse supplémentaire que la courbure sectionnelle est négative. Dans [Lef19], nous arrivons à faire l'économie de l'hypothèse de courbure sur les surfaces.

Théorème I (L., '18)**.** *Soit (M, g) une surface compacte connexe simple avec topologie. Alors I_m est s-injective pour tout $m \geq 0$.*

Notons qu'en dimension ≥ 3, il n'existe pas encore de résultat général permettant de s'affranchir de l'hypothèse de courbure négative. Quant à l'hypothèse de stricte convexité du bord, il est désormais connu qu'elle peut être enlevée : elle pose des

7. Au sens où la seconde forme fondamentale y est définie positive.

problèmes de régularité due aux trajectoire rasantes qui ne quittent pas la variété mais Guillarmou-Mazzucchelli-Tzou [GMT17] ont quasiment réussi à s'en affranchir dans un récent article, reprenant des idées de Stefanov-Uhlmann [SU09]. En revanche, il semble difficile de se passer de l'hypothèse d'absence de points conjugués : dans ce cas, l'opérateur normal $I_m^* I_m$ — dont l'analyse est cruciale comme nous le verrons par la suite — n'est plus un opérateur pseudodifférentiel mais seulement un Fourier intégral dont la relation canonique n'est plus un difféomorphisme symplectique et ses propriétés sont moins évidentes. Nous renvoyons vers l'article de survol de Ilmavirta-Monard [IM18] pour plus de précisions.

Outre son intérêt en tant que tel, nous allons voir que la transformée en rayons X apparaît naturellement après linéarisation de certains opérateurs, ce qui rend son étude d'autant plus précieuse. Elle joue un rôle-clé dans les problèmes dits de *rigidité*.

1.3.2 La rigidité (marquée) du bord des variétés compactes

Si (M, g) est une variété simple, il est bien connu qu'il existe une unique géodésique $\gamma_{x,y}$ entre chaque paire de points au bord $(x, y) \in \partial M \times \partial M$ qui réalise la distance riemannienne entre x et y, i.e. $d_g(x, y) = \ell_g(\gamma_{x,y})$. On appelle *fonction de distance au bord* l'application

$$d_g : \partial M \times \partial M \to \mathbb{R}_+, \qquad d_g(x, y) = \ell_g(\gamma_{x,y}) \tag{1.3.1}$$

Il a été conjecturé par Michel [Mic82] en 1982 que, dans le cas où la métrique g est simple, la fonction d_g *détermine la métrique* ou encore que (M, g) est *rigide au bord*, au sens suivant :

Conjecture I (Michel '81)**.** *La fonction de distance au bord d_g détermine la métrique au sens où si g et g' sont deux métriques simples sur M telles que $d_g = d_{g'}$, alors il existe un difféomorphisme $\phi : M \to M$ fixant le bord ∂M tel que $\phi^* g' = g$.*

Cette conjecture a été démontrée en dimension deux par Pestov-Uhlmann [PU05]. Mentionnons également les travaux de Gromov [Gro83] pour les sous-domaines de $\mathbb{R}^n$ et de Burago-Ivanov [BI10] pour les métriques au voisinage de la métrique euclidienne, l'article de Besson-Courtois-Gallot [BCG95] qui implique la rigidité des sous-domaines de $\mathbb{H}^n$. Sous certaines hypothèses, Croke-Dairbekov-Sharafutdinov [CDS00] et Stefanov-Uhlmann [SU04] ont établi la *rigidité locale du bord* (au sens où g' est choisie dans un certain voisinage de la métrique g). Enfin, Stefanov-Uhlmann-Vasy [SUV17] ont montré la rigidité du bord des variétés satisfaisant l'hypothèse de feuilletage évoquée au paragraphe précédent : pour montrer la conjecture de Michel, il suffirait donc de montrer que de telles variétés satisfont la condition de feuilletage.

On peut généraliser la notion de distance au bord aux variétés simples avec topologie. Pour $(x, y) \in \partial M \times \partial M$, on notera $\mathcal{P}_{x,y}$ l'ensemble des classes d'homotopie de courbes joignant x à y et

$$\Omega := \{(x, y, [\gamma]), (x, y) \in \partial M \times \partial M, [\gamma] \ \in \mathcal{P}_{x,y}\}$$

Sous les hypothèses faites, étant donné $(x, y) \in \partial M \times \partial M, [\gamma] \ \in \mathcal{P}_{x,y}$, il existe une unique géodésique $\gamma_{x,y} \in [\gamma]$ qui réalise la distance riemannienne entre x et y au sein de la classe d'homotopie $[\gamma]$ (voir [GM18]). On définit alors la *fonction de distance marquée au bord* par

$$d_g : \Omega \to \mathbb{R}_+, \qquad d_g(x, y, [\gamma]) = \ell_g(\gamma_{x,y}) \tag{1.3.2}$$

Cette définition généralise (1.3.1) au cas d'une variété non simplement connexe. Il est conjecturé que sous l'hypothèse que g est simple avec topologie, la fonction de distance marquée au bord détermine la métrique. On parlera alors de la *conjecture de Michel étendue*. Peu de résultats sont connus quant à cette conjecture. Guillarmou [Gui17b] a montré que, dans le cas d'une telle surface, la distance marquée déterminait la classe conforme de la métrique : il resterait à montrer que le facteur conforme est bien égal à 1 mais c'est une question ouverte. Guillarmou-Mazzucchelli [GM18] ont établi un résultat de *rigidité marquée du bord*, que nous mentionnons après notre contribution [Lef18b], qui est une version locale de la conjecture de Michel étendue :

Théorème II (L. '18)**.** *Soit (M, g) une variété compacte connexe $(n+1)$-dimensionnelle à bord strictement convexe et courbure strictement négative. On définit $N := \left\lfloor \frac{n+2}{2} \right\rfloor + 1$. Alors (M, g) est localement rigide au bord pour la distance marquée, au sens où il existe $\varepsilon > 0$ tel que pour tout autre métrique g' ayant même fonction de distance au bord marqué que g et telle que $\|g' - g\|_{C^N} < \varepsilon$, il existe un difféomorphisme lisse $\phi : M \to M$ préservant le bord ∂M et tel que $\phi^* g' = g$.*

Le Théorème II est plus généralement valide sur les variétés simples avec topologie sur lesquelles la transformée en rayons X sur les 2-tenseurs I_2 est s-injective : cela tient au fait que I_2 apparaît comme le linéarisé de la fonction de distance marquée au bord. Par suite, les Théorèmes I et II permettent alors de retrouver le récent résultat de Guillarmou-Mazzucchelli [GM18] :

Théorème III (Guillarmou-Mazzucchelli '18, L. '18)**.** *Soit (M, g) une surface compacte connexe simple avec topologie. Alors (M, g) est localement rigide au bord pour la distance marquée.*

1.3.3 La rigidité (marquée) du bord des variétés asymptotiquement hyperboliques.

On peut généraliser la discussion conduite au paragraphe précédent aux contextes des variétés conformellement compactes. Soit $\overline{M}$ une variété lisse compacte à bord. On dit que $\rho : \overline{M} \to \mathbb{R}_+$ est une fonction *définissant le bord* si $\rho > 0$ sur M, $\rho = 0$ et $d\rho \neq 0$ sur ∂M (que l'on définit comme étant $\partial \overline{M}$). On dit que (M, g) est *asymptotiquement hyperbolique* (i) si la métrique $\overline{g} = \rho^2 g$ s'étend en une métrique lisse sur $\overline{M}$ et (ii) si $|d\rho|_{\rho^2 g} = 1$ sur ∂M, cette dernière condition assurant que les courbures sectionnelles de g tendent uniformément vers -1 lorsque l'on s'approche du bord. Précisons que ces deux conditions sont indépendantes du choix de ρ ; la métrique $\overline{g}|_{\partial M}$, en revanche, ne l'est pas, mais sa classe conforme l'est. On appellera cette classe conforme de métriques sur ∂M l'*infini conforme* de (M, g).

Une telle variété admet une structure de produit local au voisinage du bord (voir [Gra00]). Autrement dit, si h_0 est un choix de métrique sur ∂M dans l'infini conforme de M, il existe un jeu de coordonnées (ρ, y) (où ρ est une fonction définissant le bord) tel que $|d\rho|_{\rho^2 g} = 1$ sur un voisinage de ∂M, $\rho^2 g|_{T\partial M} = h_0$, et sur un voisinage annulaire de ∂M, la métrique s'écrit dans ces coordonnées

$$g = \frac{d\rho^2 + h(\rho)}{\rho^2}, \tag{1.3.3}$$

où $h(\rho)$ est une famille lisse de métrique sur ∂M telle que $h(0) = h_0$.

Puisque (M, g) n'est ni compacte, ni même de volume fini, deux points $(x, y) \in \partial M \times \partial M$ ne sont pas situés à distance finie l'un de l'autre, mais on peut définir une notion de *distance renormalisée* entre ces points par un procédé de régularisation à la Hadamard. Si $\gamma_{x,y}$ est une géodésique joignant x à y, le terme principal dans le développement asymptotique de $\ell_g(\gamma_{x,y} \cap \{\rho > \varepsilon\})$ (lorsque $\varepsilon \to 0^+$) est $2|\log(\varepsilon)|$ et le second terme est une constante. On définit la longueur renormalisée $L_g(\gamma_{x,y})$ de $\gamma_{x,y}$ comme étant l'exponentielle de cette constante. Il est à noter que la valeur de cette longueur renormalisée *dépend du choix de la métrique dans l'infini conforme de* (M, g).

Si g est partout à courbure strictement négative, on peut montrer qu'à l'instar des variétés simples avec topologie, la variété asymptotiquement hyperbolique (M, g) vérifie la propriété suivante : entre chaque paire de points $(x, y) \in \partial M \times \partial M$, il existe une unique géodésique $\gamma_{x,y} \in [\gamma]$ dans chaque classe d'homotopie $[\gamma] \in \mathcal{P}_{x,y}$ de courbes joignant x à y (les notations font suite à celles employées au paragraphe précédent). On définit alors la *distance renormalisée marquée* par

$$D_g : \Omega \to \mathbb{R}_+, \qquad D_g(x, y, [\gamma]) = L_g(\gamma_{x,y}) \tag{1.3.4}$$

Bien sûr, si la variété ne présente pas de topologie, il suffit d'oublier le marquage par l'homotopie. Cette notion a été introduite par Graham-Guillarmou-Stefanov-Uhlmann [GGSU17]. Dans [GGSU17], un certain nombre de résultats de rigidité du bord sont démontrés, similaires à ceux déjà connus dans le cas compact. Dans le cas d'une surface, nous avons établi le résultat suivant, qui est à comparer aux résultats d'Otal [Ota90] et de Croke [Cro90], dans le cas des surfaces compacts.

Théorème IV (L., 2018)**.** *Soient (M, g) et (M, g') deux surfaces asymptotiquement hyperboliques de courbure strictement négative. On suppose que pour un certain choix h et h' de représentants conformes dans les infinis conformes de g et g', les fonctions renormalisées de distance au bord marqué coïncident i.e. $D_g = D_{g'}$. Alors il existe un difféomorphisme lisse $\phi : \overline{M} \to \overline{M}$ tel que $\phi^* g' = g$ sur M et $\phi|_{\partial M} = Id$.*

1.4 Principaux résultats sur les variétés fermées

1.4.1 La transformée en rayons X

On suppose à présent que (M, g) est une variété riemannienne fermée, c'est-à-dire compacte sans bord. On note $(\varphi_t)_{t\in\mathbb{R}}$ le flot géodésique sur le fibré unitaire tangent SM que l'on suppose Anosov — on dit alors que (M, g) est *une variété Anosov*. On note $\mathcal{G}$ l'ensemble des géodésiques périodiques, et pour $\gamma \in \mathcal{G}$, $\ell(\gamma)$ la longueur de la géodésique γ. On note $\mathcal{C}$ l'ensemble des classes d'homotopie libre (qui est en correspondance biunivoque avec l'ensemble des classes de conjugaison du groupe fondamental). Il est connu (voir [Kli74]) que dans le cas où le flot géodésique est Anosov, il existe une unique géodésique fermée $\gamma(c) \in c$ dans chaque classe d'homotopie libre $c \in \mathcal{C}$.

On peut définir de façon analogue au cas ouvert la transformée en rayons X comme étant l'application

$$I : C^0(SM) \to \ell^\infty(\mathcal{C}), \qquad If : \gamma \mapsto \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t z) dt \tag{1.4.1}$$

où $z \in \gamma$ est un point quelconque. De la même façon, on définit la transformée en rayons X des tenseurs symétriques d'ordre m via le tiré-en-arrière par π_m en posant $I_m :=$

$I \circ \pi_m^*$. La décomposition des tenseurs symétriques $f = h + Dp$ en partie solénoïdale h (i.e. telle que $D^*h = 0$) et potentielle est encore valide, les tenseurs potentiels sont dans le noyau de I_m. Comme dans le cas ouvert, on dira que I_m est s-injective si I_m restreinte à $C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ est injective.

De façon générale, il est conjecturé que I_m est s-injective si la variété (M, g) est Anosov. Comme dans le cas ouvert, le manque d'hyperbolicité — le plongement d'un cyclindre euclidien dans la variété par exemple — semble prévenir la s-injectivité de I_m. Dans le cas Anosov, l'injectivité est connue si

- $m = 0$ ou $m = 1$ (voir Dairbekov-Sharafutdinov [DS03, Theorem 1.1 and 1.3]),
- $m \in \mathbb{N}$ et $\dim(M) = 2$ (voir Guillarmou [Gui17a, Theorem 1.4]),
- $m \in \mathbb{N}$ et g est à courbure négative (voir Croke-Sharafutdinov [CS98, Theorem 1.3]).

Notons qu'en dimension 2, le cas $m = 2$ avait été préalablement établi par Paternain-Salo-Uhlmann [PSU14a].

La s-injectivité de I_m étant connue, on peut également s'intéresser à des estimées de stabilité pour cette application. Ce problème a longtemps été ouvert et nous y avons apporté une première réponse avec Colin Guillarmou dans [GL19d], puis une réponse plus quantitative avec Sébastien Gouëzel dans [GL19a]. Il n'est pas certain que l'on puisse obtenir une estimée *linéaire* pour I_m mais nous avons obtenu le

Théorème V (Guillarmou-L. '18, Gouëzel-L. '18)**.** *Pour tout $0 < \beta < \alpha$, il existe des constantes $C := C(\alpha, \beta), \theta_1 := \theta_1(\alpha, \beta) > 0$ telle que :*

$$\forall f \in C^\alpha_{\mathrm{sol}}(M, \otimes^m_S T^*M) \text{ avec } \|f\|_{C^\alpha} \leq 1, \qquad \|f\|_{C^\beta} \leq C \|I_m f\|_{\ell^\infty}^{\theta_1}$$

Bien sûr, il est possible d'énoncer le théorème précédent dans d'autres régularités. On peut aussi vouloir caractériser plus finement l'injectivité de la transformée en rayons X. Par exemple, que dire d'un tenseur dont les intégrales le long des géodésiques fermées seraient nulles, seulement jusqu'à une certaine longueur $L > 0$ (suffisamment grande) de géodésique ? C'est ce que donne le résultat suivant :

Corollaire I (Gouëzel-L. '18)**.** *Pour tout $0 < \beta < \alpha$, il existe des constantes $C := C(\alpha, \beta), \theta_2 := \theta_2(\alpha, \beta) > 0$ telle que pour $L > 0$ suffisamment grand : étant donné $f \in C^\alpha_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, un m-tenseur symétrique solénoïdal tel que $\|f\|_{C^\alpha} \leq 1$ et $I_m f(\gamma) = 0$ pour toutes les géodésiques fermées $\gamma \in \mathcal{G}$ telles que $\ell(\gamma) \leq L$, on a $\|f\|_{C^\beta} \leq CL^{-\theta_2}$.*

Même dans le cas où f est un 0-tenseur, c'est-à-dire une fonction provenant de la base, il semblerait que le résultat précédent soit inédit.

1.4.2 Le spectre marqué des longueurs des variétés compactes

Comme nous l'avons mentionné au paragraphe précédent, il existe une unique géodésique fermée dans chaque classe d'homotopie libre. Cela permet de définir le *spectre marqué des longueurs* par la donnée

$$L_g : \mathcal{C} \to \mathbb{R}^+, \qquad L_g(c) := \ell_g(\gamma(c)), \tag{1.4.2}$$

où ℓ_g désigne la longueur riemannienne relative à la métrique g. On rappelle la conjecture suivante, formulée par Burns et Katok en 1985, et depuis largement restée ouverte :

Conjecture II. *[BK85, Problem 3.1] Si (M,g) et (M,g') sont deux variétés fermées à courbure sectionnelle strictement négative et même spectre marqué des longueurs, i.e $L_g = L_{g'}$, alors elles sont isométriques au sens où il existe un difféomorphisme lisse $\phi : M \to M$ tel que $\phi^* g' = g$.*

Il est vraisemblable que la conjecture soit encore vraie dans le cadre plus général des variétés Anosov. La conjecture de Burns-Katok est l'équivalent de la conjecture de Michel généralisée aux variétés fermées. Les seuls résultats connus sont les suivants :

- Croke [Cro90] et Otal [Ota90] ont prouvé indépendamment la conjecture en dimension deux,
- Katok [Kat88] a prouvé le cas où g et g' sont conformes,
- Les résultats de [BCG95] et Hamensdtädt [Ham99] établissent le cas où (M,g) est un espace localement symétrique.

Le problème *linéarisé* ou encore *infinitésimal* consiste à étudier la question suivante : si $(g_s)_{s\in(-1,1)}$ est une famille lisse de métriques telle que $g_0 = g$ et $L_{g_s} = L_g$, existe t-il une isotopie $(\phi_s)_{s\in(-1,1)}$ telle que $\phi_s^* g_s = g$? On peut montrer que la rigidité infinitésimale du spectre marqué des longueurs est impliquée par la s-injectivité de la transformée en rayons X sur les 2-tenseurs symétriques I_2, qui est déjà connue dans un certain nombre de cas (voir §1.4.1). Ce lien a été pour la première fois compris par Guillemin-Kazhdan dans leur travail pionnier [GK80a]. Quant au problème non-linéaire, hormis les résultats évoqués un peu plus haut, le seul résultat général en dimension ≥ 3 est celui que nous avons obtenu avec Colin Guillarmou [GL19d] :

Théorème VI (Guillarmou-L. '18)**.** *Soit (M,g) :*

- *une surface fermée lisse dont le flot géodésique est Anosov,*
- *ou une variété fermée lisse de dimension $n+1 \geq 3$, de courbure négative, dont le flot géodésique est Anosov.*

On fixe une constante $N > 3(n+1)/2 + 8$. Il existe $\varepsilon > 0$ tel que pour toute autre métrique g' de même spectre marqué des longueurs que g et telle que $\|g - g'\|_{C^N(M)} < \varepsilon$, il existe un difféomorphisme lisse $\phi : M \to M$ tel que $\phi^* g' = g$.

Notre théorème est une version locale non-linéaire de la conjecture de Burns-Katok. Dans les faits, il s'obtient comme corollaire d'un théorème plus général de stabilité sur le spectre marqué des longueurs permettant de quantifier la distance entre les classes d'isométries de deux métriques par le ratio de leur spectre marqué. Entre autres résultats, on obtient également la finitude du nombre de classes d'isométries de métriques ayant même spectre marqué des longueurs, et vérifiant certaines conditions de bornitude.

Corollaire II (Guillarmou-L. '18)**.** *Soit M une variété fermée admettant une métrique à courbure strictement négative. Alors pour tout $a > 0$ et toute suite $B = (B_k)_{k\in\mathbb{N}}$ de réels positifs, il existe un nombre fini de classes d'isométries de métriques ayant même spectre marqué des longueurs, dont la courbure sectionnelle est majorée par $-a^2 < 0$, le volume uniformément borné, et le tenseur de courbure est borné par B au sens où $\|\nabla_g^k \mathcal{R}_g\|_{L^\infty,g} \leq B_k$, pour tout $k \in \mathbb{N}$.*

La preuve des Théorèmes V et VI reposent sur de nouvelles techniques impliquant l'analyse microlocale sur les variétés fermées. En particulier, l'introduction par Guillarmou [Gui17b] d'un opérateur noté Π_m (et dont l'analyse sera conduite en détail au

Chapitre 2) agissant sur les tenseurs symétriques d'ordre m a été cruciale. Cet opérateur présente d'excellentes propriétés analytiques : il est pseudodifférentiel d'ordre -1, elliptique sur les tenseurs solénoïdaux (et donc Fredholm, et même Fredholm d'indice 0 car formellement autoadjoint) et la s-injectivité de I_m est équivalente à l'invertibilité de Π_m sur les tenseurs solénoïdaux. Couplé à diverses variantes du *théorème de Livsic* (voir également le Chapitre 2), les théorèmes et corollaires énoncés précédemment en découleront facilement.

1.4.3 Le spectre marqué des longueurs des variétés à pointes hyperboliques.

Comme pour la distance (marquée) au bord, la discussion précédente peut se généraliser à un cadre non-compact qui est celui des *variétés à pointes hyperboliques.* On dira que la variété riemannienne $(n+1)$-dimensionnelle complète connexe sans bord (M, g) est une variété à pointes hyperboliques si M se décompose en une partie compacte M_0 et un nombre fini κ de pointes hyperboliques $Z_i \simeq [a, +\infty[_y \times (\mathbb{R}^n/\Lambda_\ell)_\theta$, $i = 1, \ldots, \kappa$, où $\Lambda_i \subset \mathbb{R}^n$ est un réseau unimodulaire, et la métrique g sur Z_i a l'expression particulière

$$g_{|Z_i} = \frac{dy^2 + d\theta^2}{y^2}. \tag{1.4.3}$$

La courbure sectionnelle dans la pointe est constante égale à -1. On suppose également que la courbure est strictement négative (mais possiblement variable) dans la partie compacte de la variété. En particulier, le flot géodésique sur le fibré unitaire tangent SM est uniformément hyperbolique. Contrairement au cas compact, il n'est pas vrai qu'il existe une unique géodésique fermée dans chaque classe d'homotopie libre : en fait, c'est le cas, sauf dans les classes d'homotopie libre qui s'enroulent uniquement autour des pointes. Le groupe fondamental $\pi_1(M)$ admet κ copies de $\mathbb{Z}^d$ comme sous-groupes, notées $G_i \simeq \mathbb{Z}^d$, correspondant aux classes de courbes (à point base fixé) s'enroulant uniquement autour de la même pointe. On note $\mathcal{C}$ l'ensemble des classes de conjugaison du $\pi_1(M)$ privé des classes de conjugaison des G_i.

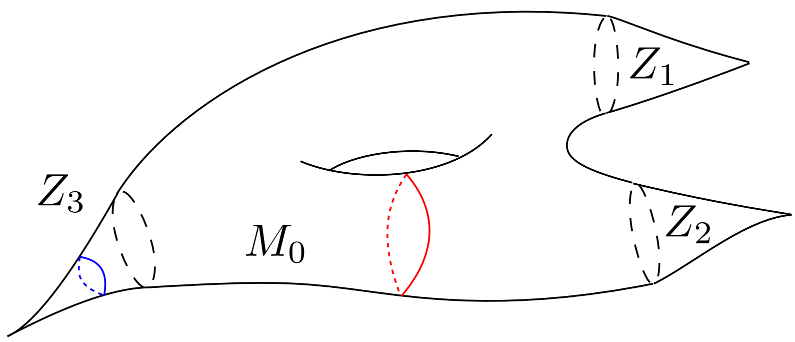

FIGURE 1.5 – Une surface à trois pointes. En rouge : une géodésique s'enroulant autour de la partie torale. En bleu : une courbe fermée autour d'une pointe ne peut pas être une géodésique fermée.

On définit alors le spectre marqué des longueurs de (M, g) comme au paragraphe §1.4.2. Il est vraisemblable que la conjecture de Burns-Katok [BK85] soit encore valide dans le cas des variétés à pointes sur lesquelles le flot géodésique est Anosov. La non-compacité de la variété pose de nombreux problèmes analytiques. Avec Yannick Bonthonneau, nous avons établi le

Théorème VII (Bonthonneau-L. '18)**.** *Soit* (M, g) *une variété sans bord à pointes hyperboliques et courbure strictement négative. On fixe une constante* $N \in \mathbb{N}$ *suffisamment grande. Il existe* $\varepsilon > 0$ *et* $\mathcal{N}_{\text{iso}}$*, une sous-variété de l'espace des classes d'isométries de codimension* 1*, tels que pour toute autre métrique* $g' \in \mathcal{N}_{\text{iso}}$ *de même spectre marqué des longueurs que* g *et telle que* $\|y^N(g - g')\|_{C^N(M)} < \varepsilon$*, il existe un difféomorphisme* $\phi : M \to M$ *tel que* $\phi^* g' = g$*.*

Il est très vraisemblable que l'hypothèse de codimension soit un artefact de la preuve que nous ne savons pour le moment pas traiter. L'énoncé du théorème est détaillé au Chapitre 6. On obtient également un théorème de stabilité duquel découle le précédent résultat. Tout comme dans le cas compact, l'idée de la preuve repose sur l'étude d'un opérateur pseudodifférentiel d'ordre -1, noté Π_2 par la suite, qui généralise en un certain sens l'opérateur I_2 de transformée en rayons X. Comme évoqué dans le cas compact, cet opérateur est elliptique et inversible sur les tenseurs solénoïdaux : c'est essentiellement cet argument qui permet d'obtenir des estimées de stabilité satisfaisantes dans le cas fermé, le lien entre Π_2 et I_2 étant ensuite obtenu via un *théorème de Livsic positif ou approché* (voir Théorèmes 2.1.2 et 2.1.3). Dans le cas non compact, ces arguments demandent à être raffinés car l'analyse elliptique devient beaucoup plus difficile, essentiellement parce que le théorème d'injection compacte $H^s \hookrightarrow H^{s'}$ (pour $s > s'$) de Kato-Rellich n'est plus valide. Les méthodes adéquates relèvent de la théorie du b-calcul telle qu'elle a été formulée par Melrose (voir [Mel93]). L'idée est de comprendre précisément la raison de la non-compacité dans les espaces fonctionnels (en l'occurrence, dans les pointes hyperboliques, la non-compacité provient des modes nuls en θ, c'est-à-dire des sections qui sont indépendantes de la variable θ), et d'inverser *exactement* l'opérateur elliptique en question sur des espaces à poids. Pour calculer les poids qui vont fonctionner, il faut étudier l'opérateur sur le *modèle à l'infini*, c'est-à-dire sur le cusp entier, sans tenir compte de la variété compacte.

Première partie

X-ray transform and marked length spectrum on closed manifolds

Chapitre 2

Classical and microlocal analysis of the X-ray transform on Anosov manifolds

« *J'ai tort, ou j'ai raison.* »

Alceste, *Le Misanthrope*, Molière

This chapter reviews some ideas developed by Faure-Sjostrand [FS11] and Guillarmou [Gui17a] and contains the article *Classical and microlocal analysis of the X-ray transform on Anosov manifolds*, written in collaboration with Sébastien Gouëzel and published in **Analysis & PDE**.

Sommaire

We review the construction of the operators Π_m — the *generalized X-ray transforms* or *normal operators* — on smooth Anosov manifolds as they were introduced by Guillarmou [Gui17a] and study their microlocal properties. Using variations of the Livsic Theorem, namely an approximate and a positive Livsic Theorem, and the operators Π_m, we prove stability estimates for the classical X-ray transform.

2.1 Introduction

Let $\mathcal{M}$ be a smooth closed $(n+1)$-dimensional manifold endowed with a vector field X generating a complete flow $(\varphi_t)_{t\in\mathbb{R}}$. We assume that there exists a smooth invariant (by the flow) probability measure $d\mu$ and that the flow $(\varphi_t)_{t\in\mathbb{R}}$ is Anosov in the sense that there exists a continuous flow-invariant splitting

$$T_x(\mathcal{M}) = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x), \tag{2.1.1}$$

where $E_s(x)$ (resp. $E_u(x)$) is the *stable* (resp. *unstable*) vector space at $x \in \mathcal{M}$, and a smooth Riemannian metric g such that

$$\begin{aligned} |d\varphi_t(x) \cdot v|_{\varphi_t(x)} &\leq Ce^{-\nu t}|v|_x, \quad \forall t > 0, v \in E_s(x) \\ |d\varphi_t(x) \cdot v|_{\varphi_t(x)} &\leq Ce^{-\nu |t|}|v|_x, \quad \forall t < 0, v \in E_u(x), \end{aligned} \tag{2.1.2}$$

for some uniform constants $C, \nu > 0$. The norm, here, is $|\cdot|_x := g_x(\cdot,\cdot)$. The dimension of E_s (resp. E_u) is denoted by n_s (resp. n_u). As a consequence, $n+1 = 1 + n_s + n_d$ (where the 1 stands for the *neutral* direction, that is the direction of the flow). The case we will have in mind will be that of a geodesic flow on the unit tangent bundle $SM =: \mathcal{M}$ of a smooth Riemannian manifold (M, g) with negative sectional curvatures, where the probability measure is the normalized Liouville measure.

2.1.1 A spectral description of X

Let us first consider the case of an Anosov geodesic flow. The dynamical properties of such a flow are now well understood : in particular, it is ergodic, mixing and even exponentially mixing, following the work of Liverani [Liv04]. But the spectral properties are less obvious. Actually, the infinitesimal generator $P := -iX$ is selfadjoint on its domain in $L^2(\mathcal{M})$ but its L^2-spectrum is equal to $\mathbb{R}$: it consists of the isolated eigenvalue 0 of multiplicity 1 associated to $\mathbb{R} \cdot \mathbf{1}$ (the constant functions) and of absolutely continuous spectrum. This spectral description is not satisfactory but the main difficulty comes from the fact that P, seen as a differential operator of order 1, is not elliptic : its principal symbol is given by

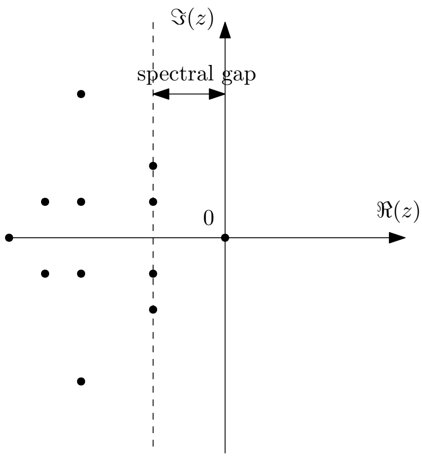

FIGURE 2.1 – A spectral gap implies the exponential decay of correlations. The black dots are the Pollicott-Ruelle resonances.

$$\sigma_P(x,\xi) = \lim_{h\to 0} he^{-iS(x)/h} \underbrace{Pe^{iS(x)/h}}_{=h^{-1}dS(X)e^{iS(x)/h}} = \langle \xi, X(x)\rangle,$$

where $\xi = dS(x) \neq 0$, and it is immediate that this operator has a non-trivial characteristic set $\Sigma = \{\langle \xi, X \rangle = 0\}$.

But it is still possible to prove that the resolvent of this operator can be meromorphically extended through the real axis. This is done by making P act on *anisotropic Sobolev spaces* $\mathcal{H}^s$, which are adapted to the dynamics. The poles of the resolvent are intrinsic (i.e. they do not depend on the choices made in the construction of the spaces $\mathcal{H}^s$) and are called the *Pollicott-Ruelle resonances*. Actually, most of the arguments are valid in the general context of an Anosov flow (not necessarily geodesic) on a closed manifold and we will state them with this degree of generality, unless explicitly mentioned.

The description of the Pollicott-Ruelle spectrum for the generator X has various interests. For instance, for $f_1, f_2 \in C^\infty(\mathcal{M})$, we define the *correlation function* $t \mapsto C_t(f_1, f_2)$ of f_1 and f_2 by

$$C_t(f_1, f_2) = \int_{\mathcal{M}} f_1(\varphi_{-t}(x)) f_2(x) d\mu(x) - \int_{\mathcal{M}} f_1(x) f_2(x) d\mu(x) \qquad (2.1.3)$$

By definition, the flow is mixing if and only if $C_t(f_1, f_2) \rightarrow_{t \rightarrow +\infty} 0$. We will say that the flow is *exponentially mixing* if C_t converges exponentially fast to 0. Note that there are Anosov flows which are not mixing, but it is conjectured that, *generically*, an Anosov flow that is mixing is actually exponentially mixing. The asymptotic properties of the correlation function C_t are governed by the Pollicott-Ruelle resonances of X. For instance, if there exists a *spectral gap* (see Figure 2.1), then it is a well-known fact that the flow is exponentially mixing.

2.1.2 X-ray transform on $\mathcal{M}$

In this paragraph, we only assume that the flow is transitive and make no assumptions as to the existence of an invariant measure. We denote by $\mathcal{G}$ the set of closed orbits of the flow and for $f \in C^0(\mathcal{M})$, its X-ray transform If is defined by :

$$\mathcal{G} \ni \gamma \mapsto If(\gamma) := \langle \delta_\gamma, f \rangle = \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} f(\varphi_t x) dt,$$

where $x \in \gamma$, $\ell(\gamma)$ is the length of γ.

The Livsic Theorem characterizes the kernel of the X-ray transform for a hyperbolic flow : if $If = 0$ then f is a coboundary that is $f = Xu$, where u is a function defined on $\mathcal{M}$ whose regularity is prescribed by that of f. In the following result, $H(\mathcal{M}) \in \{H^s(\mathcal{M}), C^\alpha(M), C^\infty(\mathcal{M}) \mid s > (n+1)/2, \alpha \in (0,1)\}$.

Theorem 2.1.1 ([Liv72, dlLMM86, Gui17a]). *Let $f \in H(\mathcal{M})$ such that $If = 0$. Then, there exists $u \in H(\mathcal{M})$, differentiable in the flow direction, such that $f = Xu$.*

For the sake of completeness, we will give the proof of this result : it was initially obtained by Livsic [Liv72] in Hölder regularity. The version of the Livsic theorem in smooth regularity is due to De la Llave-Marco-Moriyon [dlLMM86]. Much more recently, Guillarmou [Gui17a, Corollary 2.8] proved it in Sobolev regularity which implies the theorem of [dlLMM86]. We will call these results (independently of the regularity considered) an *exact Livsic theorem*.

It is also rather natural to expect other versions of the Livsic theorem to hold. For instance, if we modify the condition $If = 0$ by $If \geq 0$, is it true that $f \geq Xu$, for some well-chosen function u (*positive Livsic theorem*) ? And if $\|If\|_{\ell^\infty} := \sup_{\gamma\in\mathcal{G}} |If(\gamma)| \leq \varepsilon$, can one write $f = Xu + h$, where some norm of h is controlled by a power of ε (*approximate Livsic theorem*) ? Eventually, what can be said if $If(\gamma) = 0$ for all closed orbits γ of length $\leq L$ (finite Livsic theorem) ?

The positive Livsic theorem for Anosov flows was proved by Lopes-Thieullen [LT05] with an explicit control of a Hölder norm of the coboundary Xu in terms of a norm of f.

Theorem 2.1.2 (Lopes-Thieullen)**.** *Let $0 < \alpha \leq 1$. There exist $0 < \beta \leq \alpha$, $C > 0$ such that : for all functions $f \in C^\alpha(\mathcal{M})$, there exist $u \in C^\beta(\mathcal{M})$, differentiable in the flow-direction with $Xu \in C^\beta(\mathcal{M})$ and $h \in C^\beta(\mathcal{M})$, such that $f = Xu + h + m(f)$, with $h \geq 0$ and $m(f) = \inf_{\gamma\in\mathcal{G}} If(\gamma)$. Moreover, $\|Xu\|_{C^\beta} \leq C\|f\|_{C^\alpha}$.*

In this chapter, we will also prove a finite approximate version of the Livsic theorem. It was combined with other results in the paper [GL19a].

Theorem 2.1.3. *Let $0 < \alpha \leq 1$. There exist $0 < \beta \leq \alpha$ and $\tau, C > 0$ with the following property. Let $\varepsilon > 0$. Consider a function $f \in C^\alpha(\mathcal{M})$ with $\|f\|_{C^\alpha(\mathcal{M})} \leq 1$ such that $|If(\gamma)| \leq \varepsilon$ for all γ with $\ell(\gamma) \leq \varepsilon^{-1/2}$. Then there exist $u \in C^\beta(\mathcal{M})$ differentiable in the flow-direction with $Xu \in C^\beta(\mathcal{M})$ and $h \in C^\beta(\mathcal{M})$, such that $f = Xu + h$. Moreover, $\|u\|_{C^\beta} \leq C$ and $\|h\|_{C^\beta} \leq C\varepsilon^\tau$.*

We note that a rather similar result had already been obtained by S. Katok [Kat90] in the particular case of a contact Anosov flow on a 3-manifold.

The assumptions of Theorem 2.1.3 hold in particular if $\|If\|_{\ell^\infty} = \sup_{\gamma\in\mathcal{G}} |If(\gamma)| \leq \varepsilon$. Under the assumptions of the theorem (only mentioning the closed orbits of length at most $\varepsilon^{-1/2}$), the decomposition $f = Xu + h$ also gives a global control on $\|If\|_{\ell^\infty}$, of the form

$$\|If\|_{\ell^\infty} \leq C\varepsilon^\tau. \tag{2.1.4}$$

Indeed, if one integrates $f = Xu + h$ along a closed orbit of any length, the contribution of Xu vanishes and one is left with a bound $\|h\|_{C^0} \leq C\varepsilon^\tau$. The bound (2.1.4) holds in particular if $If(\gamma) = 0$ for all γ with $\ell(\gamma) \leq \varepsilon^{-1/2}$. This statement illustrates quantitatively the fact that the quantities $If(\gamma)$ for different γ are far from being independent.

Remark 2.1.1. In Theorem 2.1.3, the constants β, C, τ depend on the Anosov flow under consideration, but in a locally uniform way : given an Anosov flow, one can find such parameters that work for any flow in a neighborhood of the initial flow. The local uniformity can be checked either directly from the proof, or using a (Hölder-continuous) orbit-conjugacy between the initial flow and the perturbed one.

Remark 2.1.2. It could be interesting to extend the positive and the finite approximate Livsic theorems to other regularities like H^s spaces for $s > \frac{n+1}{2}$ but we were unable to do so.

2.1.3 X-ray transform for the geodesic flow

If (M, g) is a smooth closed Riemannian manifold, we set $\mathcal{M} := SM$, the unit tangent bundle, and denote by X the geodesic vector field on SM. We will always assume that the geodesic flow is Anosov on SM and we say that (M, g) is an *Anosov Riemannian manifold*. It is a well-known fact that a negatively-curved manifold has Anosov geodesic flow. We will denote by $\mathcal{C}$ the set of free homotopy classes on M : they are in one-to-one correspondence with the set of conjugacy classes of $\pi_1(M)$. If (M, g) is Anosov, we know by [Kli74] that given a free homotopy class $c \in \mathcal{C}$, there exists a unique closed geodesic $\gamma \in \mathcal{G}$ belonging to the free homotopy class c. In other words, $\mathcal{G}$ and $\mathcal{C}$ are in one-to-one correspondence. As a consequence, we will rather see the X-ray transform as a map $I^g : C^0(SM) \to \ell^\infty(\mathcal{C})$ and we will drop the index g if the context is clear.

If $f \in C^\infty(M, \otimes_S^m T^*M)$ is a symmetric tensor, then by Appendix B, we can see f as a function $\pi_m^* f \in C^\infty(SM)$, where $\pi_m^* f(x, v) := f_x(v, ..., v)$. The X-ray transform I_m of f is simply defined by $I_m f := I \circ \pi_m^* f$. In other words, it consists in integrating the tensor f along closed geodesics by plugging m-times the speed vector in f. This map I_m may appear in different contexts. In particular, I_2 is well-known to be the differential of the marked length spectrum and it was studied in [GL19d] to prove its rigidity, thus partially answering the conjecture of Burns-Katok [BK85]. This will be studied in the next chapter.

The natural operator of derivation of symmetric tensors is $D := \sigma \circ \nabla$, where ∇ is the Levi-Civita connection and σ is the operator of symmetrization of tensors (see Appendix B). Any smooth tensor $f \in C^\infty(M, \otimes_S^m T^*M)$ can be uniquely decomposed as $f = Dp + h$, where $p \in C^\infty(M, \otimes_S^{m-1} T^*M)$ and $h \in C^\infty(M, \otimes_S^m T^*M)$ is a *solenoidal tensor* i.e., a tensor such that $D^*h = 0$, where D^* is the formal adjoint of D. We say that Dp is the *potential part* of the tensor f and we have $I_m(Dp) = 0$. In other words, the potential tensors are always in the kernel of the X-ray transform. We will say that I_m is *solenoidal injective* or in short s-injective if injective when restricted to

$$C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M) := C^\infty(M, \otimes_S^m T^*M) \cap \ker(D^*)$$

Note that we will often add an index sol to a functional space on tensors to denote the fact that we are considering the intersection with $\ker D^*$.

It is conjectured that I_m is s-injective for all Anosov Riemannian manifolds, in any dimension and without any assumption on the curvature. This is only known to hold when

1. $m = 0$ or $m = 1$, see [DS03, Theorem 1.1 and 1.3],
2. $m \in \mathbb{N}$ and $\dim(M) = 2$, see [Gui17a, Theorem 1.4],
3. $m \in \mathbb{N}$ and g_0 has non-positive curvature, see [CS98, Theorem 1.3].

The case (2) with $m = 2$ was first proved in [PSU14a, Theorem 1.1]. The s-injectivity of I_m for $m \geq 2$ is an open question in dimension ≥ 3 without any assumption on the curvature. However, it is already known that $C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M) \cap \ker(I_m)$ is finite-dimensional (see also Proposition 2.5.1 for a proof). We refer to Appendix B for further details.

We will prove the following stability estimate on I_m.

Theorem 2.1.4. *Assume I_m is s-injective. Then for all $0 < \beta < \alpha < 1$, there exists $\theta_1 := \theta(\alpha, \beta) > 0$ and $C := C(\alpha, \beta) > 0$ such that : if $f \in C^\alpha_{\mathrm{sol}}(M, \otimes_S^m T^*M)$ is a solenoidal symmetric m-tensor such that $\|f\|_{C^\alpha} \leq 1$, then $\|f\|_{C^\beta} \leq C\|I_m f\|_{\ell^\infty}^{\theta_1}$.*

Actually, if I_m is not known to be injective, one still has the previous estimate by taking f solenoidal and orthogonal to the kernel of I_m. Combining this estimate with Theorem 2.1.3 (and more specifically (2.1.4)), we immediately obtain the following

Theorem 2.1.5. *Assume I_m is s-injective. Then for all $0 < \beta < \alpha < 1$, there exists $\theta_2 := \theta(\alpha, \beta) > 0$ and $C := C(\alpha, \beta) > 0$ such that for any $L > 0$ large enough : if*

$f \in C^\alpha_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is a solenoidal symmetric m-tensor such that $\|f\|_{C^\alpha} \leq 1$, and $I_m f(\gamma) = 0$ for all closed geodesics $\gamma \in \mathcal{C}$ such that $\ell(\gamma) \leq L$, then $\|f\|_{C^\beta} \leq CL^{-\theta_2}$.

Even in the case where $f \in C^\alpha(M)$ is a function on M, this result seemed to be previously unknown.

2.2 Properties of Anosov flows

We refer to the exhaustive [KH95] and the forthcoming book [HF] for an introduction to hyperbolic dynamics.

2.2.1 Classical properties

Stable and unstable manifolds. The *global stable* and *unstable manifolds* $W^s(x), W^u(x)$ are defined by :

$$\begin{aligned} W^s(x) &= \{x' \in \mathcal{M}, d(\varphi_t(x), \varphi_t(x')) \to_{t\to+\infty} 0\} \\ W^u(x) &= \{x' \in \mathcal{M}, d(\varphi_t(x), \varphi_t(x')) \to_{t\to-\infty} 0\} \end{aligned}$$

For $\varepsilon > 0$ small enough, we define the *local stable* and *unstable manifolds* $W^s_\varepsilon(x) \subset W^s(x), W^u_\varepsilon(x) \subset W^u(x)$ by :

$$\begin{aligned} W^s_\varepsilon(x) &= \{x' \in W^s(x), \forall t \geq 0, d(\varphi_t(x), \varphi_t(x')) \leq \varepsilon\} \\ W^u_\varepsilon(x) &= \{x' \in W^u(z), \forall t \geq 0, d(\varphi_{-t}(x), \varphi_{-t}(x')) \leq \varepsilon\} \end{aligned}$$

For all $\varepsilon > 0$ small enough, there exists $t_0 > 0$ such that :

$$\forall x \in \mathcal{M}, \forall t \geq t_0, \varphi_t(W^s_\varepsilon(x)) \subset W^s_\varepsilon(\varphi_t(x)), \varphi_{-t}(W^u_\varepsilon(x)) \subset W^u_\varepsilon(\varphi_{-t}(x)) \qquad (2.2.1)$$

And :

$$T_x W^s_\varepsilon(x) = E_s(x), \quad T_x W^u_\varepsilon(x) = E_u(x)$$

Specification lemma. The main tool we will use to construct suitable periodic orbits is the following classical shadowing property of Anosov flows. Part of the proof can be found in [KH95, Corollary 18.1.8] and [HF, Theorem 5.3.2]. The last bound is a consequence of hyperbolicity and can be found in [HF, Proposition 6.2.4]. For the sake of simplicity, we will write $\gamma = [xy]$ if γ is an orbit segment with endpoints x and y.

Theorem 2.2.1. *There exist $\varepsilon_0 > 0$, $\theta > 0$ and $C > 0$ with the following property. Consider $\varepsilon < \varepsilon_0$, and a finite or infinite sequence of orbit segments $\gamma_i = [x_i y_i]$ of length T_i greater than 1 such that for any n, $d(y_n, x_{n+1}) \leq \varepsilon$. Then there exists a genuine orbit γ and times τ_i such that γ restricted to $[\tau_i, \tau_i + T_i]$ shadows γ_i up to $C\varepsilon$. More precisely, for all $t \in [0, T_i]$, one has*

$$d(\gamma(\tau_i + t), \gamma_i(t)) \leq C\varepsilon e^{-\theta \min(t, T_i - t)}. \qquad (2.2.2)$$

Moreover, $|\tau_{i+1} - (\tau_i + T_i)| \leq C\varepsilon$. Finally, if the sequence of orbit segments γ_i is periodic, then the orbit γ is periodic.

Remark 2.2.1. In this theorem, we could also allow the first orbit segment γ_i to be infinite on the left, and the last orbit segment γ_j to be infinite on the right. In this case, (2.2.2) should be replaced by its obvious reformulation : assuming that γ_i is defined on $(-\infty, 0]$ and γ_j on $[0, +\infty)$, we would get for some $\tilde{\tau}_{i+1}$ within $C\varepsilon$ of τ_{i+1}, and all $t \geq 0$

$$d(\gamma(\tilde{\tau}_{i+1} - t), \gamma_i(-t)) \leq C\varepsilon e^{-\theta t} \tag{2.2.3}$$

and

$$d(\gamma(\tau_j + t), \gamma_j(t)) \leq C\varepsilon e^{-\theta t}.$$

In particular, if γ_0 is an orbit segment $[xy]$ with $d(y, x) \leq \varepsilon_0$, then applying the above theorem to $\gamma_i := \gamma_0$ for all $i \in \mathbb{Z}$, one gets a periodic orbit that shadows γ_0 : this is the *Anosov closing lemma*. We will also use thoroughly the version with two orbit segments that are repeated to get a periodic orbit.

Cover by parallelepipeds. We will now fix ε_0 small enough so that the previous propositions are guaranteed. For $\varepsilon \leq \varepsilon_0$, we define the set $W_\varepsilon(x) := \bigcup_{y \in W^u_\varepsilon(x)} W^s_\varepsilon(x)$. We can cover the manifold $\mathcal{M}$ by a finite union of flowboxes $\mathcal{U}_i := \cup_{t \in (-\delta,\delta)} \varphi_t(\Sigma_i)$, where $\Sigma_i := W_{\varepsilon_0}(x_i)$ and $x_i \in \mathcal{M}$.

We denote by $\pi_i : \mathcal{U}_i \to \Sigma_i$ the projection by the flow on the transverse section and we define $\mathfrak{t}_i : \mathcal{U}_i \to \mathbb{R}$ such that $\pi_i(x) = \varphi_{\mathfrak{t}_i(x)}(x)$ for $x \in \mathcal{U}_i$. We will need the following lemma :

Lemma 2.2.1. *$\pi_i, \mathfrak{t}_i$ are Hölder-continuous.*

Proof. This is actually a general fact related to the Hölder regularity of the foliation and the smoothness of the flow.

For the sake of simplicity, we drop the index i in this proof. Let us first prove the Hölder continuity for x close to Σ and x' close to x. We fix $p \in \Sigma$ and choose smooth local coordinates $\psi : B(p, \eta) \to \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$ around p (and centered at 0) so that $d\psi_p(X) = \partial_{x_0}$. This choice guarantees that in a neighborhood of 0, the flow is transverse to the hyperplane $\{0\} \times \mathbb{R}^{n_s+n_u}$. We still denote by Σ_η its image $\psi(\Sigma_\eta) \subset \mathbb{R}^{n+1}$, which is a submanifold of Hölder regularity (the index η indicates that we consider the same objects intersected with the ball $B(x, \eta)$). Moreover, there exists a Hölder-continuous homeomorphism $\Phi : S \to \Sigma_\eta$, where $S \subset \{0\} \times \mathbb{R}^{n_s+n_u}$ (since Σ_η is a submanifold of M with Hölder regularity). We consider $\hat{\varphi} : (-\delta, \delta) \times S \to \varphi_{(-\delta,\delta)}(S) =: V \supset \Sigma_\eta$ defined by $\hat{\varphi}(t, z) = \varphi_t(0, z)$, which is a smooth diffeomorphism. Remark that $\mathfrak{t}$ satisfies for $(0, z) \in S$, $(\mathfrak{t}(z), z) = \hat{\varphi}^{-1}(\Phi(z))$. So it is Hölder-continuous on S. Then $z \mapsto \pi(0, z) = \varphi_{\mathfrak{t}(z)}(0, z)$ is Hölder-continuous on S too.

We denote by $\pi_S : V \to S$ the projection and by $\mathfrak{t}_S : V \to S$ the time such that $\pi_S(x) = \varphi_{\mathfrak{t}_S(x)}(x)$. These two maps are smooth by the implicit function theorem since the flow is transverse to S. Moreover, we have : $\pi(x) = \pi|_S(\pi_S(x))$ so π is Hölder-continuous. And $\mathfrak{t}(x) = \mathfrak{t}_S(x) + \mathfrak{t}|_S(\pi_S(x))$ so $\mathfrak{t}$ is Hölder-continuous too. Note that by compactness of Σ, this procedure can be done with only a finite number of charts, thus ensuring the uniformity of the constants. Thus, $\pi_i, \mathfrak{t}_i$ are Hölder-continuous in a neighborhood of Σ. Now, in order to obtain the continuity on the whole cube $\mathcal{U}$, one can repeat the same argument i.e., write the projection as the composition of a first projection on a smooth small section S defined in a neighborhood of Σ with the actual projection on Σ. This provides the sought result.

□

2.2.2 Proof of the usual Livsic theorem in Hölder regularity

This is a classical theorem in hyperbolic dynamics, but we provide the proof for the reader's convenience. We also refer to the proof of Guillemin-Kazhdan [GK80a, Appendix] and to [KH95, Theorem 19.2.4]. The idea is to define u as the integral of f over a dense orbit in the manifold and then to compute the Hölder regularity.

Proof of Theorem 2.1.1. We consider a point x_0 whose orbit $\mathcal{O}(x_0)$ is dense in $\mathcal{M}$ and we define $u(\varphi_t x_0) := \int_0^t f(\varphi_s x_0) ds$ (remark that $Xu = f$ on $\mathcal{O}(x_0)$ by construction). Let us prove that u is C^α on $\mathcal{O}(x_0)$. We pick $x, y \in \mathcal{O}(x_0)$ such that $d(x,y) < \varepsilon_0$ (in particular, the Anosov closing lemma is satisfied at this scale). We write $x = \varphi_t x_0, y = \varphi_{t+T} x_0$ and we assume that $T \geq 1$ which is always possible since the orbit is dense. Let p be the periodic point of period $T + \tau$ (with $|\tau| \leq Cd(x,y)$) closing the segment of orbit $[xy]$. We have :

$$u(x) - u(y) = \int_0^T f(\varphi_s x) ds = \underbrace{\int_0^T f(\varphi_s x) - f(\varphi_s p) ds}_{=\text{(I)}} + \underbrace{\int_0^{T+\tau} f(\varphi_s p) ds}_{=\text{(II)}} - \underbrace{\int_T^{T+\tau} f(\varphi_s p) ds}_{=\text{(III)}}$$

And :

$$|\text{(I)}| \leq \int_0^T \|f\|_{C^\alpha} d(\varphi_s x, \varphi_s p)^\alpha ds \leq C \|f\|_{C^\alpha} d(x,y)^\alpha \int_0^T e^{-\alpha\theta \min(s, T-s)} ds \lesssim d(x,y)^\alpha$$

By hypothesis, we know that (II) $= 0$. And $|\text{(III)}| \leq \|f\|_\infty |\tau| \lesssim d(x,y)$. As a consequence, u is C^α on $\mathcal{O}(x_0)$ (and its C^α norm is controlled by that of f). Since $\mathcal{O}(x_0)$ is dense in M, u admits a unique C^α-extension to M and it satisfies $Xu = f$. □

2.3 Proof of the approximate Livsic Theorem

The following proof of Theorem 2.1.3 was communicated to us by Sébastien Gouëzel.

2.3.1 A key lemma.

The following lemma states that we can find a sufficiently dense and yet separated orbit in the manifold $\mathcal{M}$. The separation can only hold transversally to the flow direction, and is defined as follows. Recall that $W_\varepsilon(x) := \bigcup_{y \in W^u_\varepsilon(x)} W^s_\varepsilon(x)$. Then we say that a set S is *ε-transversally separated* if, for any $x \in S$, we have $S \cap W_\varepsilon(x) = \{x\}$.

Lemma 2.3.1. *Consider a transitive Anosov flow on a compact manifold. There exist $\beta_s, \beta_d > 0$ such that the following holds. Let $\varepsilon > 0$ be small enough. There exists a periodic orbit $\mathcal{O}(x_0) := (\varphi_t x_0)_{0 \leq t \leq T}$ with $T \leq \varepsilon^{-1/2}$ such that this orbit is ε^{β_s}-transversally separated and $(\varphi_t x_0)_{0 \leq t \leq T-1}$ is ε^{β_d}-dense. If $\kappa > 0$ is some fixed constant, then one can also require that there exists a piece of $\mathcal{O}(x_0)$ of length $\leq C(\kappa)$ which is κ-dense in the manifold.*

Proof. Let us fix two periodic points p_1 and p_2 with different orbits $\mathcal{O}(p_1)$ and $\mathcal{O}(p_2)$ of respective lengths ℓ_1 and ℓ_2. By the shadowing theorem and transitivity, there exists an orbit γ_- which is asymptotic to $\mathcal{O}(p_1)$ in negative time and to $\mathcal{O}(p_2)$ in positive time.

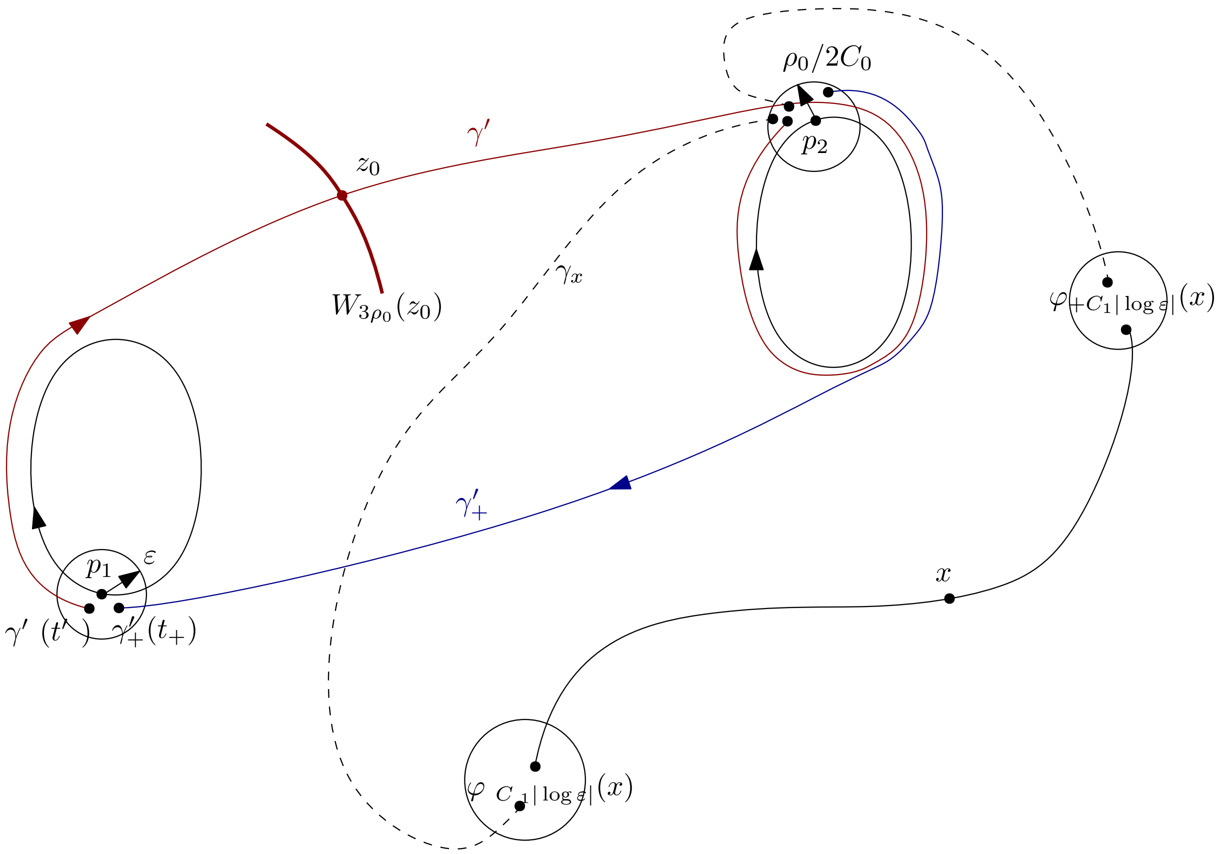

FIGURE 2.2 – γ'_- joins p_1 to p_2, γ'_+ joins p_2 to p_1 and γ_x has size $6C_1|\log\varepsilon| + O(1)$.

Also, there exists an orbit γ_+ which is asymptotic to $\mathcal{O}(p_2)$ in negative time and to $\mathcal{O}(p_1)$ in positive time. On γ_-, fix a point z_0, and $\rho_0 > 0$ small enough so that $\gamma_- \cup \gamma_+$ meets $W_{3\rho_0}(z_0)$ only at z_0, and $\mathcal{O}(p_1)$ and $\mathcal{O}(p_2)$ are at distance $> 3\rho_0$ of z_0. Denote by C_0 the constant C in the Shadowing Theorem 2.2.1. Reducing ρ_0 if necessary, we can assume $\rho_0 < \varepsilon_0$ where ε_0 is given by Theorem 2.2.1. Let us also fix a large constant C_1, on which our construction will depend.

We truncate γ_- in positive time, stopping it at a fixed time where it is within distance $\rho_0/(2C_0)$ of p_2, to get an orbit γ'_-. Let t_- be the largest time in $(-\infty, -2C_1|\log\varepsilon|]$ where $\gamma'_-(t)$ is within distance ε of p_1. As the orbit γ'_- converges exponentially quickly in negative time to $\mathcal{O}(p_1)$ by hyperbolicity, one has $d(\gamma'_-(t), \mathcal{O}(p_1)) \leq \varepsilon$ for $t \leq -2C_1|\log\varepsilon|$, if C_1 is large enough. Hence, one needs to wait at most ℓ_1 before being ε-close to p_1. This shows that the time t_- satisfies $t_- = -2C_1|\log\varepsilon| + O(1)$.

In the same way, we truncate γ_+ in negative time at a fixed time for which it is within distance $\rho_0/(2C_0)$ of p_2, obtaining an orbit γ'_+. We denote by t_+ the smallest time in $[2C_1|\log\varepsilon|, +\infty)$ with $d(\gamma'_+(t), p_1) \leq \varepsilon$. It satisfies $t_+ = 2C_1|\log\varepsilon| + O(1)$.

As the flow is transitive, it has a dense orbit. Therefore, for any x, y, there exists an orbit $\gamma_{x,y}$ starting from a point within distance $\rho_0/(2C_0)$ of x, ending at a point within distance $\rho_0/(2C_0)$ of y, and with length $\in [1, T_0]$ where T_0 is fixed and independent of x and y.

To any x, we associate an orbit as follows. Start with γ'_-, then follow $\gamma_{p_2,\varphi_{-C_1|\log\varepsilon|}x}$, then follow the orbit of x between times $-C_1|\log\varepsilon|$ and $C_1|\log\varepsilon|$, then follow $\gamma_{\varphi_{C_1|\log\varepsilon|}x,p_2}$, then follow γ'_+. In this sequence, the distance between an endpoint of a piece and the starting point of the next one is always less than ρ_0/C_0. Hence, Theorem 2.2.1 applies and yields an infinite orbit γ'_x, that follows the above pieces of orbits up to ρ_0. If C_1 is large enough, (2.2.2) implies that x is within distance at most ε of γ'_x. The inequality (2.2.3) shows that $\gamma'_-(t_-)$ and the corresponding point x_- on γ'_x are within distance $e^{-\theta t_-}$. If C_1 is large enough, this is bounded by ε since $t_- = -2C_1|\log \varepsilon| + O(1)$. Therefore, $d(x_-, p_1) \le 2\varepsilon$. In the same way, the point x_+ on γ'_x matching $\gamma'_+(t_+)$ is within distance ε of $\gamma'_+(t_+)$, and therefore within distance 2ε of p_1. Let us truncate γ'_x between x_- and x_+, to get an orbit segment γ_x of length $6C_1|\log \varepsilon| + O(1)$, starting and ending within 2ε of p_1, and passing within ε of x.

Let $\beta_d = 1/(3\dim(\mathcal{M}))$. We define a sequence of points of $\mathcal{M}$ as follows. Let x_1 be an arbitrary point for which the $C(\kappa)$-beginning of its orbit is $\kappa/2$-dense, to guarantee in the end that the last condition of the lemma is satisfied. If γ_{x_1} is not $\varepsilon^{\beta_d}/2$-dense, we choose another point x_2 which is not in the $\varepsilon^{\beta_d}/2$-neighborhood of γ_{x_1}. Then $\gamma_{x_1} \cup \gamma_{x_2}$ contain both x_1 and x_2 in their ε-neighborhood, and therefore in their $\varepsilon^{\beta_d}/2$-neighborhood. If $\gamma_{x_1} \cup \gamma_{x_2}$ is still not $\varepsilon^{\beta_d}/2$-dense, then we add a third piece of orbit γ_{x_3}, and so on. By compactness, this process stops after finitely many steps, giving a finite sequence $x_1, \dots, x_N$.

As all γ_{x_i} start and end with p_1 up to 2ε, we can glue the sequence

$$\dots, \gamma_{x_N}, \gamma_{x_1}, \gamma_{x_2}, \dots, \gamma_{x_N}, \gamma_{x_1}, \dots$$

thanks to Theorem 2.2.1. We get a periodic orbit γ which shadows them up to $2C_0\varepsilon$. We claim this orbit satisfies the requirements of the lemma. We should check its length, its density, and its separation.

Let us start with the length. The points x_i are separated by at least $\varepsilon^{\beta_d}/3$. The balls of radius $\varepsilon^{\beta_d}/6$ are disjoint, and each has a volume $\ge c\varepsilon^{\beta_d \cdot \dim(\mathcal{M})} = c\varepsilon^{1/3}$. We get that the number N of points x_i is bounded by $C\varepsilon^{-1/3}$. As each piece γ_{x_i} has length at most $C|\log \varepsilon|$, it follows that the total length of γ is bounded by $C|\log \varepsilon|\varepsilon^{-1/3} \le \varepsilon^{-1/2}$.

Let us check the density. By construction, the union of the γ_{x_i} is $\varepsilon^{\beta_d}/2$-dense. As γ approximates each γ_{x_i} within $2C_0\varepsilon$, it follows that γ is $2C_0\varepsilon + \varepsilon^{\beta_d}/2$ dense, and therefore ε^{β_d}-dense. In the statement of the lemma, we require the slightly stronger statement that if one removes a length 1 piece at the end of the orbit it remains ε^{β_d}-dense. Such a length 1 piece in γ_{x_N} consists of points that are within 2ε of $\mathcal{O}(p_1)$. They are approximated within ε^{β_d} by the start and end of all the other γ_{x_i}.

Finally, let us check the more delicate separation, which has motivated the finer details of the construction as we will see now. Let β_s be suitably large. We want to show that any two points x, y of γ within distance ε^{β_s} are on the same local flow line. Since the expansion of the flow is at most exponential, for any $t \le 20C_1|\log \varepsilon|$, we have $d(\varphi_t x, \varphi_t y) \le \varepsilon$ if β_s is large enough. In the piece of γ of length $10C_1|\log \varepsilon|$ starting at x, there is an interval $[t_1, t_2]$ of length $4C_1|\log \varepsilon| + O(1)$ during which $\varphi_t x$ is within distance at most $\rho_0/2$ of $\mathcal{O}(p_1)$, corresponding to the junction between the orbits γ_{x_i} and $\gamma_{x_{i+1}}$ where i is such that x belongs to the shadow of $\gamma_{x_{i-1}}$. For $t \in [t_1, t_2]$, one also has $d(\varphi_t y, \mathcal{O}(p_1)) \le \rho_0$ as the orbits follow each other up to ε. Note that in each γ_j the consecutive time spent close to $\mathcal{O}(p_1)$ is bounded by $2C_1|\log \varepsilon|$ as we have forced a passage close to p_2 (and therefore far away from $\mathcal{O}(p_1)$)) after this time in the construction. It follows that also for y the time interval $[t_1, t_2]$ has to correspond to a junction between two orbits γ_{x_j} and $\gamma_{x_{j+1}}$. Consider the smallest times t and t' after the junctions for which $\varphi_t(x)$ and $\varphi_{t'}(y)$ are $2\rho_0$-close to z_0. Since the orbit γ'_- meets $W_{3\rho_0}(z_0)$ at the single point z_0, these times have to correspond to each other, i.e., the orbits are synchronized up to an error $O(\varepsilon)$. To conclude, it remains to show that $i = j$. Suppose by contradiction $i < j$ for instance. The orbit of x follows γ_{x_i} up to $2C_0\varepsilon$, the orbit of y follows γ_{x_j} up to $2C_0\varepsilon$, and the orbits of x and y are within ε of each other. We deduce that γ_{x_i} and γ_{x_j} follow each other up to $(4C_0+1)\varepsilon$. Since x_j is within ε of γ_{x_j}, it follows that x_j is at within $(4C_0+2)\varepsilon$ of γ_{x_i}. This is a contradiction with the construction, as we could have added the point x_j only if it was not in the ε^{β_d}-neighborhood of γ_{x_i}, and $\varepsilon^{\beta_d} > (4C_0+2)\varepsilon$ if ε is small enough. □

2.3.2 Construction of the approximate coboundary.

Let us now prove the approximate Livsic Theorem that is Theorem 2.1.3. The result is obvious if ε is bounded away from 0, by taking $u=0$ and $h=f$. Hence, we can assume that ε is small enough to apply Lemma 2.3.1, with $\kappa=\varepsilon_0$. On the orbit $\mathcal{O}(x_0)$ given by this lemma, we define a function $\tilde{u}$ by $\tilde{u}(\varphi_t x_0) = \int_0^t f(\varphi_s x_0)ds$. Note that it may not be continuous at x_0. As a consequence, we will rather denote by $\mathcal{O}(x_0)$ the set $(\varphi_t x_0)_{0\leq t\leq T-1}$ (which satisfies the required properties of density and transversality) in order to avoid problems of discontinuity.

Lemma 2.3.2. *There exist $\beta_1, C>0$ independent of ε such that $\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))}\leq C$.*

Proof. We first study the Hölder regularity of $\tilde{u}$, namely we want to control $|\tilde{u}(x)-\tilde{u}(y)|$ by $Cd(x,y)^{\beta_1}$ for some well-chosen exponent β_1, when $d(x,y)\leq\varepsilon_0$ (where ε_0 is the scale under which the Shadowing Theorem 2.2.1 holds). If x and y are on the same local flow line, then the result is obvious since f is bounded by 1, so we are left to prove that $\tilde{u}$ is transversally C^{β_1}. Consider $x=\varphi_{t_0}x_0\in\mathcal{O}(x_0)$ and $y=\varphi_{t_0+t}\in W_{\varepsilon_0}(x)$. By transversal separation of $\mathcal{O}(x_0)$, these points satisfy $d(x,y)\geq\varepsilon^{\beta_s}$. We can close the segment $[xy]$ i.e., we can find a periodic point p such that $d(p,x)\leq Cd(x,y)$ with period $t_p=t+\tau$, where $|\tau|\leq Cd(x,y)$ which shadows the segment. Then :

$$|\tilde{u}(y)-\tilde{u}(x)| \leq \underbrace{\left|\int_0^t f(\varphi_s x)ds - \int_0^{t_p} f(\varphi_s p)ds\right|}_{=(\mathrm{I})} + \underbrace{\left|\int_0^{t_p} f(\varphi_s p)ds\right|}_{=(\mathrm{II})}$$

The first term (I) is bounded by $Cd(x,y)^{\beta_1'}$ for some $\beta_1'>0$ depending on the dynamics, whereas the second term (II) is bounded — by assumption — by εt_p. But $\varepsilon t_p \lesssim \varepsilon t \lesssim \varepsilon T \lesssim \varepsilon^{1/2} \lesssim d(x,y)^{1/2\beta_s}$. We thus obtain the sought result with $\beta_1 := \min(\beta_1', 1/2\beta_s)$.

We now prove that $\tilde{u}$ is bounded for the C^0-norm. We know that there exists a segment of the orbit $\mathcal{O}(x_0)$ — call it S — of length $\leq C$ which is ε_0-dense in $\mathcal{M}$. In particular, for any $x\in\mathcal{O}(x_0)$, there exists $x_S\in S$ with $d(x,x_S)\leq\varepsilon_0$, and therefore $|\tilde{u}(x)-\tilde{u}(x_S)|\leq Cd(x,x_S)^{\beta_1}\leq C\varepsilon_0^{\beta_1}$ thanks to the Hölder control of the previous paragraph. Using the same argument with x_0, we get as $\tilde{u}(x_0)=0$

$$|\tilde{u}(x)| = |\tilde{u}(x)-\tilde{u}(x_0)| \leq |\tilde{u}(x)-\tilde{u}(x_S)| + |\tilde{u}(x_S)-\tilde{u}((x_0)_S)| + |\tilde{u}(x_0)-\tilde{u}((x_0)_S)|.$$

The first and last term are bounded by $C\varepsilon_0^{\beta_1}$, and the middle one is bounded by C as S has a bounded length and $\|f\|_{C^0}\leq 1$. □

For each i, we extend the function $\tilde{u}$ (defined on $\mathcal{O}(x_0)$) to a Hölder function u_i on Σ_i, by the formula $u_i(x)=\sup\tilde{u}(y)-\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))}d(x,y)^{\beta_1}$, where the supremum is taken over all $y\in\mathcal{O}(x_0)$. With this formula, it is classical that the extension is Hölder continuous, with $\|u_i\|_{C^{\beta_1}(\Sigma_i)}\leq\|\tilde{u}\|_{C^{\beta_1}(\mathcal{O}(x_0))}$. We then push the function u_i by the flow in order to define it on $\mathcal{U}_i$ by setting for $x\in\Sigma_i, \varphi_t x\in\mathcal{U}_i : u_i(\varphi_t x) = u_i(x)+\int_0^t f(\varphi_s x)ds$. Note that by Lemma 2.2.1, the extension is still Hölder with the same regularity. We now set $u := \sum_i u_i\theta_i$ and $h := f - Xu = -\sum_i u_i X\theta_i$. The functions $X\theta_i$ are uniformly bounded in C^∞, independently of ε so the functions $u_i X\theta_i$ are in C^{β_1} with a Hölder norm independent of $\varepsilon > 0$ and thus $\|h\|_{C^{\beta_1}} \leq C$.

Lemma 2.3.3. $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}$

Proof. We claim that h vanishes on $\mathcal{O}(x_0)$: indeed, on $\mathcal{U}_i \cap \mathcal{O}(x_0)$ one has $u_i \equiv \tilde{u}$ and thus $h = -\tilde{u}\sum_i X\theta_i = -\tilde{u}X\sum_i \theta_i = -\tilde{u}X\mathbf{1} = 0$. Since $\mathcal{O}(x_0)$ is ε^{β_d}-dense and $\|h\|_{C^{\beta_1}} \leq C$, we get that $\|h\|_{C^0} \leq C\varepsilon^{\beta_1\beta_d} = C\varepsilon^{\beta_3}$, where $\beta_3 = \beta_1\beta_d$. By interpolation, we eventually obtain that $\|h\|_{C^{\beta_1/2}} \leq \varepsilon^{\beta_3/2}$. □

Proof of Theorem 2.1.3. The previous lemma provides the sought estimate on the remainder h. This completes the proof of Theorem 2.1.3. □

2.4 Resolvent of the flow at 0

From now on, we will rather use the dual decomposition of the cotangent space $T^*\mathcal{M} = E_0^* \oplus E_u^* \oplus E_s^*$, where $E_0^*(E_u \oplus E_s) = 0, E_s^*(E_s \oplus \mathbb{R}X) = 0, E_u^*(E_u \oplus \mathbb{R}X) = 0$. If $A^{-\top}$ denotes the inverse transpose of a linear operator A, then the dual estimates to (2.1.2) are :

$$\begin{array}{ll} |d\varphi_t^{-\top}(x) \cdot \xi|_{\varphi_t(x)} \leq Ce^{-\lambda t}|\xi|_x, & \forall t > 0, \xi \in E_s^*(x) \\ |d\varphi_t(x) \cdot \xi|_{\varphi_t(x)} \leq Ce^{-\lambda|t|}|\xi|_x, & \forall t < 0, \xi \in E_u^*(x), \end{array} \tag{2.4.1}$$

where $|\cdot|_x$ is now g^{-1}, the dual metric to g (which makes the musical isomorphism $\flat : T\mathcal{M} \to T^*\mathcal{M}$ an isometry).

2.4.1 Meromorphic extension of the resolvent

Consider any smooth measure μ on $\mathcal{M}$. The unbounded operator $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \subset L^2(\mathcal{M}, d\mu) =: L^2(\mathcal{M})$ is a differential operator of order 1 and thus admits a unique closed extension on $L^2(\mathcal{M})$ (see [FS11, Lemma 29] for instance) with domain $\mathcal{D}_{L^2}(X) = \{u \in L^2(\mathcal{M}), Xu \in L^2(\mathcal{M})\}$. When the flow is *not selfadjoint*, the semigroup $e^{tX} : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is continuous but not unitary. As a consequence, there exist constants $C_0, \omega > 0$ such that $\|e^{tX}\|_{\mathcal{L}(L^2,L^2)} \leq C_0e^{\omega t}$. If X preserves the smooth measure μ, $-iX$ is selfadjoint on $\mathcal{D}_{L^2}(X)$. The following theorem gives a satisfactory spectral description of the operator X.

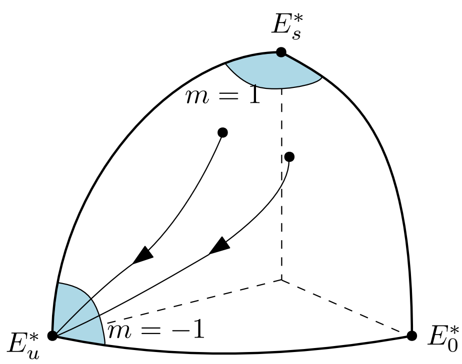

FIGURE 2.3 – The projective flow on the unit cosphere $S^*\mathcal{M}$.

Theorem 2.4.1 (Faure-Sjöstrand)**.** *There exists a constant $c > 0$ such that for any $s > 0$, there exists a Hilbert space $\mathcal{H}_+^s$, such that on the half-space $\{\Re(\lambda) > -cs\}$, $X + \lambda : \mathcal{D}_{\mathcal{H}_+^s}(X) \to \mathcal{H}_+^s(\mathcal{M})$ is an analytic family of unbounded operators with domain $\mathcal{D}_{\mathcal{H}_+^s}(X) = \{u \in \mathcal{H}_+^s(\mathcal{M}), Xu \in \mathcal{H}_+^s(\mathcal{M})\}$ which are Fredholm of index 0*[1]*. Moreover, $X + \lambda$ is invertible for $\Re(\lambda) \gg 0$ large enough. As a consequence, the operator has discrete discrete spectrum in the half-space $\{\Re(\lambda) > \omega - cs\}$.*

1. $\mathcal{D}_{\mathcal{H}_+^s}(X)$ equipped with the norm $\|u\|_{\mathcal{D}_{\mathcal{H}_+^s}} := \|u\|_{\mathcal{H}_+^s} + \|Xu\|_{\mathcal{H}_+^s}$ is a Hilbert space and $X + \lambda$ becomes bounded when the vector spaces are equipped with this norm, so it makes sense to talk about a Fredholm operator.

We recall that the principal symbol of $P = -iX$ is $\sigma_P : (x,\xi) \mapsto \langle \xi, X(x)\rangle$. We denote by $\mathbf{X}$ the Hamiltonian vector field on the symplectic manifold $T^*\mathcal{M}$ induced by the Hamiltonian σ_P and by $(\Phi_t)_{t\in\mathbb{R}}$ the symplectic flow generated. A quick computation shows that $\Phi_t = (\varphi_t, d\varphi_t^{-\top})$. Note that since $(\Phi_t)_{t\in\mathbb{R}}$ is 1-homogeneous in the ξ variable, it induces a flow $(\Phi_t^{(1)})_{t\in\mathbb{R}}$ on the unit sphere $S^*\mathcal{M}$. If $\kappa : T^*\mathcal{M} \to S^*\mathcal{M}$ denotes the canonical projection, then $\kappa(E_s^*)$ is a hyperbolic repeller/source and $\kappa(E_u^*)$ is a hyperbolic attractor/sink for the dynamics of $(\Phi_t^{(1)})_{t\in\mathbb{R}}$ (see Figure 2.3) in the sense of Definition A.4.1. The following lemma asserts the existence of an *escape function* which will be the crucial tool in the proof of the meromorphic extension of the resolvent $(X+\lambda)^{-1}$.

Lemma 2.4.1 (Faure-Sjöstrand)**.** *There exists a* 0*-homogenous* order function $m \in C^\infty(T^*\mathcal{M} \setminus \{0\}, [-1,1])$ *such that* $\mathbf{X}\cdot m \leq 0$*,* $m \equiv 1$ *in a conic neighborhood of* E_s^**,* $m \equiv -1$ *in a conic neighborhood of* E_u^* *and there exists an* escape function $G_m \in S^0_{\rho,1-\rho}(T^*\mathcal{M})$*, for all* $\rho < 1$*, constructed from* m*, such that :*

- *There exist constants* $C_1, R > 0$ *such that on* $|\xi| \geq R$ *intersected with a conic neighborhood of* $\Sigma := E_s^* \oplus E_u^*$*, one has* $\mathbf{X}\cdot G_m \leq -C_1 < 0$*.*
- *For* $|\xi| \geq R$*,* $\mathbf{X}\cdot G_m \leq C_2$ *for some constant* $C_2 > 0$*.*

An important remark is that $G_m \in S^0_{\rho,1-\rho}$ and $e^{G_m} \in S^m_{\rho,1-\rho}$ for any $\rho < 1$ (these are the anisotropic classes introduced in Appendix A) and we will sometimes write this as S^{m+}. In other words, G_m narrowly misses the usual class $S^0_{1,0}$. This will not be a problem when working in Sobolev regularity (that is when working with spaces from from L^2) but may (and actually will) induce complications when using other spaces like Hölder-Zygmund spaces. More precisely, e^{G_m} satisfies the following symbolic estimates in coordinates :

$$\forall (x,\xi) \in T^*\mathcal{M}, \qquad |\partial_\xi^\alpha \partial_x^\beta e^{G_m}(x,\xi)| \leq C_{\alpha,\beta} (\log\langle\xi\rangle)^{|\alpha|+|\beta|} \langle\xi\rangle^{m(x,\xi)-|\alpha|},$$

where $\alpha, \beta \in \mathbb{N}^{n+1}$.

Proof of Theorem 2.4.1. The computation rules of symbols in anisotropic classes enjoy the same properties (composition rules, ellipticity, etc.) as symbols in the usual classes (see [FRS08]) ; we leave it as an exercise to the reader to check that all the symbols and pseudodifferential operators are in the right anisotropic classes.

We consider a cutoff function $\chi \in C_c^\infty([0,+\infty))$ such that $\chi \equiv 1$ on $[0,1/2]$ and $\chi \equiv 0$ outside $[0,1]$. We then define for $T > 0$ the function $\chi_T(t) := \chi(t/T)$. We have :

$$(X+\lambda)\int_0^{+\infty} \chi_T(t) e^{-t(X+\lambda)} \mathrm{d}t = \mathbb{1} + \int_0^{+\infty} \chi_T'(t) e^{-t(X+\lambda)} \mathrm{d}t$$

Note that the integral on the right-hand side is actually performed for $t \in [0,T]$, that is on a finite time interval, as will be all the integrals in the following. Let $P := \mathrm{Op}(p)$, where $p \in S^0(T^*M)$ and $p \equiv 1$ in a conic neighborhood of $\Sigma := E_s^* \oplus E_u^*$ and $p \equiv 0$ outside this conic neighborhood. We define $A_s := \mathrm{Op}(e^{sG_m}) \in \Psi_h^{sm+}(M)$, where $s > 0$ is some fixed number. Up to a lower order modification, we can assume that A_s is invertible. We introduce $H + \lambda := A_s(X+\lambda){A_s}^{-1}$. Then :

$$(H+\lambda)\underbrace{A_s\int_0^{+\infty} \chi_T(t) e^{-t(X+\lambda)} {A_s}^{-1} \mathrm{d}t}_{:=Q(\lambda)} = \mathbb{1} + \underbrace{A_s\int_0^{+\infty} \chi_T'(t) e^{-t(X+\lambda)} \mathrm{d}t {A_s}^{-1}}_{:=R(\lambda)} \tag{2.4.2}$$

Note that $\|R(\lambda)\|_{\mathcal{L}(L^2,L^2)} = O(\langle\Re(\lambda)\rangle^{-\infty})$ for $\Re(\lambda) \gg 0$. In particular, for $\Re(\lambda) \gg 0$, $\mathbb{1} + R(\lambda)$ is invertible on L^2.

Then, we write :

$$R(\lambda) = A_s \int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t P {A_s}^{-1} + A_s \int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P){A_s}^{-1} \quad (2.4.3)$$

By elementary wavefront set arguments (see Example A.3.1) we have that

$$\int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P) \in \Psi^{-\infty}$$

As a consequence

$$\mathbb{C} \ni \lambda \mapsto A_s \int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t(\mathbb{1}-P){A_s}^{-1} \in \Psi^{-\infty}$$

is a holomorphic family of compact operators on L^2. Then, we deal with the first term in (2.4.3). First, notice that by Egorov's Theorem (see [Zwo12, Theorem 11.1])

$$e^{tX}A_s e^{-tX} = e^{tX}\,\mathrm{Op}(e^{sG_m})e^{-tX} = \mathrm{Op}(e^{sG_m\circ\Phi_t}) + K_t,$$

where $e^{sG_m\circ\Phi_t} \in S^{sm\circ\Phi_t+}$ and thus

$$\mathrm{Op}(e^{sG_m\circ\Phi_t}) \in \Psi^{sm\circ\Phi_t+}, \qquad K_t \in \Psi^{sm\circ\Phi_t-1+}$$

Thus :

$$\begin{aligned}
A_s \int_0^{+\infty} \chi_T'(t)e^{-t(X+\lambda)}\mathrm{d}t P{A_s}^{-1} &= \int_0^{+\infty} \chi_T'(t)e^{-t\lambda}A_s e^{-tX}P{A_s}^{-1}\mathrm{d}t \\
&= \int_0^{+\infty} \chi_T'(t)e^{-t\lambda}e^{-tX}e^{tX}A_s e^{-tX}P{A_s}^{-1}\mathrm{d}t \\
&= \int_0^{+\infty} \chi_T'(t)e^{-t\lambda}e^{-tX}\left(\mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}p) + K_t'P{A_s}^{-1}\right)\mathrm{d}t
\end{aligned}$$

But on the support of p, we have $\mathbf{X}\cdot m \le 0$, so

$$e^{G_m\circ\Phi_t-G_m}p \in S^{m\circ\Phi_t-m}_{\rho,1-\rho} \subset S^0_{\rho,1-\rho},$$

for all $\rho < 1$. Thus $\mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}p) \in \Psi^0_{\rho,1-\rho}(M)$ for all $\rho < 1$ and this is bounded on L^2. Moreover, $K_t'P{A_s}^{-1} \in \Psi^{-1+}(M)$ and is thus compact on L^2. Since e^{-tX} is bounded on L^2, we deduce that

$$\int_0^{+\infty} \chi'(t)e^{-t\lambda}e^{-tX}K_t'P{A_s}^{-1}\mathrm{d}t$$

is compact on L^2. We now need to study the norm of the operator in $\Psi^0_{\rho,1-\rho}$. Let $q \in C^\infty(T^*M)$ be a smooth cutoff function such that $q(x,\xi) \equiv 0$ for $|\xi| \le R$ and $q(x,\xi) = 1$ for $|\xi| \ge R+1$. We write

$$\mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}p) = \mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}pq) + \mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}p(1-q))$$

The last operator is in $\Psi^{-\infty}$ and is thus compact on L^2. We are left with the operator $\mathrm{Op}(e^{s(G_m\circ\Phi_t-G_m)}pq)$. Note that $e^{s(G_m\circ\Phi_t-G_m)}pq \le e^{-C_1 sT/2}$ since $\mathbf{X}\cdot G_m \le -C_1 < 0$ on the support of pq. By the Calderon-Vaillancourt Theorem (see [Shu01, Theorem

6.4] for instance), for $t \in [0,T]$, we can write $\mathrm{Op}(e^{s(G_m\circ\Phi_t - G_m)}pq) = A_t + L_t$, where $A_t \in \Psi^0_{\rho,1-\rho}$, $L_t \in \Psi^{-\infty}$ and $\|A_t\|_{\mathcal{L}(L^2,L^2)} \leq e^{-C_1 st/2}$. Since the operator L_t contributes to a compact operator in (2.4.2), we can forget it.

In (2.4.2), we thus obtain that

$$\mathbb{1} + R(\lambda) = \mathbb{1} + B(\lambda) + K(\lambda),$$

with $K(\lambda)$ holomorphic (on $\mathbb{C}$) family of compact operators on L^2 and using $\|e^{-tX}\|_{\mathcal{L}(L^2,L^2)} \leq C_0 e^{\omega t}$:

$$\begin{aligned}\|B(\lambda)\|_{\mathcal{L}(L^2,L^2)} &= \| \int_0^T \chi_T'(t) e^{-t\lambda} e^{-tX} A_t \mathrm{d}t \|_{\mathcal{L}(L^2,L^2)} \\ &\leq C_0 \int_0^T |\chi_T'(t)| e^{-t\Re(\lambda)} e^{-C_1 st/2} e^{\omega t} \mathrm{d}t \\ &\leq \frac{C_0 \|\chi'\|_{L^\infty}}{T} \int_0^T e^{-(C_1 s/2 + \Re(\lambda) - \omega)t} \mathrm{d}t \leq \frac{C_0 \|\chi'\|_{L^\infty}}{T(C_1 s/2 + \Re(\lambda) - \omega)}\end{aligned} \tag{2.4.4}$$

This can be made smaller than 1 for some well-chosen constants. Indeed, choose $T > 0$ large enough so that $C_0 \|\chi'\|_{L^\infty}/T < C_1 s/8$. Then, for $\Re(\lambda) > \omega - C_1 s/4$, one obtains :

$$\frac{\|\chi'\|_{L^\infty}}{T(C_1 s/2 + \Re(\lambda) - \omega)} < \frac{\|\chi'\|_{L^\infty}}{TC_1 s/4} < 1/2$$

Therefore, by (2.4.4), $\|B(\lambda)\|_{\mathcal{L}(L^2,L^2)} < 1$. In fine, we obtain that $\mathbb{1} + B(\lambda)$ is invertible by Neumann series and thus in (2.4.2), we obtain that $\mathbb{1} + B(\lambda) + K(\lambda)$ is a holomorphic family of Fredholm operators on $\Re(\lambda) > \omega - cs$ (where $c := C_1/4$) with index 0. We then conclude by the analytic Fredholm Theorem. The sought space is $\mathcal{H}^s_+(\mathcal{M}) := A_s^{-1}(L^2(\mathcal{M}))$.

□

The poles of the meromorphic extension of $(X+\lambda)^{-1} : \mathcal{H}^s_+(\mathcal{M}) \rightarrow \mathcal{H}^s_+(\mathcal{M})$ to the half-space $\{\Re(\lambda) > -cs\}$ are called the *Pollicott-Ruelle resonances*. They are intrinsic to the operator X, namely they do not depend on the choice of the spaces $\mathcal{H}^s_+(\mathcal{M})$ as they can be seen as the poles of the meromorphic extension of $(X+\lambda)^{-1} : C^\infty(\mathcal{M}) \rightarrow C^{-\infty}(\mathcal{M})$ to the whole complex plane. Here $C^{-\infty}(\mathcal{M}) := \cup_{s\in\mathbb{R}} H^s(\mathcal{M})$. One geometric way of seeing this independence of the resonances with respect to the spaces is to relate them to the *dynamical determinant*. Indeed, one can prove (see [DZ16] for instance) that

$$\zeta_0(\lambda) := \exp\left(- \sum_{\gamma\in\mathcal{G}} \sum_{n\geq 1} \frac{e^{i\lambda n \ell(\gamma)}}{n|\det(1-\mathcal{P}_\gamma)|} \right),$$

where $\mathcal{P}_\gamma$ is the Poincaré return map for closed orbits $\gamma \in \mathcal{G}$, $\mathcal{G}$ being the set of primitive closed orbits, admits a holomorphic extension to the whole complex plane and that the zeros of this function on $\{\Re(\lambda) > -cs\}$ are exactly the poles of the meromorphic extension $(X+\lambda)^{-1} : \mathcal{H}^s_+(\mathcal{M}) \rightarrow \mathcal{H}^s_+(\mathcal{M})$. Obviously, since ζ_1 does not depend on $\mathcal{H}^s_+(\mathcal{M})$, this shows that the resonances are intrinsic.

Note that the positive resolvent $R_+(\lambda)$ is bounded as an operator $R_+(\lambda) : L^2(\mathcal{M}) \rightarrow L^2(\mathcal{M})$ for $\Re(\lambda) > \omega$ and its expression is given by

$$R_+(\lambda) = (X+\lambda)^{-1} = \int_0^{+\infty} e^{-\lambda t} e^{-tX} dt, \tag{2.4.5}$$

where $e^{-tX}f(z) = f(\varphi_{-t}(z))$ for $f \in C^\infty(\mathcal{M}), z \in \mathcal{M}$. In particular, it has no poles in the region $\{\Re(\lambda) > \omega\}$. When the vector field preserve the measure μ, $\omega = 0$ and we will study its poles on $i\mathbb{R}$ in the next paragraph. In the same fashion, the negative resolvent $R_-(\lambda) : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ is bounded for $\Re(\lambda) > \omega$, given by

$$R_-(\lambda) = (X - \lambda)^{-1} = -\int_{-\infty}^{0} e^{\lambda t} e^{-tX} dt, \tag{2.4.6}$$

it can be meromorphically extended to $\Re(\lambda) > -cs$ when acting on adapted spaces $\mathcal{H}^s_- \to \mathcal{H}^s_-$.

If λ_0 is a pole of say R_-, then $R_-(\lambda)$ has a Laurent expansion in a neighborhood of λ_0, namely

$$R_-(\lambda) = R_-^{\mathrm{hol}}(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(X-\lambda_0)^{j-1}\Pi_{\lambda_0}}{(\lambda-\lambda_0)^j}$$

where $R_-^{\mathrm{hol}}(\lambda_0) : \mathcal{H}^s_-(\mathcal{M}) \to \mathcal{H}^s_-(\mathcal{M})$ is bounded, $\Pi_{\lambda_0} : \mathcal{H}^s_-(\mathcal{M}) \to \mathcal{H}^s_-(\mathcal{M})$ is the commuting projection onto $\ker((X-\lambda_0)^{J(\lambda_0)})$. We call *generalized eigenstates* the elements of $\ker((X-\lambda_0)^{J(\lambda_0)})$. A priori, there may be Jordan blocks and these may not be real eigenvectors. Note that the generalized eigenstates are themselves intrinsic insofar $\ker((X-\lambda_0)^{J(\lambda_0)}) = \Pi_{\lambda_0}(\mathcal{H}^s_+(\mathcal{M})) = \Pi_{\lambda_0}(C^\infty(\mathcal{M}))$, by density of $C^\infty(\mathcal{M})$ in $\mathcal{H}^s_+(\mathcal{M})$.

2.4.2 Elements of spectral theory

We now assume that X preserves a smooth measure. As mentioned in §2.1.1, we prove that the L^2-spectrum of $-iX$ is $\mathbb{R}$.

Lemma 2.4.2. $\sigma(-iX) = \mathbb{R}$

The proof we give is that of Guillemin [Gui77, Lemma 3], following Helton.

Proof. We argue by contradiction. Assume $\sigma(-iX) \neq \mathbb{R}$, then since $\sigma(-iX)$ is closed, there exists an interval I of $\mathbb{R}$ such that $I \cap \sigma(-iX) = \emptyset$. Let $f \in C_c^\infty(I)$, $f \neq 0$. Then $f(-iX) = 0$ and this operator is given by [2]

$$f(-iX) = \int_{-\infty}^{+\infty} \hat{f}(t) e^{tX} dt$$

Given $a \in C^\infty(\mathcal{M})$, $f(-iX)a$ is continuous. Moreover, it is given at $x_0 \in \mathcal{M}$ by :

$$f(-iX)a(x_0) = \int_{-\infty}^{+\infty} \hat{f}(t) a(\varphi_t x_0) dt$$

We now consider g, a smooth function on $\mathbb{R}$ with compact support and a constant $A > 0$. If $x_0 \in \mathcal{M}$ is not periodic, then we can construct $a \in C^\infty(\mathcal{M}), h \in C^\infty(\mathbb{R})$ such that $a(\varphi_t x_0) = g(t) + h(t)$ for all $t \in \mathbb{R}$, where $\|h\|_\infty \leq \|g\|_\infty$ and $\mathrm{supp}(h) \cap [-A, A] = \emptyset$

2. Formally, this follows from the following computation, where $dP(\lambda)$ is the spectral measure of $-iX$:

$$f(-iX) = \int_{-\infty}^{+\infty} f(\lambda) dP(\lambda) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\lambda t}\hat{f}(t) dP(\lambda) dt = \int_{-\infty}^{+\infty} \hat{f}(t) e^{tX} dt$$

The justification of the permutation is not difficult since f has compact support.

(define a by $a(\varphi_t x_0)$ on a sufficiently large segment of the orbit of x_0 and then extend to a sufficiently small tubular neighborhood in order to obtain a smooth function). Then :

$$f(-iX)a(x_0) = 0 = \int_{-\infty}^{+\infty} \hat{f}(t)g(t)dt + \int_{-\infty}^{+\infty} \hat{f}(t)h(t)dt$$

As $A \to +\infty$, the second integral converges to 0 since $\hat{f}$ is Schwartz. We thus obtain that $\int_{-\infty}^{+\infty} \hat{f}(t)g(t)dt = 0$ for any smooth function g with compact support, thus $\hat{f} \equiv 0$ and $f \equiv 0$. □

We assume from now on that the flow is mixing.

Lemma 2.4.3. *The resolvents $R_\pm$ have a unique pole on $i\mathbb{R}$: it is the point* 0, *with rank* 1 *residue given by $\pm\mathbf{1} \otimes \mathbf{1}$, the projection on the constants.*

Proof. We will argue on R_+ since all the arguments are similar for R_-. First, remark that if $i\lambda_0$ is a pole on $i\mathbb{R}$, then it is of order 1 since by the spectral theorem, for $f_1, f_2 \in C^\infty(\mathcal{M})$, $\lambda > 0$, $|\langle R_+(i\lambda_0 + \lambda)f_1, f_2\rangle| \leq \lambda^{-1}\|f_1\|_{L^2}\|\|f_2\|_{L^2}$.

We fix $\varepsilon > 0$. Since the flow is mixing, there exists a time T_ε such that for all $T > T_\varepsilon$, $|C_t(f_1, f_2)| < \varepsilon$. Moreover, for $\Re(\lambda) > 0$:

$$\begin{aligned}\lambda\langle R_+(\lambda)f_1, f_2\rangle = &\underbrace{\int_0^{T_\varepsilon}\int_{\mathcal{M}} \lambda e^{-\lambda t}\langle f_1 \circ \varphi_{-t}, f_2\rangle_{L^2(\mathcal{M})}}_{\leq(1-e^{-\lambda T_\varepsilon})\|f_1\|_{L^2}\|f_2\|_{L^2}} + \underbrace{\int_{T_\varepsilon}^{+\infty} \lambda e^{-\lambda t}\langle f_1, \mathbf{1}\rangle\langle f_2, \mathbf{1}\rangle dt}_{=e^{-\lambda T_\varepsilon}\langle f_1,\mathbf{1}\rangle\langle f_2,\mathbf{1}\rangle} \\ &+ \underbrace{\int_{T_\varepsilon}^{+\infty} \lambda e^{-\lambda t} C_t(f_1, f_2)dt}_{\leq \varepsilon e^{-\lambda T_\varepsilon}}\end{aligned}$$

As $\lambda \to 0$, we obtain that

$$\lim_{\lambda \to 0^+} \lambda\langle R_+(\lambda)f_1, f_2\rangle = \langle f_1, \mathbf{1}\rangle\langle f_2, \mathbf{1}\rangle + \mathcal{O}(\varepsilon)$$

and since $\varepsilon > 0$ was chosen arbitrarily small, we obtain that 0 is a pole of order 1 with residue $\mathbf{1} \otimes \mathbf{1}$, the projection on the constants. The same arguments also immediately show that for $\lambda_0 \in i\mathbb{R} \setminus \{0\}$,

$$\lim_{\lambda \to \lambda_0^+} (\lambda - \lambda_0)\langle R_+(\lambda)f_1, f_2\rangle = 0$$

As to R_-, the same arguments apply and the residue at 0 is $-\mathbf{1} \otimes \mathbf{1}$. □

We recall that $\sigma(-iX)$ denotes the L^2-spectrum of the operator, $\sigma_{\mathrm{ac}}(-iX)$ its absolutely continuous spectrum.

Lemma 2.4.4. *$\sigma_{ac}(-iX) = \mathbb{R}$ and $\sigma(-iX) = \sigma_{ac}(-iX) \cup \{0\}$,* 0 *is a discrete embedded eigenvalue, associated to the* 1*-dimensional subspace $\mathbb{R} \cdot \mathbf{1}$, the constant functions.*

Proof. We first show that $\lambda_0 \in \sigma_{\mathrm{p}}(-iX)$ (the point spectrum) if and only if λ_0 is a pole of the resolvent. By contradiction, if λ_0 is not a pole of the resolvent, Stone's formula gives that for $\delta > 0$:

$$\begin{aligned}\frac{1}{2}(\mathbf{1}_{[\lambda_0-\delta,\lambda_0+\delta]} + \mathbf{1}_{(\lambda_0-\delta,\lambda_0+\delta)}) &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi}\int_{\lambda_0-\delta}^{\lambda_0+\delta} R_+(\varepsilon - i\lambda) - R_-(\varepsilon + i\lambda)d\lambda \\ &= \frac{1}{2\pi}\int_{\lambda_0-\delta}^{\lambda_0+\delta} R_+(-i\lambda) - R_-(i\lambda)d\lambda,\end{aligned} \tag{2.4.7}$$

where the convergence is in the weak sense[3], that is by applying the expression to $f_1 \in C^\infty(\mathcal{M})$ and testing against $f_2 \in C^\infty(\mathcal{M})$ — the permutation of the limit and the integral being guaranteed by the holomorphy of the integrand. Taking the limit $\delta \to 0$ in (2.4.7), the left-hand side converges (in the weak sense) to Π_{λ_0}, the spectral projection on $\ker(-iX - \lambda_0)$, whereas the right-hand side converges to 0. This is a contradiction so λ_0 is not in the point spectrum. Conversely, 0 is the only pole of the resolvent and it is clear that 0 is a discrete eigenvalue with $\ker_{L^2}(-iX) = \mathbb{R} \cdot \mathbf{1}$ (by ergodicity of the geodesic flow). So the pure point spectrum is reduced to $\{0\}$.

Formula (2.4.7) also allows to show that there is no singular continuous spectrum, the spectral measure being $dP(\lambda) = \frac{1}{2\pi}(R_+(-i\lambda) - R_-(i\lambda))d\lambda$. Since $\sigma(-iX) = \mathbb{R}$ and the only discrete eigenvalue is 0 and the absolutely continuous spectrum is closed, $\sigma_{\mathrm{ac}}(-iX) = \mathbb{R}$.

□

Remark 2.4.1. Actually the flow is mixing if and only if 0 is the only pole of the resolvent. The converse is obtained from the fact that the spectrum on $(\mathbb{R} \cdot \mathbf{1})^\perp$ is absolutely continuous (by the previous proof) and this implies that the flow is mixing (see [RS80, Theorem VII.15]). Indeed, for $f_1, f_2 \in C^\infty(\mathcal{M})$, orthogonal to the constants, one has :

$$\begin{aligned} \langle e^{tX} f_1, f_2 \rangle &= \int_{-\infty}^{+\infty} e^{it\lambda} \langle dP(\lambda) f_1, f_2 \rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it\lambda} \langle (R_+(-i\lambda) - R_-(i\lambda)) f_1, f_2 \rangle d\lambda \\ &= \frac{1}{2\pi} \widehat{T}(-t), \end{aligned}$$

where $T(\lambda) := \langle (R_+(-i\lambda) - R_-(i\lambda)) f_1, f_2 \rangle$. By the spectral theorem, $T \in L^1(\mathbb{R})$ (and $\int \langle (R_+(-i\lambda) - R_-(i\lambda)) f_1, f_2 \rangle d\lambda = \langle f_1, f_2 \rangle$) so by the Riemann-Lebesgue theorem, one has

$$\lim_{t \to +\infty} \langle e^{tX} f_1, f_2 \rangle = \lim_{t \to +\infty} \frac{1}{2\pi} \widehat{T}(-t) = 0,$$

that is the flow is mixing.

2.4.3 The operator Π

In a neighborhood of 0, we can thus write the Laurent expansions

$$R_+(\lambda) = R_0^+ + \frac{\mathbf{1} \otimes \mathbf{1}}{\lambda} + \mathcal{O}(\lambda), \qquad R_-(\lambda) = R_0^- - \frac{\mathbf{1} \otimes \mathbf{1}}{\lambda} + \mathcal{O}(\lambda), \tag{2.4.8}$$

where $R_0^+ : \mathcal{H}_+^s \to \mathcal{H}_+^s, R_0^- : \mathcal{H}_-^s \to \mathcal{H}_-^s$ are bounded. Since $H^s \subset \mathcal{H}_\pm^s \subset H^{-s}$, we obtain that $R_0^\pm : H^s \to H^{-s}$ are bounded and thus $(R_0^+)^* : H^s \to H^{-s}$ is bounded too. Moreover, it is easy to check that formally $(R_0^+)^* = -R_0^-$ (i.e. the operators coincide on $C^\infty(\mathcal{M})$), in the sense that for all $f_1, f_2 \in C^\infty(\mathcal{M})$, $\langle R_0^- f_1, f_2 \rangle_{L^2(\mathcal{M})} = \langle f_1, -R_0^+ f_2 \rangle_{L^2(\mathcal{M})}$. This follows from the fact that we formally have $R_+(\lambda)^* = -R_-(\overline{\lambda})$ for $\Re(\lambda) > 0$. Since $C^\infty(\mathcal{M})$ is dense in $H^s(\mathcal{M})$, we obtain that $(R_0^+)^* = -R_0^-$ on $H^s(\mathcal{M})$, in the sense that for all $f_1, f_2 \in H^s(\mathcal{M})$, $\langle R_0^- f_1, f_2 \rangle_{L^2(\mathcal{M})} = \langle f_1, -R_0^+ f_2 \rangle_{L^2(\mathcal{M})}$.

3. The limit in Stone's formula is in the strong sense but we here want to inverse limit and integration.

Also remark that, as operators $C^\infty(\mathcal{M}) \to C^{-\infty}(\mathcal{M})$, one has :

$$XR_0^+ = R_0^+X = \mathbb{1} - \mathbf{1}\otimes\mathbf{1}, XR_0^- = R_0^-X = \mathbb{1} - \mathbf{1}\otimes\mathbf{1} \tag{2.4.9}$$

For the sake of simplicity, we will write $R_0 := R_0^+$. We are now interested in studying the wavefront set of the Schwartz kernel of R_0. The following result can be found in [DZ16].

Proposition 2.4.1 (Dyatlov-Zworski)**.**

$$\mathrm{WF}'(R_0) = [\Delta(T^*\mathcal{M}\setminus\{0\}) \times \Omega_+ \times (E_u^* \times E_s^*)] \setminus \{0\}\,,$$

with $\Omega_+ = \{(\Phi_t(x,\xi), x, \xi) \mid \langle \xi, X(x)\rangle = 0, \xi \neq 0, t > 0\}$.

In other words, singularities are propagated forward. Note that by Lemma A.2.6, $R_0 f$ makes sense for any distribution $f \in C^{-\infty}(SM)$ as long as $\mathrm{WF}(f)\cap E_s^* = \emptyset$ and $R_0 f$ may have wavefront set in E_u^* even if $f \in C^\infty(\mathcal{M})$. So R_0 creates singularities from scratch (the set $\mathrm{WF}(K_{R_0})_1$ in Lemma A.2.6 is not trivial). We also obtain

$$\mathrm{WF}'(R_0^-) = [\Delta(T^*\mathcal{M}\setminus\{0\}) \times \Omega_- \times (E_s^* \times E_u^*)] \setminus \{0\}\,,$$

with $\Omega_- = \{(\Phi_t(x,\xi), x, \xi) \mid \langle \xi, X(x)\rangle = 0, \xi \neq 0, t < 0\}$.

We will admit this proposition which is not an easy result, although it may appear rather natural. One of the main difficulty is that there is no *characterization lemma* for the wavefront set of the Schwartz kernel of an operator in microlocal analysis [4]. So one has to resort to semiclassical analysis — where such a lemma is available — but we do not want to introduce semiclassical notations in order not to flood the discussion. In particular, the proof in [DZ16] of this proposition relies on the radial source/sink estimates (see Theorems A.4.2 and A.4.3) in their semiclassical versions (which are more accurate).

We now assume that the flow is exponentially mixing (polynomially mixing is actually sufficient). We introduce the operator

$$\Pi := R_0 + R_0^*, \tag{2.4.10}$$

the sum of the two holomorphic parts of the resolvent. An easy computation, using (2.4.8), proves that $\Pi(\mathbf{1}) = 0$ and the image $\Pi(C^\infty(\mathcal{M}))$ is orthogonal to the constants. There exists two other characterizations of the operator Π that are more tractable and which we detail in the next proposition. We set $\Pi_\lambda := \mathbf{1}_{(-\infty,\lambda]}(-iX)$.

Proposition 2.4.2. *For $f_1, f_2 \in C^\infty(\mathcal{M})$ such that $\int_\mathcal{M} f_i d\mu = 0$:*

1. $\langle \Pi f_1, f_2\rangle = 2\pi\partial_\lambda|_{\lambda=0}\langle \Pi_\lambda f_1, f_2\rangle$,
2. $\langle \Pi f_1, f_2\rangle = \int_{-\infty}^{+\infty}\langle f_1\circ\varphi_t, f_2\rangle dt$.

4. Actually, this is a drawback that is intrinsic to microlocal analysis. Given a linear operator A, in order to characterize $\mathrm{WF}'(A)$, one would like to consider f with wavefront set at (y,η) and study the microlocalization of the wavefront set of Af. But Af does not always make sense by Lemma A.2.6 so we would need to know *a priori* the set $\mathrm{WF}(K_A)_1 = \{(x,\xi) \mid \exists y \in M, (x,\xi,y,0) \in K_A\}$ ... but this is precisely what we are looking for ! The semiclassical notion of the wavefront set allows to bypass this problem.

Proof. (1) For $f_1, f_2 \in C^\infty(\mathcal{M})$ such that $\int_{\mathcal{M}} f_i d\mu = 0$, we have like in (2.4.7), for $\delta > 0$:

$$\begin{aligned}\langle \Pi_{\lambda+\delta} f_1, f_2 \rangle - \langle \Pi_{\lambda-\delta} f_1, f_2 \rangle &= \langle \mathbf{1}_{[\lambda-\delta,\lambda+\delta]} f_1, f_2 \rangle \\ &= \frac{1}{2\pi} \int_{\lambda-\delta}^{\lambda+\delta} \langle (R_+(-i\lambda) - R_-(i\lambda)) f_1, f_2 \rangle d\lambda\end{aligned}$$

Dividing by 2δ and passing to the limit $\delta \to 0^+$, we obtain $\partial_\lambda|_{\lambda=0} \langle \Pi_\lambda f_1, f_2 \rangle = \frac{1}{2\pi} \langle (R_0^+ - R_0^-) f_1, f_2 \rangle = \frac{1}{2\pi} \langle \Pi f_1, f_2 \rangle$.

(2) Thanks to the exponential decay of correlations (see [Liv04]), one can apply Lebesgue's dominated convergence theorem in the limit $\lambda \to 0^+$ in the following expression

$$\langle \Pi f_1, f_2 \rangle = \lim_{\lambda \to 0^+} \int_{-\infty}^{+\infty} e^{-\lambda |t|} \langle f_1 \circ \varphi_{-t}, f_2 \rangle dt,$$

and the result is then immediate. □

The quantity $\langle \Pi f, f \rangle$ is sometimes referred to in the literature as the *variance* of the flow. We refer to the paragraph §3.3.2 for a more exhaustive discussion. In particular, it enjoys the following positivity property :

Lemma 2.4.5. *The operator $\Pi : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ is positive in the sense of quadratic forms, namely $\langle \Pi f, f \rangle_{L^2} \geq 0$ for all $f \in H^s(\mathcal{M})$.*

There are different ways of proving this lemma, related to the different characterizations of the operator Π. We only detail one of them and provide some hints for the other proofs :

- Since $\Pi(\mathbf{1}) = 0$, we can always assume that $\int_{\mathcal{M}} f d\mu = 0$. Then, by using the fact that the flow is exponentially mixing, one can prove that :

$$\frac{1}{T} \int_{\mathcal{M}} \left(\int_0^T f \ \circ \varphi_t \cdot f dt \right)^2 d\mu = \langle \Pi f, f \rangle + \mathcal{O}(1/T),$$

  which provides the sought result.
- A more immediate way of obtaining the positivity, is to use the characterization of Π as the derivative of the spectral measure. If $\int_{\mathcal{M}} f d\mu = 0$, that is $f \in (\mathbb{R} \cdot \mathbf{1})^\perp$, then $\lambda \mapsto \langle \mathbf{1}_{(-\infty,\lambda]}(-iX) f, f \rangle$ is non-decreasing. Thus its derivative is nonnegative.

We provide a more dynamical proof of this result.

Proof. By density, it is sufficient to prove the lemma for $f \in C^\infty(\mathcal{M})$. We will actually show that for $\Re(\lambda) > 0$:

$$\langle \left(R_+(\lambda) - \frac{1 \otimes 1}{\lambda} \right) f, f \rangle = \langle R_+(\lambda) f, f \rangle - \frac{1}{\lambda} \left(\int_{\mathcal{M}} f d\mu \right)^2 \geq 0$$

The same arguments being valid for $R_-(\lambda)$, we will deduce the result by taking the limit $\lambda \to 0^+$. By Parry' formula [Par88, Paragraph 3], we know that :

$$\langle R_+(\lambda) f, f \rangle = \lim_{T \to \infty} \frac{1}{N(T)} \sum_{\ell(\gamma) \leq T} e^{\int_\gamma J^u} \frac{1}{\ell(\gamma)} \int_0^{\ell(\gamma)} R_+(\lambda) f(\varphi_t z) f(\varphi_t z) dt, \qquad (2.4.11)$$

where $z \in \gamma$ and $N(T) = \sum_{\ell(\gamma)\leq T} e^{\int_\gamma J^u}$ is a normalizing coefficient, and J^u is the unstable Jacobian (or the *geometric potential*) associated to the measure μ. Let us fix a closed orbit γ and a base point $z \in \gamma$. We set $\bar{f}(t) := f(\varphi_t z)$ which we see as a smooth function, ℓ-periodic on $\mathbb{R}$ (with $\ell := \ell(\gamma)$). Since $R_+(\lambda)$ commutes with X, $R_+(\lambda)$ acts as a Fourier multiplier on functions defined on γ. As a consequence, if we decompose $\bar{f}(t) = \sum_{n \in\mathbb{Z}} c_n e^{2i\pi nt/\ell}$, we have :

$$\begin{aligned} R_+(\lambda)\bar{f}(t) &= \int_0^{+\infty} e^{-\lambda s}\bar{f}(t+s)ds \\ &= \sum_{n\in\mathbb{Z}} c_n e^{2i\pi nt/\ell}\int_0^{+\infty} e^{-(\lambda-2i\pi n/\ell)s}ds \\ &= \sum_{n\in\mathbb{Z}} \frac{c_n(\lambda+2i\pi n/\ell)}{\lambda^2+4\pi^2n^2/\ell^2}e^{2i\pi nt/\ell} \end{aligned}$$

Then :

$$\langle R_+(\lambda)\bar{f},\bar{f}\rangle_{L^2} = \frac{1}{\ell}\int_0^\ell R_+(\lambda)\bar{f}(t)\bar{f}(t)dt = \sum_{n\in\mathbb{Z}}\frac{|c_n|^2(\lambda+2i\pi n/\ell)}{\lambda^2+4\pi^2n^2/\ell^2} = \lambda\sum_{n\in\mathbb{Z}}\frac{|c_n|^2}{\lambda^2+4\pi^2n^2/\ell^2},$$

by oddness of the imaginary part of the sum. In particular :

$$\frac{1}{\ell}\int_0^\ell R_+(\lambda)\bar{f}(t)\bar{f}(t)dt \geq \frac{|c_0|^2}{\lambda} = \frac{1}{\lambda}\left(\frac{1}{\ell}\int_0^\ell \bar{f}(t)dt\right)^2 \tag{2.4.12}$$

Inserting (2.4.12) into (2.4.11), then applying Jensen's convexity inequality :

$$\begin{aligned} \langle R_+(\lambda)f,f\rangle \geq\ & \lambda^{-1}\lim_{T\to\infty}\frac{1}{N(T)}\sum_{\ell(\gamma)\leq T} e^{\int_\gamma J^u}\left(\frac{1}{\ell(\gamma)}\int_0^{\ell(\gamma)} f(\varphi_t z)dt\right)^2 \\ &\geq \lambda^{-1}\lim_{T\to\infty}\left(\frac{1}{N(T)}\sum_{\ell(\gamma)\leq T} e^{\int_\gamma J^u}\frac{1}{\ell(\gamma)}\int_0^{\ell(\gamma)} f(\varphi_t z)dt\right)^2 = \frac{1}{\lambda}\left(\int_{SM} fd\mu\right)^2 \end{aligned}$$

□

Theorem 2.4.2. *[Gui17a, Theorem 1.1] The operator $\Pi : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ is bounded, for any $s > 0$, selfadjoint and satisfies :*

1. *$\forall f \in H^s(\mathcal{M}), X\Pi f = 0$,*
2. *$\forall f \in H^s(\mathcal{M})$ such that $Xf \in H^s(\mathcal{M})$, $\Pi X f = 0$.*

If $f \in H^s(\mathcal{M})$ with $\langle f, \mathbf{1}\rangle_{L^2} = 0$, then $f \in \ker \Pi$ if and only if there exists a solution $u \in H^s(\mathcal{M})$ to the cohomological equation $Xu = f$, and u is unique modulo constants.

Remark 2.4.2. Actually, by slightly changing the previous constructions (the definition of m), we could have obtained that $\Pi : H^s(\mathcal{M}) \to H^{-r}(\mathcal{M})$ is bounded for any $s, r > 0$. It is possible to choose the escape function with a lot of freedom. For instance, concerning $R_+(\lambda)$, we could have taken an espace function m, like the one designed on Figure 2.4. In particular, such a choice guarantees that for $f \in H^s(\mathcal{M})$, $R_0 f$ is microlocally H^s everywhere, except in conic neighborhood of E_u^*.

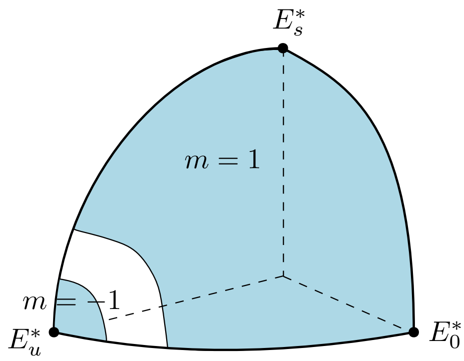

FIGURE 2.4 – Another choice for the escape function m.

Proof. The fact that $\Pi : H^s(\mathcal{M}) \to H^{-s}(\mathcal{M})$ is bounded and selfadjoint is immediate thanks to the previous constructions and the two identities follow from (2.4.9). This also proves one implication in the last sentence.

Assume $f \in H^s(\mathcal{M})$ and $f = Xu$ for some function $u \in H^s(\mathcal{M})$. The fact that u is unique modulo constants follows from the ergodicity of the flow. Indeed, assume $u' \in H^s(M)$ is such that $f = Xu' = Xu$, then $X(u' - u) = 0$ so $u' - u$ is constant.

Now, assume $f \in H^s(\mathcal{M})$, $\langle f, \mathbf{1} \rangle_{L^2} = 0$ and $\Pi f = 0 = R_0 f + R_0^* f$. We set $u_+ = R_0 f, u_- = -R_0^* f \in H^{-s}(\mathcal{M})$ (so that *formally*, $Xu_+ = Xu_- = f$) and $u_+ = u_-$. By Remark 2.4.2, $u_+ = R_0 f$ is microlocally H^s everywhere, except in conic neighborhood of $V_u \subset T^*M$ of E_u^* (where it is, at worst, H^{-s}). The same occurs for u_-, but reversing the time : $u_- = -R_0^* f = R_0^- f$ is microlocally H^s everywhere except in a conic neighborhood $V_s \subset T^*M$ of E_s^*. Since $u_+ = u_-$ and we can always choose V_u, V_s so that $V_s \cap V_u = \emptyset$, we obtain that $u := u_+ \in H^s(\mathcal{M})$. Then, $Xu = XR_0 f = f$, since $\langle f, \mathbf{1} \rangle = 0$. ☐

As a corollary, we obtain the proof of Guillarmou [Gui17a] of the Livsic theorem in Sobolev regularity (see Theorem 2.1.1) :

Proof. We fix $s > \frac{n+1}{2}$ and assume $f \in H^s(\mathcal{M})$ satisfies $If = 0$. Then $f \in C^\alpha$ for some $\alpha > 0$ so by the classical Livsic theorem in Hölder regularity, $f = Xu$ for some $u \in C^\alpha$. In particular, $f \in H^{\alpha/2}, u \in H^{\alpha/2}$ and $\Pi f = \Pi X u = 0$ i.e. $\ker \Pi = 0$ (and $\langle f, \mathbf{1} \rangle = 0$ since $If = 0$). By Theorem 2.4.2, u is actually in $H^s(\mathcal{M})$. ☐

2.5 The normal operator

We will apply the results of the previous sections to the case where $\mathcal{M} := SM$, the unit tangent bundle of a Riemannian manifold (M, g), whose geodesic flow is assumed to be Anosov. In particular, a geodesic flow is a contact flow (the contact form is the Liouville 1-form α) and is thus exponentially mixing by the result of Liverani [Liv04]. Most of the results of this section can be found in [Gui17a, GL19d, GL19a].

Recall from Appendix B that

$$T(SM) = \mathbb{R} \cdot X \oplus^\perp \mathbb{H} \oplus^\perp \mathbb{V},$$

where $\mathbb{V} = \ker d\pi_0$, $\pi_0 : SM \to M$ the projection and $d\pi_0 : \mathbb{R}X \oplus^\perp \mathbb{H} \to TM$ is an isometry. Here, the metric on SM is the Sasaki metric induced by the metric g on M. We define the dual spaces $\mathbb{V}^*, \mathbb{H}^*$ by $\mathbb{V}^*(\mathbb{V}) = 0, \mathbb{H}^*(\mathbb{R}X \oplus \mathbb{H}) = 0$. Note that $d\pi_0^\top : T^*M \to \mathbb{V}^*$ is an isometry. By [Pat99, Theorem 2.50], $\mathbb{R}X \oplus E_s \oplus \mathbb{V} = T(SM) = \mathbb{R}X \oplus E_u \oplus \mathbb{V}$

and thus $\mathbb{V}^* \oplus E_s^* = \mathbb{V}^* \oplus E_u^* = T^*(SM)$. Recall that (M, g) has *no conjugate points* if for all $t \neq 0$, $d\varphi_t(\mathbb{V}) \cap \mathbb{V} = \{0\}$. By [Kli74], an Anosov Riemannian manifold has no conjugate points and on the cotangent bundle, this implies that $d\varphi_t^\top(\mathbb{V}^*) \cap \mathbb{V}^* \cap \Sigma = \{0\}$ for all $t \neq 0$, where $\Sigma = E_u^* \oplus E_s^*$ is the characteristic set.

2.5.1 Definition and first properties

We introduce the normal operator

$$\Pi_m := \pi_{m*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_m^*. \tag{2.5.1}$$

Recall from §B.1.3 that given $(x, \xi) \in T^*M$, the space $\otimes_S^m T_x^*M$ decomposes as the direct sum

$$\begin{aligned}\otimes_S^m T_x^*M &= \operatorname{ran}\left(\sigma_D(x,\xi)|_{\otimes_S^{m-1}T_x^*M}\right) \oplus \ker\left(\sigma_{D^*}(x,\xi)|_{\otimes_S^m T_x^*M}\right) \\ &= \operatorname{ran}\left(\sigma j_\xi|_{\otimes_S^{m-1}T_x^*M}\right) \oplus \ker\left(i_\xi|_{\otimes_S^m T_x^*M}\right)\end{aligned}$$

The projection on the right space parallel to the left space is denoted by $\pi_{\ker i_\xi}$ and $\operatorname{Op}(\pi_{\ker i_\xi}) = \pi_{\ker D^*} + \mathcal{O}(\Psi^{-1})$ by Lemma B.1.6. The following theorem will be crucial in the sequel. We recall that M is $(n+1)$-dimensional.

Theorem 2.5.1. *Π_m is a pseudodifferential operator of order -1 with principal symbol*

$$\sigma_m := \sigma_{\Pi_m} : (x,\xi) \mapsto \frac{2\pi}{C_{n,m}} |\xi|^{-1} \pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi},$$

with $C_{n,m} = \int_0^\pi \sin^{n-1+2m}(\varphi) d\varphi$.

We have the following

Lemma 2.5.1. *One has :*

$$\mathrm{WF}'(\pi_m^*) \subset \left\{ \left(((x,v), (\underbrace{d\pi_0^\top \xi}_{\in \mathbb{V}^*}, \underbrace{0}_{\in \mathbb{H}^*})), (x,\xi) \right) \mid (x,\xi) \in T^*M \setminus \{0\} \right\}$$

In particular, if $u \in C^{-\infty}(M, \otimes_S^m T^*M)$ then, $\mathrm{WF}(\pi_m^* u) \subset \mathbb{V}^*$.

Proof. The case $m = 0$ is rather immediate and follows from Lemma A.2.5, since $d\pi_0(\mathbb{V}) = 0$. We have for $z = (x, v) \in SM$:

$$\mathrm{WF}(\pi_0^* u) \subset \left\{ (z, d\pi_0(z)^T \eta), (\pi_0(z), \eta) \in \mathrm{WF}(u) \right\} \subset \mathbb{V}^*$$

As to the case $m \geq 1$, it actually boils down to the case $m = 0$. Indeed, consider a point $x_0 \in M$ and a local smooth orthonormal basis $(e_1(x), ..., e_{N(m)}(x))$ of $\otimes_S^m T^*M$ in a neighborhood V_{x_0} of x_0, where $N(m) = \binom{n+m}{m}$ denotes the rank of $\otimes_S^m T^*M$. Consider a smooth cutoff function χ such that $\chi \equiv 1$ in a neighborhood $W_{x_0} \subset V_{x_0}$ of x_0 and $\operatorname{supp}(\chi) \subset V_{x_0}$. Any smooth section ψ of $\otimes_S^m T^*M$ can be decomposed in V_{x_0} as :

$$\psi(x) = \sum_{j=1}^{N(m)} \langle \psi(x), e_j(x) \rangle_g e_j(x)$$

Thus :

$$\pi_m^*(\chi\psi) = \sum_{j=1}^{N(m)} \pi_0^*\left(\langle\psi(x), \chi e_j(x)\rangle_g\right)\pi_m^* e_j = \sum_{j=1}^{N(m)} \pi_0^*\left(A_j\psi\right)\pi_m^* e_j,$$

where the $A_j : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, \mathbb{R})$ are pseudodifferential operators of order 0 with support in $\mathrm{supp}(\chi)$. This expression still holds for a distribution u. Note that $\pi_m^* e_j$ is a smooth function on SM, thus the wavefront is given by the $\pi_0^*(A_j\psi)$ and by our previous remark for $m = 0$:

$$\mathrm{WF}(\pi_m^*(\chi u)) \subset \mathbb{V}^*$$

□

In other words, π_m^* localizes the wavefront set in $\mathbb{V}^*$. Moreover, since π_{m*} consists in integrating in the fibers S_xM, one has by Lemma A.2.3

$$\mathrm{WF}(\pi_{m*}u) \subset \left\{(x,\xi) \mid \exists v \in S_xM, ((x,v), \underbrace{d\pi^\top\xi}_{\in\mathbb{V}^*}, \underbrace{0}_{\in\mathbb{H}^*}) \in \mathrm{WF}(f)\right\}, \tag{2.5.2}$$

so π_{m*} only selects the wavefront set in $\mathbb{V}^*$ and kills the wavefront set in the other directions.

For $\varepsilon > 0$, we consider a smooth cutoff function χ such that $\chi \equiv 1$ on $[0, \varepsilon]$, and $\chi \equiv 0$ on $[2\varepsilon, +\infty)$. For $\Re(\lambda) > 0$, we write

$$\begin{aligned} R_+(\lambda) &= \int_0^{2\varepsilon} \chi(t)e^{-\lambda t}e^{-tX}dt + \int_\varepsilon^{+\infty}(1-\chi(t))e^{-\lambda t}e^{-tX}dt \\ &= \int_0^{2\varepsilon} \chi(t)e^{-\lambda t}e^{-tX}dt + \int_\varepsilon^{T}(1-\chi(t))e^{-\lambda t}e^{-tX}dt + e^{-T\lambda}e^{-TX}R_+(\lambda), \end{aligned}$$

where $T > 2\varepsilon$. Note that this expression can be meromorphically extended to the whole complex plane since $R_+(\lambda)$ can by Theorem 2.4.1. Taking the finite part at 0, we obtain :

$$R_0 = \int_0^{2\varepsilon} \chi(t)e^{-tX}dt + \int_\varepsilon^{T}(1-\chi(t))e^{-tX}dt + e^{-TX}R_0 - T \times \mathbf{1} \otimes \mathbf{1}$$

Note that the last operator is obviously smoothing. We will write

$$\Delta_T(M \times M) = \{(x,x') \in M \times M, d(x,x') = T\}.$$

By the previous computation, we obtain :

$$\begin{aligned} \pi_{m*}R_0\pi_m^* = \pi_{m*}\int_0^{2\varepsilon}\chi(t)e^{-tX}dt\pi_m^* + \pi_{m*}\int_\varepsilon^T(1-\chi(t))e^{-tX}dt\pi_m^* \\ + \pi_{m*}e^{-TX}R_0\pi_m^* + \text{ smoothing} \end{aligned}$$

Lemma 2.5.2. $\mathrm{suppsing}\left(\pi_{m*}e^{-TX}R_0\pi_m^*\right), \mathrm{suppsing}\left(\pi_{m*}\int_\varepsilon^T(1-\chi(t))e^{-tX}dt\pi_m^*\right) \subset \Delta_T$

Proof. By Lemma A.2.7,

$$\begin{aligned} \mathrm{WF}'(e^{-TX}R_0) \subset & \underbrace{\{(\Phi_T(z,\xi),(z,\xi)) \mid (z,\xi) \in T^*(SM)\}}_{=C_1} \\ & \cup \underbrace{\{(\Phi_t(z,\xi),(z,\xi)) \mid t \geq T, \langle\xi, X(z)\rangle = 0\}}_{=C_2} \cup \underbrace{E_u^* \times E_s^*}_{=C_3} \end{aligned}$$

Since $\mathbb{V}^* \cap E_s^*, \mathbb{V}^* \cap E_u^* = \{0\}$, using (2.5.2) together with Lemma 2.5.1, and applying Lemma A.2.7, we see that C_3 does not contribute to the wavefront set of $\pi_{m*}e^{-TX}R_0\pi_m^*$. Since there are no conjugate points (i.e. $d\varphi_t^\top(\mathbb{V}^*) \cap \mathbb{V}^* = \{0\}$ for all $t \neq 0$), C_2 does not contribute neither. Only C_1 contributes to the wavefront set and the sought result follows. We leave it as an exercise for the reader to prove that suppsing $\left(\pi_{m*}\int_\varepsilon^T (1-\chi(t))e^{-tX}dt\pi_m^*\right) \subset \Delta_T$. □

Here is what we have proved : if we go back to the decomposition

$$R_+(\lambda) = \int_0^{2\varepsilon} \chi(t)e^{-\lambda t}e^{-tX}dt + \int_\varepsilon^{+\infty}(1-\chi(t))e^{-\lambda t}e^{-tX}dt,$$

take the finite part at 0 and pre/post-compose with π_{m*}/π_m^*, we obtain that

$$\pi_{m*}R_0\pi_m^* = \pi_{m*}\int_0^{2\varepsilon}\chi(t)e^{-tX}dt\pi_m^* + R_T,$$

where $\mathrm{suppsing}(K_{R_T}) \subset \Delta_T(M\times M)$. Since $T > 2\varepsilon$ was chosen arbitrary, if we go back to the operator Π_m, then we obtain that for any $\varepsilon > 0$:

$$\Pi_m = \pi_{m*}\int_{-\varepsilon}^{+\varepsilon}\chi(t)e^{-tX}dt\pi_m^* + \text{ smoothing},$$

where χ is a cutoff function chosen to be equal to 1 at 0 and 0 outside $(-\varepsilon,\varepsilon)$.

We can now prove Theorem 2.5.1. We will only deal with the case of Π_0 since it is rather similar for higher order tensors but complications arise due to the fact that the rank of $\otimes_S^m T^*M \to M$ is strictly bigger than 1. However, the computation for the principal symbol will be carried out in full generality.

Proof. By the previous discussion, we have to prove that $\pi_{0*}\int_{-\varepsilon}^{\varepsilon}e^{tX}dt\pi_0^*$ is a pseudo-differential operator of order 0, where we can choose $\varepsilon > 0$ small enough, less than the injectivity radius of (M,g). Note that π_{0*} is simply the integration in the fibers S_xM. We fix a local chart (U,φ) and compute everything in this chart. If χ is a cutoff function with support in $\varphi(U)$ such that $e^{tX}(\mathrm{supp}(\chi)) \subset \varphi(U)$ for all $t \in (-\varepsilon,\varepsilon)$, then for $f \in C_c^\infty(\varphi(U))$:

$$\begin{aligned}\left(\chi\pi_{0*}\int_{-\varepsilon}^{\varepsilon}e^{tX}dt\pi_0^*\chi\right)f(x) &= \int_{S_xM}\chi(x)\int_{-\varepsilon}^{\varepsilon}\pi_0^*\chi f(\varphi_t(x,v))dtdv \\ &= 2\int_{S_xM}\chi(x)\int_0^{\varepsilon}\pi_0^*\chi f(\varphi_t(x,v))dtdv\end{aligned}$$

For fixed x, since $\varepsilon > 0$ is smaller than the injectivity radius of (M,g), the map $(t,v) \mapsto \pi_0(\varphi_t(x,v)) = \exp_x(tv)$ is a diffeomorphism from $[0,\varepsilon)\times S_xM$ onto $B(x,\varepsilon)$. By making a change of variable in the previous integral, we obtain

$$\chi\pi_{0*}\int_{-\varepsilon}^{\varepsilon}e^{tX}dt\pi_0^*\chi f(x) = \int_{\varphi(U)}K(x,y)f(y)dy,$$

with $K(x,y) = 2\chi(x)\chi(y)|\det d(\exp_x^{-1})_y|\sqrt{\det g(y)}/d^n(x,y)$. We compute the left symbol

$$p(x,\xi) = \int_{\mathbb{R}^{n+1}}e^{-iz\cdot\xi}K(x,x-z)dz,$$

and we want to prove that $p \in S^{-1}(\mathbb{R}^{n+1}_x \times \mathbb{R}^{n+1}_\xi)$. We write $F(x,z) = K(x, x-z)$. By [Tay11b, Proposition 2.7], this amounts to proving that $F \in S^{-n}_{(0)}(\mathbb{R}^{n+1}_x \times \mathbb{R}^{n+1}_z)$ (see Appendix A), i.e.

$$\forall \alpha, \beta,\ \exists C_{\alpha\beta} > 0, \forall x \in \varphi(U), \forall z \neq 0, \qquad |\partial_x^\beta \partial_z^\alpha F(x,z)| \leq C_{\alpha\beta}|z|^{-n-|\alpha|} \tag{2.5.3}$$

The singularity of F is induced by $(x,z) \mapsto d^{-n}(x, x-z)$ (remark that $F(x,z) \sim_{|z|\to 0} 2\chi(x)^2\sqrt{\det g(x)}|z|^{-n}$) so this boils down to proving (2.5.3) for this function. But by the usual argument relying on Leibniz formula for the derivative of a product, this amounts to proving

$$\forall \alpha, \beta,\ \exists C_{\alpha\beta} > 0, \forall x \in \varphi(U), \forall z \neq 0, \qquad |\partial_x^\beta \partial_z^\alpha d^n(x,z)| \leq C_{\alpha\beta}|z|^{n-|\alpha|}.$$

But this is a rather immediate consequence of the fact that in local coordinates, there exist smooth functions $(G^{ij})_{1\leq i,j\leq n+1}$ defined in the patch $\varphi(U)$ such that $d^2(x, x-z) = \sum_{i,j} G^{ij}(x, x-z) z_i z_j$. Combining everything, we obtain that $p \in S^{-1}(\mathbb{R}^{n+1}_x \times \mathbb{R}^{n+1}_\xi)$ so Π_0 is a pseudodifferential operator of order -1. The same arguments allow to show that Π_m is also a ΨDO of order -1 for any $m \geq 0$.

We now compute the principal symbol of Π_m. Let us consider a smooth section $f_1 \in C^\infty(M, \otimes_S^m T^*M)$ defined in a neighborhood of $x \in M$ and $f_2 \in \otimes_S^m T_x^*M$, then :

$$\begin{aligned} \langle \sigma_m(x_0,\xi) f_1, f_2\rangle_{x_0} &= \lim_{h\to 0} h^{-1} e^{-iS(x_0)/h} \langle \Pi_m(e^{iS(x)/h} f_1), f_2\rangle_{x_0} \\ &= \lim_{h\to 0} h^{-1} e^{-iS(x_0)/h} \langle \Pi \pi_m^*(e^{iS(x)/h} f_1), \pi_m^* f_2\rangle_{L^2(S_{x_0}M)}, \end{aligned}$$

where $\xi = dS(x) \neq 0$. Here, it is assumed that $\mathrm{Hess}_x S$ is non-degenerate. According to the previous paragraph, we can only consider the integral in time between $(-\varepsilon, \varepsilon)$, where $\varepsilon > 0$ is chosen small enough (less than the injectivity radius at the point x), namely :

$$\begin{aligned} &\langle \sigma_m(x,\xi) f_1, f_2\rangle_{x_0} \\ &= \lim_{h\to 0} h^{-1} \int_{\mathbb{S}^n} \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t)) - S(x))} \pi_m^* f_1(\gamma(t), \dot\gamma(t)) \pi_m^* f_2(x_0, v)\chi(t) dt dv \\ &= \lim_{h\to 0} h^{-1} \int_{\mathbb{S}^{n-1}} \left(\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t)) - S(x))} \pi_m^* f_1(\gamma(t), \dot\gamma(t)) \pi_m^* f_2(x_0, v) \sin^{n-1}(\varphi)\chi(t) dt d\varphi \right) du \end{aligned}$$

where χ is a cutoff function with support in $(-\varepsilon,\varepsilon)$, γ is the geodesic such that $\gamma(0) = x, \dot\gamma(0) = v$ and we have decomposed $v = \cos(\varphi) w + \sin(\varphi) u$ with $w = \xi^\sharp/|\xi| = dS(x)^\sharp/|dS(x)|$, $u \in \mathbb{S}^{n-1}$. We apply the stationnary phase lemma [Zwo12, Theorem 3.13] uniformly in the $u \in \mathbb{S}^{n-1}$ variable. For fixed u, the phase is $\Phi : (t,\varphi) \mapsto S(\gamma(t)) - S(x)$ so $\partial_t \Phi(t,\varphi) = dS(\dot\gamma(t))$. More generally if $\tilde\Phi : (t,v) \mapsto S(\gamma(t)) - S(x)$, then

$$\partial_v \tilde\Phi(t,v) \cdot V = d\pi(d\varphi_t(x,v) \cdot V), \qquad \forall V \in \mathbb{V}.$$

Since (M,g) has no conjugate points, $d\pi(d\varphi_t(x,v)) \cdot V \neq 0$ as long as $t \neq 0$ and $V \in \mathbb{V} \setminus \{0\}$. And $dS(\dot\gamma(0)) = dS(\cos(\varphi) w + \sin(\varphi) u) = \cos(\varphi)|dS(x)| = 0$ if and only if $\varphi = \pi/2$. So the only critical point of Φ is $(t = 0, \varphi = \pi/2)$. Let us also remark that

$$\mathrm{Hess}_{(0,\pi/2)} \Phi = \begin{pmatrix} \mathrm{Hess}_x S(u,u) & -|dS(x)| \\ -|dS(x)| & 0 \end{pmatrix}$$

is non-degenerate with determinant $-|\xi|^2$, so the stationnary phase lemma can be applied and we get :

$$\int_0^\pi \int_{-\varepsilon}^{+\varepsilon} e^{i/h(S(\gamma(t))-S(x_0))} \pi_m^* f_1(\gamma(t),\dot\gamma(t)) \pi_m^* f_2(x_0,v) \sin^{n-1}(\varphi) dt d\varphi$$
$$\sim_{h\to 0} 2\pi h |\xi|^{-1} \pi_m^* f_1(x_0,u) \pi_m^* f_2(x_0,u).$$

Eventually, we obtain :

$$\langle \sigma_m(x,\xi) f_1, f_2 \rangle_{x_0} = \frac{2\pi}{|\xi|} \int_{\{\langle \xi, v\rangle = 0\}} \pi_m^* f_1(v) \pi_m^* f_2(v) dS_\xi(v),$$

where dS_ξ is the canonical measure induced on the $n-1$-dimensional sphere $\mathbb{S}_x M \cap \{\langle \xi, v \rangle = 0\}$. The sought result then follows from Lemma B.1.1. □

2.5.2 Properties of the normal operator on solenoidal tensors

Ellipticity. The crucial property of the normal operator Π_m is that it is elliptic on solenoidal tensors. This was the reason for Guillarmou to introduce it in [Gui17a].

Lemma 2.5.3. *The operator Π_m is elliptic on solenoidal tensors, that is there exists pseudodifferential operators Q and R of respective order 1 and $-\infty$ such that :*

$$Q\Pi_m = \pi_{\ker D^*} + R$$

Proof. We define

$$\tilde q(x,\xi) = \begin{cases} 0, & \text{on } \operatorname{ran}(\sigma j_\xi) \\ \dfrac{C_{n,m}}{2\pi} |\xi| (\pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi})^{-1}, & \text{on } \ker(i_\xi) \end{cases}$$

and $q(x,\xi) = (1-\chi(x,\xi))\tilde q(x,\xi)$ for some cutoff function $\chi \in C_c^\infty(T^*M)$ around the zero section. By construction, $\mathrm{Op}(q)\Pi_m = \pi_{\ker D^*} - R'$ with $R' \in \Psi^{-1}$. Let $r' = \sigma_{R'}$ and define $a \sim \sum_{k=0}^\infty r'^k$. Then $\mathrm{Op}(a)$ is a microlocal inverse for $\mathbb{1} - R'$ that is $\mathrm{Op}(a)(\mathbb{1}-R') \in \Psi^{-\infty}$. Since $R'D = 0$, we obtain that $R' = R'\pi_{\ker D^*}$ and thus

$$\underbrace{\mathrm{Op}(a)\,\mathrm{Op}(q)}_{=Q} \Pi_m = \mathrm{Op}(a)(\mathbb{1} - R')\pi_{\ker D^*} = \pi_{\ker D^*} + R,$$

where R is a smoothing operator. □

Injectivity. The next lemma shows that the s-injectivity of the X-ray transform is equivalent to that of the normal operator Π_m :

Lemma 2.5.4. *I_m is solenoidal injective if and only if Π_m is injective on the space $H^s_{\mathrm{sol}}(M, \otimes_S^m T^*M)$, for all $s \in \mathbb{R}$.*

Proof. There is a trivial implication : s-injectivity of Π_m implies that of I_m. Indeed, assume $f \in C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M)$ is such that $I_m f = 0$, then $\pi_m^* f = Xu$ for some $u \in C^\infty(SM)$ by the smooth Livsic theorem. But then $\Pi_m f = \pi_{m*}\Pi\pi_m^* f = \pi_{m*}\Pi X u = 0$ by Theorem 2.4.2. Thus $f = 0$.

Let us now prove the converse. We fix $s \in \mathbb{R}$. We assume that $\Pi_m f = 0$ for some $f \in H^s_{\mathrm{sol}}(M, S^m(T^*M))$. By ellipticity of the operator, we get that $f \in C^\infty_{\mathrm{sol}}(M, S^m(T^*M))$. And :

$$\begin{aligned}\langle \Pi_m f, f\rangle_{L^2} &= \langle \Pi \pi_m^* f, \pi_m^* f\rangle_{L^2} + \left(\int_{SM} \pi_m^* f d\mu\right)^2 \\ &= \langle (-\Delta+1)^{-s} \Pi \pi_m^* f, \pi_m^* f\rangle_{H^s} + \left(\int_{SM} \pi_m^* f d\mu\right)^2 = 0.\end{aligned}$$

By Lemma 2.4.5, since $\langle \Pi \pi_m^* f, \pi_m^* f\rangle \geq 0$, we obtain that $\int_{SM} \pi_m^* f d\mu = 0$. Moreover, $(-\Delta+1)^{-s}\Pi$ is bounded and positive on H^s so there exists a square root $R : H^s \to H^s$, that is a bounded positive operator satisfying $(-\Delta+1)^{-s}\Pi = R^*R$, where R^* is the adjoint on H^s. Then :

$$\langle (-\Delta+1)^{-s} \Pi \pi_m^* f, \pi_m^* f\rangle_{H^s} = 0 = \|R\pi_m^* f\|^2_{H^s}$$

This yields $(-\Delta+1)^{-s}\Pi\pi_m^* f = 0$ so $\Pi \pi_m^* f = 0$. By Theorem 2.4.2, there exists $u \in C^\infty(SM)$ such that $\pi_m^* f = Xu$ so $f \in \ker I_m \cap \ker D^*$. By s-injectivity of the X-ray transform, we get $f \equiv 0$. □

In particular, the previous lemma directly implies the following

Proposition 2.5.1. *Let (M, g) be a smooth Anosov Riemannian manifold. Then, the kernel of I_m on $C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is finite dimensional.*

Proof. By Lemma 2.5.4, s-injectivity of I_m is equivalent to that of Π_m, which is elliptic on solenoidal tensors. □

Another direct consequence of Lemma 2.5.4 and Theorem 2.5.3 is the

Theorem 2.5.2. *If I_m is solenoidal injective, then there exists a pseudodifferential operator Q' of order 1 such that : $Q'\Pi_m = \pi_{\ker D^*}$.*

Proof. The operator Π_m is elliptic of order -1 on $\ker D^*$, thus Fredholm as an operator $H^s_{\mathrm{sol}}(M, \otimes^m_S T^*M) \to H^{s+1}_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ for all $s \in \mathbb{R}$. It is selfadjoint on the space $H^{-1/2}_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, thus Fredholm of index 0 (the index being independent of the Sobolev space considered, see [Shu01, Theorem 8.1]), and injective, thus invertible on $H^s_{\mathrm{sol}}(M, \otimes^m_S T^*M)$. We multiply the equality $Q\Pi_m = \pi_{\ker D^*} + R$ on the right by $Q' := \pi_{\ker D^*} \Pi_m^{-1} \pi_{\ker D^*}$:

$$Q\Pi_m Q' = Q \underbrace{\Pi_m \pi_{\ker D^*}}_{=\Pi_m} \Pi_m^{-1} \pi_{\ker D^*} = Q\pi_{\ker D^*} = Q' + RQ'$$

As a consequence, $Q' = Q\pi_{\ker D^*} + \text{smoothing}$ so it is a pseudodifferential operator of order 1. And $Q'\Pi_m = \pi_{\ker D^*}$. □

This yields the following stability estimate :

Lemma 2.5.5. *If I_m is solenoidal injective, then for all $s \in \mathbb{R}$, there exists a constant $C := C(s) > 0$ such that :*

$$\forall f \in H^s_{\mathrm{sol}}(M, \otimes^m_S T^*M), \qquad \|f\|_{H^s} \leq C\|\Pi_m f\|_{H^{s+1}}$$

We also obtain a coercivity lemma on the operator Π_m.

Lemma 2.5.6. *If I_m is solenoidal injective, then there exists a constant $C > 0$ such that :*

$$\forall f \in H^{-1/2}(M, \otimes_S^m T^*M), \qquad \langle \Pi_m f, f\rangle \geq C\|\pi_{\ker D^*} f\|^2_{H^{-1/2}}.$$

Proof. The operator $\pi_{m*}\pi_m^* : \otimes_S^m T_x^*M \to \otimes_S^m T_x^*M$ is positive definite and thus admits a square root $S(x) : \otimes_S^m T_x^*M \to \otimes_S^m T_x^*M$, self-adjoint and such that $S^m(x) = \pi_{m*}\pi_m^*$. We introduce the symbol $b \in C^\infty(T^*M)$ of order $-1/2$ defined by $b : (x,\xi) \mapsto \chi(x,\xi)|\xi|^{-1/2}S(x)$, where $\chi \in C^\infty(T^*M)$ vanishes near the 0 section in T^*M and equal to 1 for $|\xi| > 1$ and define $B := \mathrm{Op}(b) \in \Psi^{-1/2}(M, \otimes_S^m T^*M)$, where Op is a quantization on M. Using that the principal symbol of $\pi_{\ker D^*}$ is $i_{\xi^\sharp}$, the inner product with $\xi^\sharp$, we observe that $\Pi_m = \pi_{\ker D^*}B^*B\pi_{\ker D^*} + R$, where $R \in \Psi^{-2}(M, \otimes_S^m T^*M)$. Thus, given $f \in H^{-1/2}(M, \otimes_S^m T^*M)$:

$$\langle \Pi_m f, f\rangle_{L^2} = \|B\pi_{\ker D^*} f\|^2_{L^2} + \langle Rf, f\rangle_{L^2} \tag{2.5.4}$$

By ellipticity of B, there exists a pseudodifferential operator Q of order $1/2$ such that $QB\pi_{\ker D^*} = \pi_{\ker D^*} + R'$, where $R' \in \Psi^{-\infty}(M, \otimes_S^m T^*M)$ is smoothing. Thus there is $C > 0$ such that for each $f \in C^\infty(M, \otimes_S^m T^*M)$

$$\|\pi_{\ker D^*} f\|^2_{H^{-1/2}} \leq \|QB\pi_{\ker D^*} f\|^2_{H^{-1/2}} + \|R'f\|^2_{H^{-1/2}} \leq C\|B\pi_{\ker D^*} f\|^2_{L^2} + \|R'f\|^2_{H^{-1/2}}.$$

Since Lemma 2.5.6 is trivial on potential tensors, we can already assume that f is solenoidal, that is $\pi_{\ker D^*} f = f$. Reporting in (2.5.4), we obtain that

$$\begin{aligned} \|f\|^2_{H^{-1/2}} &\leq C\langle \Pi_m f, f\rangle_{L^2} - C\langle Rf, f\rangle_{L^2} + \|R'f\|^2_{H^{-1/2}} \\ &\leq C\langle \Pi_m f, f\rangle_{L^2} + C\|Rf\|_{H^{1/2}}\|f\|_{H^{-1/2}} + \|R'f\|^2_{H^{-1/2}}. \end{aligned} \tag{2.5.5}$$

Now, assume by contradiction that the statement in Lemma 2.5.6 does not hold, that is we can find a sequence of tensors $f_n \in C^\infty(M, \otimes_S^m T^*M)$ such that $\|f_n\|_{H^{-1/2}} = 1$ with $D^* f_n = 0$ and

$$\|\sqrt{\Pi_m} f_n\|^2_{L^2} = \langle \Pi_m f_n, f_n\rangle_{L^2} \leq \|f_n\|^2_{H^{-1/2}}/n = 1/n \to 0.$$

Up to extraction, and since R is of order -2, we can assume that $Rf_n \to v_1$ in $H^{1/2}$ for some v_1, and $R'f_n \to v_2$ in $H^{-1/2}$. Then, using (2.5.5), we obtain that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1/2}$ which thus converges to an element $v_3 \in H^{-1/2}$ such that $\|v_3\|_{H^{-1/2}} = 1$ and $D^*v_3 = 0$. By continuity, $\Pi_m f_n \to \Pi_2 v_3$ in $H^{1/2}$ and thus $\langle \Pi_2 v_3, v_3\rangle = 0$. Since v_3 is solenoidal, we get $\sqrt{\Pi_m} v_3 = 0$, thus $\Pi_2 v_3 = 0$. Note that I_m is s-injective by assumption, thus Π_m is also injective by Lemma 2.5.4. This implies that $v_3 \equiv 0$, thus contradicting $\|v_3\|_{H^{-1/2}} = 1$. □

Now, assume that the operator $\Pi_m = \Pi_m^g$ is a continuous family as a map between Fréchet spaces

$$g \in \mathcal{U}_{g_0} \mapsto \Pi_m^g \in \Psi^{-1}(M, \otimes_S^m T^*M)$$

where $\mathcal{U}_{g_0} \subset C^\infty(M, \otimes_S^m T^*M)$ is a neighborhood of g_0, a fixed Anosov metric, and the topology on the Fréchet space $\Psi^{-1}(M, \otimes_S^m T^*M)$ is that detailed in (A.1.5) (the topology on $\Psi^{-1}(M, \otimes_S^m T^*M)$ is given by the semi-norms of the symbols, as usual). This will be proved in Section §2.6 (see Theorem 2.6.1). As a consequence, we obtain the

Lemma 2.5.7. *Let (M, g_0) be a smooth compact Anosov Riemannian manifold with nonpositive curvature. There exists a C^∞ neighborhood $\mathcal{U}_{g_0}$ of g_0 and a constant $C > 0$ such that for all $g \in \mathcal{U}_{g_0}$ and all tensors $f \in H^{-1/2}(M, \otimes_S^m T^*M)$,*

$$\langle \Pi_m^g f, f\rangle_{L^2} \geq C\|\pi_{\ker D_g^*} f\|^2_{H^{-1/2}(M)}.$$

Proof. Let g_0 be fixed Anosov metric with non-positive curvature and let $g \in \mathcal{M}$ be a smooth metric in a C^∞-neighborhood $\mathcal{U}_{g_0}$ of g_0. We choose $\mathcal{U}_{g_0}$ small enough so that g is Anosov. Let $f \in \ker D_g^*$ be a solenoidal (with respect to g) symmetric m-tensor, then $f = \pi_{\ker D_g^*} f$. Let $C_{g_0} > 0$ be the constant provided by Lemma 2.5.6 applied to the metric g_0. We choose $\mathcal{U}_{g_0}$ small enough so that $\|\Pi_m^g - \Pi_m^{g_0}\|_{H^{-1/2}\to H^{1/2}} \leq C_{g_0}/3$ (this is made possible by the continuity of $g \mapsto \Pi_m^g \in \Psi^{-1}$). Then :

$$\langle \Pi_m^g f, f\rangle = \langle (\Pi_m^g - \Pi_m^{g_0})f, f\rangle + \langle \Pi_m^{g_0} f, f\rangle \geq C_{g_0}\|\pi_{\ker D_{g_0}^*} f\|_{H^{-1/2}}^2 - C_{g_0}/3\|f\|_{H^{-1/2}}^2$$

But the map $\mathcal{U}_{g_0} \ni g \mapsto \pi_{\ker D_g^*} = \mathbb{1} - D_g\Delta_g^{-1}D_g^* \in \Psi^0$ is continuous, where $\Delta_g := D_g^* D_g$ is the Laplacian on $(m-1)$-tensors, and this implies that for g in a possibly smaller neighborhood $\mathcal{U}_{g_0}$ of g_0, using $f = \pi_{\ker D_g^*} f$:

$$\langle \Pi_m^g f, f\rangle \geq C_{g_0}\|\pi_{\ker D_g^*} f\|_{H^{-1/2}}^2 - 2/3 \times C_{g_0}\|f\|_{H^{-1/2}}^2 = C_{g_0}/3\|\pi_{\ker D_g^*} f\|_{H^{-1/2}}^2$$

□

Of course, more generally, the previous lemma is valid as long as g_0 is Anosov and $I_m^{g_0}$ is known to be injective. As mentioned earlier, an immediate consequence of the previous lemma is the following

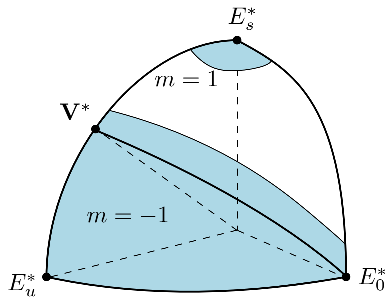

FIGURE 2.5 – The order function on $S^*\mathcal{M}$.

Proposition 2.5.2. *Let (M, g_0) be a smooth Riemannian $(n+1)$-dimensional Anosov manifold which is non-positively curved. Then, there exists a C^∞-neighborhood $\mathcal{U}_{g_0}$ of g_0 in $\mathcal{M}$ such that for any $g \in \mathcal{U}_{g_0}$, for any $m \in \mathbb{N}$, I_m^g is s-injective.*

Of course, the result is only interesting when $n+1 \geq 3, m \geq 2$, the other cases being covered in full generality by the literature.

Proof. By Lemma 2.5.4, the s-injectivity of I_m^g is equivalent to that of Π_m^g and the previous Lemma allows to conclude. □

Surjectivity. The normal operator Π_m is formally self-adjoint, elliptic on solenoidal tensors and is thus Fredholm of index 0. As a consequence, Π_m is injective on solenoidal tensors if and only if it is surjective. We can even be more precise on this statement :

Lemma 2.5.8. *I_m is solenoidal injective if and only if*

$$\pi_{m*} : C^{-\infty}_{\mathrm{inv}}(SM) \to C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M)$$

is surjective.

Here, $C^{-\infty}_{\mathrm{inv}}(SM) = \cup_{s\leq 0}, H^{-s}_{\mathrm{inv}}(SM)$ denotes the distributions which are invariant by the geodesic flow. We note that this lemma was first stated in the literature in the case of simple manifolds [PZ16].

Proof. Assume that $\pi_{m*} : C^{-\infty}_{\mathrm{inv}}(SM) \to C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M)$ is surjective. Let $f \in C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^*M)$ be such that $I_m f = 0$. Then $\pi_m^* f = Xu$ for some $u \in C^\infty(SM)$ by the smooth Livsic theorem and $f = \pi_{m*}h$ for some $h \in C^{-\infty}_{\mathrm{inv}}(SM)$ by assumption. Then :

$$0 = \langle Xh, u\rangle = -\langle h, Xu\rangle = -\langle h, \pi_m^* f\rangle = -\langle \pi_{m*}h, f\rangle = -\|f\|^2$$

Thus $f \equiv 0$.

We now prove the converse. If I_m is s-injective, then Π_m is s-injective and thus surjective on solenoidal tensors. Thus, given $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, there exists $u \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ such that $f = \Pi_m u = \pi_{m*}\Pi\pi_m^* u$, that is $f = \pi_{m*}h$ for $h = \Pi\pi_m^* u \in \cap_{s>0} H^{-s}(SM)$. □

The surjectivity of π_{m*} is described with more details in Appendix B. In particular, in the case of Anosov Riemannian manifolds with nonpositive curvature, it is known to be surjective by some construction known as the Beurling transform. It is still an open question in full generality without any assumption on the curvature, just like is the s-injectivity of I_m.

Boundedness. Eventually, we will need this last lemma :

Lemma 2.5.9. $\Pi\pi_m^* : H^{-s}(M, \otimes^m_S T^*M) \to H^{-s}(SM)$ *is bounded, for any* $s > 0$. *By duality,* $\pi_{m*}\Pi : H^s(SM) \to H^s(M, \otimes^m_S T^*M)$ *is bounded too, for any* $s > 0$.

Idea of proof. We only prove the first part of the statement since the other follows by duality and we actually restrict ourselves to proving that $R_0\pi_m^* : H^{-s}(M, \otimes^m_S T^*M) \to H^{-s}(SM)$ is bounded. The proof can be done directly by constructing a relevant escape function.

Indeed, the order function m introduced in Lemma 2.4.1 can always be changed so that $m \equiv -1$ on a slightly larger domain, namely on a domain encapsulating $\kappa(\mathbb{V}^*) \cup \kappa(E_u^*)$ (see Figure 2.5), where $\kappa : T^*(SM) \to S^*(SM)$ is the projection. If $f \in H^{-s}(M, \otimes^m_S T^*M)$, then by Lemma 2.5.1, $\mathrm{WF}(\pi_m^* f) \subset \mathbb{V}^*$ and thus $\pi_m^* f$ is microlocally H^{-s} at $\mathbb{V}^*$ and smooth outside $\mathbb{V}^*$. So $\pi_m^* f \in \mathcal{H}^s_+$, where the anisotropic space space is built by using the order function m described in Figure 2.5. Since $R_0 : \mathcal{H}^s_+ \to \mathcal{H}^s_+ \subset H^{-s}(SM)$ is bounded, we obtain the sought result.

Another way of proving this lemma is to use the radial source estimate (see Theorem A.4.2). We will rather prove that $\pi_{m*}R_0 : H^s(SM) \to H^s(M, \otimes^m_S T^*M)$ is bounded, only using the fact that $R_0 : H^s(SM) \to H^{-s}(SM)$ is bounded — which follows from Theorem 2.4.1. Since π_{m*} is smoothing outside $\mathbb{V}^*$, it is sufficient to prove that given $f \in H^s(SM)$, $R_0 f$ is microlocally H^s in a neighborhood of $\mathbb{V}^*$. In other words, if $A \in \Psi^0(SM)$ is microlocally supported near $\mathbb{V}^*$, it is sufficient to prove that $\pi_{m*}AR_0 : H^s(SM) \to H^s(SM)$ is bounded because

$$\pi_{m*}R_0 = \pi_{m*}AR_0 + \pi_{m*}(\mathbb{1} - A)R_0$$

and the last term is immediately smoothing.

Let $f \in H^s(SM)$, we set $u := R_0 f$ and thus $Xu = f$. Note that $\|u\|_{H^{-s}} \lesssim \|f\|_{H^s}$ and we already know — by construction of the anisotropic Sobolev spaces — that u is microlocally H^s in a neighborhood of E_s^*. We first apply the source estimate : let $B_1 \in \Psi^0(SM)$ be elliptic near E_s^*. Then, there exists $A_1 \in \Psi^0(SM)$, elliptic near E_s^* such that :

$$\|A_1 u\|_{H^s} \lesssim \|B_1 f\|_{H^s} + \|u\|_{H^{-s}} \lesssim \|f\|_{H^s} \tag{2.5.6}$$

Consider $A_2 \in \Psi^0(SM)$, elliptic near $\mathbb{V}^* \cap \Sigma$, B_2 elliptic in a conic neighborhood of Σ such that $\mathrm{WF}(A_2) \subset \mathrm{ell}(B_2)$. This choice can be done so that for any $(x, \xi) \in \mathrm{ell}(A_2)$, there exits a time $T \geq 0$ such that $\Phi_{-T}(x, \xi) \in \mathrm{ell}(A_1)$. Applying the classical propagation estimate (see Theorem A.4.1), we obtain :

$$\|A_2 u\|_{H^s} \lesssim \|A_1 u\|_{H^s} + \|B_2 f\|_{H^s} + \|u\|_{H^{-s}} \lesssim \|f\|_{H^s},$$

by inserting (2.5.6) in the last inequality. Consider $A_3, B_3 \in \Psi^0(SM)$ elliptic everywhere outside a neighborhood of $\sigma = \{\langle \xi, X(x)\rangle = 0\}$ such that $\mathrm{WF}(A_3) \subset \mathrm{ell}(B_3)$. By ellipticity of the operator X outside Σ, one has :

$$\|A_3 u\|_{H^s} \lesssim \|B_3 f\|_{H^{s-1}} + \|u\|_{H^{-s}} \lesssim \|f\|_{H^s}$$

If we define $A := A_2 + A_3$, the previous construction can always be done so that A is elliptic in a neighborhood of $\mathbb{V}^*$. Thus $\|AR_0 f\|_{H^s} \lesssim \|f\|_{H^s}$ and this concludes the proof.

□

2.5.3 Stability estimates for the X-ray transform

Before going on with the proof of Theorem 2.1.4, let us recall the definition *Hölder-Zygmund spaces*. Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function with support in $[-2, 2]$ and such that $\psi \equiv 1$ on $[-1, 1]$. For $j \in \mathbb{N}$, we introduce the functions $\varphi_j \in C_c^\infty(T^*M)$ defined by $\varphi_0(x, \xi) := \psi(|\xi|)$, $\varphi_j(x, \xi) := \psi(2^{-j}|\xi|) - \psi(2^{-j+1}|\xi|)$, for $j \geq 1$ with $(x, \xi) \in T^*M$, $|\cdot|$ being the norm induced by g on the cotangent bundle. Since φ_j is a symbol in $S^{-\infty}$, one observes that the operators $\mathrm{Op}(\varphi_j)$ are smoothing.

For $s \in \mathbb{R}$, we define $C_*^s(M)$, the *Hölder-Zygmund space of order s* as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{C_*^s} := \sup_{j \in \mathbb{N}} 2^{js} \| \mathrm{Op}(\varphi_j) u\|_{L^\infty},$$

and we recall (see [Tay91, Appendix A, A.1.8] for instance) that a pseudodifferential operator $P \in \Psi^m(M)$ of order $m \in \mathbb{R}$ is bounded as an operator $C_*^{s+m}(M) \to C_*^s(M)$, for all $s \in \mathbb{R}$. Note that the previous definition of Hölder-Zygmund spaces can be easily generalized to sections of a vector bundle. When $s \in (0, 1)$, it is a well-known fact that the space $C_*^s(M)$ coincide with $C^s(M)$, the space of Hölder-continuous functions, with equivalent norms $\|u\|_{C_*^s} \asymp \|u\|_{C^s}$. The Hölder-Zygmund spaces correspond to the Besov spaces $B_{q,r}^s(M)$ with $q = r = +\infty$ while the Sobolev spaces $H^s(M)$ correspond to the choice $q = r = 2$. Here :

$$\|u\|_{B_{q,r}^s} := \left(\sum_{j=0}^{+\infty} \|2^{sj} \mathrm{Op}(\varphi_j) u\|_{L^q}^r \right)^{1/r}$$

In particular, Lemma 2.5.5 can be upgraded to :

Lemma 2.5.10. *For all $s \in \mathbb{R}$, there exists a constant $C := C(s) > 0$ such that :*

$$\forall f \in C_{*,\mathrm{sol}}^s(M, \otimes_S^m T^*M), \qquad \|f\|_{C_*^s} \leq C \|\Pi_m f\|_{C_*^{s+1}}$$

Eventually, we will need this last result :

Lemma 2.5.11. *For all $s > 0$, the operator $\Pi : C_*^s(SM) \to C_*^{-s-(n+1)/2}(SM)$ is bounded.*

Proof. Fix $\varepsilon > 0$ small enough. Then :

$$C_*^s \hookrightarrow H^{s-\varepsilon} \xrightarrow{\Pi} H^{-s+\varepsilon} \hookrightarrow C_*^{-s-(n+1)/2+\varepsilon} \hookrightarrow C_*^{-s-(n+1)/2},$$

by Sobolev embeddings.

□

We can now deduce from the previous work the stability estimate of Theorem 2.1.4.

Proof of Theorem 2.1.4. We assume that $f \in C^\alpha_{\mathrm{sol}}(M, \otimes^m_S T^*M)$ is such that $\|f\|_{C^\alpha} \leq 1$. By Theorem 2.1.3, we can write $\pi_m^* f = Xu + h$, with $u, Xu, h \in C^{\alpha'}$, where $0 < \alpha' < \alpha$.
We have :

$$\begin{aligned}
\|f\|_{C_*^{-1-\alpha'-(n+1)/2}} &\lesssim \|\Pi_m f\|_{C_*^{-\alpha'-(n+1)/2}} && \text{by Lemma 2.5.10} \\
&\lesssim \|\Pi \pi_m^* f\|_{C_*^{-\alpha'-(n+1)/2}} \\
&\lesssim \|\Pi(Xu+h)\|_{C_*^{-\alpha'-(n+1)/2}} \\
&\lesssim \|\Pi h\|_{C_*^{-\alpha'-(n+1)/2}} \\
&\lesssim \|h\|_{C^{\alpha'}} && \text{by Lemma 2.5.11} \\
&\lesssim \|I_m f\|^\nu_{\ell^\infty} && \text{by Theorem 2.1.3}
\end{aligned}$$

Using $\|f\|_{C^\alpha} \leq 1$ and interpolating C^β between $C^{-1-\alpha'-(n+1)/2}$ and C^α, one obtains the sought result.

□

2.6 Continuity of the normal operator with respect to the metric

In this section, we prove that the normal operator $\Pi_2(g) \in \Psi^{-1}$ depends continuously on the metric g as an operator in Ψ^{-1}.

Theorem 2.6.1. *The map* $\mathrm{An} \ni g \mapsto \Pi_2(g) \in \Psi^{-1}$ *is continuous as a map between Fréchet spaces.*

Here $\mathrm{An} \subset \mathrm{Met} = C^\infty(M, \otimes^2_{S,+} T^*M)$ is the open subset of Anosov metrics in the cone of metrics (which is an open set of the Fréchet space $C^\infty(M, \otimes^2_S T^*M)$ endowed with the usual semi-norms in coordinates). The space Ψ^{-1} is endowed with the topology of a Fréchet space, the semi-norms being defined by (A.1.5).

We fix an Anosov metric g_0 and we work in a neighborhood $\mathcal{U}_{g_0}$ of g_0 in the C^∞ topology. In particular, we will always assume that this neighborhood $\mathcal{U}_{g_0}$ is small enough so that any $g \in \mathcal{U}_{g_0}$ has an Anosov geodesic flow that is orbit-conjugated to that of g_0. We will also see the geodesic flows $(\varphi_t^g)_{t \in \mathbb{R}}$ as acting on the unit bundle $\mathcal{M} := SM_{g_0}$ for g_0 by using the natural identification $SM_g \to SM_{g_0}$ obtained by scaling in the fibers. The operator π_2^* associated to g becomes for $(x,v) \in SM_{g_0}$

$$\pi_2^* h(x,v) = h_x(v,v)|v|_g^{-2},$$

if $h \in C^\infty(M, \otimes^2_S T^*M)$. It is just a scaling times the π_2^* associated to g_0. For the sake of simplicity, we will write π_2^* and π_{2*} without insisting on the dependence on g.

The operator $\Pi_2(g)$ can be decomposed as

$$\begin{aligned}
\Pi_2(g) = \pi_{2*} &\int_{-\infty}^{\infty} \chi(t) e^{-tX(g)} dt \pi_2^* \\
&- \pi_{2*} R_0^+(g) \int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* - \pi_{2*} R_0^-(g) \int_{-\infty}^0 \chi'(t) e^{-tX(g)} dt \pi_2^* \\
&+ \left(1 - \int_{-\infty}^{+\infty} \chi(t) dt\right) \pi_{2*}(g) \langle \cdot, \mu_g^{\mathrm{L}} \rangle \; \pi_2^*,
\end{aligned} \tag{2.6.1}$$

where $\chi \in C_c^\infty(\mathbb{R})$ is a smooth cutoff function such that $\chi \equiv 1$ around 0 and there exists $T > 0$ such that χ is supported in $[-T-1, T+1]$ and χ' is supported in $[-T-1, -T] \cup [T, T+1]$. We will choose T large enough in the end. Note that we now see the operator π_2^* as a map $C^\infty(M, \otimes_S^2 T^*M) \to C^\infty(\mathcal{M})$. The first term on the right-hand side of the equality (2.6.1) carries all the microlocal properties of the operator $\Pi_2(g)$, the three other terms on the remaining lines are all smoothing (see [Gui17a, GL19a]). We will actually prove the following propositions which, in turn, imply Theorem 2.6.1.

Proposition 2.6.1. *The map*

$$\text{Met} \ni g \mapsto \pi_{2*} \int_{-\infty}^{+\infty} \chi(t) e^{-tX(g)} dt \pi_2^* \in \Psi^{-1}$$

is continuous.

For the sake of simplicity, we will only deal with the case of the operator Π_0 acting on functions. The arguments are similar for tensors of higher order but more tedious to write.

Proposition 2.6.2. *For $N \in \mathbb{N} \setminus \{0\}$ large enough, the map*

$$\text{An} \ni g \mapsto \pi_{2*} R_0^\pm(g) \int_0^{\pm\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, H^N)$$

is continuous. In particular, this implies that

$$\text{An} \ni g \mapsto \pi_{2*} R_0^\pm(g) \int_0^{\pm\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in C^\infty(M \times M, \otimes_S^2 T^*M \otimes (\otimes_S^2 T^*M)')$$

is continuous if we identity the operator and its Schwartz kernel.

2.6.1 Continuity of the microlocal part

In this paragraph, we prove Proposition 2.6.1. As mentioned previously, for the sake of simplicity, we will only deal with the operator Π_0. We thus have to prove that the map

$$\text{Met} \ni g \mapsto \pi_{0*} \int_{-\infty}^{+\infty} \chi(t) e^{-tX(g)} dt \pi_0^* =: T(g) \in \Psi^{-1}$$

is continuous.

Proof of Proposition 2.6.1. Given $f \in C^\infty(M)$, the map $T(g)$ can be written as :

$$T(g)f(x) = 2 \int_{S_xM} \int_0^{T+1} f(\exp_x^g(tv)) \chi(t) \mathrm{d}t \mathrm{d}S_x(v)$$

We introduce a partition of unity $\sum_{i=1}^K \Psi_i = \mathbf{1}$ of $\mathcal{M} = SM_{g_0}$ and a partition of unity $\sum_{j=1}^L \Phi_j = \chi(t)$ on $[0, T+1]$ such that for all $i \in \{1, ..., K\}, j \in \{1, ..., L\}$, the map

$$\text{supp}(\Psi_i) \times \text{supp}(\Phi_j) \ni (t, (x, v)) \mapsto \exp_x^g(tv)$$

is a local diffeomorphism (which is possible since the metrics do not have conjugate points). Since this is a C^1-open condition in the metric g, this can be done uniformly for $g \in \mathcal{U}$ in a C^1-neighborhood of g_0. The function Φ_1 is chosen to have support in $[0, \varepsilon)$ where $\varepsilon > 0$ is less than a third of the injectivity radius of g_0. Thus :

$$\begin{aligned}
T(g)f(x) &= 2\sum_{i,j}\int_{S_xM}\int_0^{T+1} f(\exp_x^g(tv))\Psi_i(x,v)\Phi_j(t)\mathrm{d}t\mathrm{d}S_x(v) \\
&= 2\int_{S_xM}\int_0^{\varepsilon} f(\exp_x^g(tv))\Phi_1(t)\mathrm{d}t\mathrm{d}S_x(v) \\
&\quad + 2\sum_{i,j\neq 1}\int_{S_xM}\int_0^{T+1} f(\exp_x^g(tv))\Psi_i(x,v)\Phi_j(t)\mathrm{d}t\mathrm{d}S_x(v) \\
&= \int_M K_g(x,y)f(y)\mathrm{d}\,\mathrm{vol}_{g_0}(y) + \sum_{i,j\neq 1}\int_M K_g^{i,j}(x,y)f(y)\mathrm{d}\,\mathrm{vol}_{g_0}(y),
\end{aligned}$$

where the kernel $K_g(x,y) \in C^\infty(M \times M \setminus \Delta)$ (Δ being the diagonal in $M \times M$) writes in coordinates

$$K_g(x,y) = \frac{2\chi_1(x,y)}{J_g(x,y)d_g^{n-1}(x,y)}, \tag{2.6.2}$$

where $J_g(x,y) := |\det(\mathrm{d}_v(\exp_x^g)_{tv})|$ with $y = \exp_x(tv)$ is the determinant of the map

$$\mathrm{d}_v(\exp_x^g)_{tv} : (T_{(x,v)}S_xM, g_{\mathrm{can}}) \to (T_{\exp_x^g(tv)}, g_0),$$

(the differential is only taken with respect to the v variable) and $\chi_1 \in C^\infty(M \times M)$ is a smooth cutoff function localized near Δ. The kernels $K_g^{i,j}$ have a similar expression to (2.6.2) with a cutoff function that is not localized near Δ.

In particular, it is straightforward that the maps

$$\mathrm{Met} \ni g \mapsto K_g^{i,j} \in C^\infty(M \times M)$$

are continuous. Moreover, the map

$$\mathrm{Met} \ni g \mapsto K_g \in \Psi^{-1},$$

is continuous because the local full symbols are Fourier transforms of the integral kernel K_g in polar coordinates around the diagonal. Note that in this proof, it was crucial that the metrics do not have conjugate points in order to perform a change of variable (one could also have lifted the kernels to the universal cover, avoiding the partitions of unity) □

2.6.2 Continuity of the smooth part

Preliminary lemmas. In this paragraph, we prove a version of Egorov's Theorem (for the operator e^{tX}) with parameter $X \in C^\infty(\mathcal{M}, T\mathcal{M})$). Let Op be a fixed quantization on the manifold. We recall that $\Psi^m(\mathcal{M})$ is the class of operators of the form $A = \mathrm{Op}(a) + R$, where $a \in S^m(T^*\mathcal{M})$ is a symbol of order m in the usual Kohn-Nirenberg class, $R \in \Psi^{-\infty}(\mathcal{M})$ is a smoothing operator (it has smooth kernel $K_R \in C^\infty(\mathcal{M} \times \mathcal{M})$), the topology of this space being described in (A.1.5).

Proposition 2.6.3. *Let $A \in \Psi^m(\mathcal{M})$ be a pseudodifferential operator of order $m \in \mathbb{R}$. Then,*

$$\mathbb{R} \times C^\infty(\mathcal{M}, T\mathcal{M}) \ni (t, X) \mapsto e^{tX}Ae^{-tX} = A(t,X) \in \Psi^m(\mathcal{M})$$

is continuous as a map between Fréchet vector spaces. Moreover, $\mathrm{WF}(A(t,X)) = \Phi_t^X(\mathrm{WF}(A))$ *and* $\sigma(A(t,X)) = \sigma(A) \circ \Phi_t$.

Observe, that in our particular case, given $f \in C^\infty(\mathcal{M})$, one has the exact formula $e^{tX}f = f(\varphi_t(\cdot)) = \varphi_t^* f$ and thus the previous theorem boils down to statement that pseudodifferential operators are invariant under the action of the group of diffeomorphisms on $\mathcal{M}$. More precisely, the previous proposition is implied by the

Proposition 2.6.4. *Let E be a Fréchet vector space and $E \ni X \mapsto F_X \in \mathrm{Diff}(\mathcal{M})$ be a continuous map (between Fréchet manifolds). Then,*

$$E \ni X \mapsto F_X^* A F_X^{-1^*} = A(X) \in \Psi^m(\mathcal{M})$$

is continuous. Moreover, $\mathrm{WF}(A(X)) = \widetilde{F}_X(\mathrm{WF}(A))$ *and* $\sigma(A(X)) = \sigma(A) \circ \widetilde{F}_X$*, where $\widetilde{F}_X$ is the symplectic lift on $T^*\mathcal{M}$ of the diffeomorphism F_X.*

Proof. Without the X-dependence, this is a standard result of microlocal analysis (see [Mel03, Proposition 2.11] for instance). One simply has to follow the classical proof and check that the X-dependence is continuous. $\square$

We denote by Diff the group of smooth diffeomorphisms on the manifold (this is a Fréchet Lie group). We also have the following lemma of continuity

Lemma 2.6.1. *For all $N \in \mathbb{Z}$, the map*

$$\mathrm{Diff} \ni \psi \mapsto \psi^* \in \mathcal{L}(H^N, H^{N-1})$$

is continuous as a map from a Fréchet manifold to a Banach space.

It is actually continuous as a map from $\mathcal{L}(H^s, H^r)$ for all $s > r$ but we will not need this fact.

Proof. Of course, by the group property, this boils down to proving the lemma at the identity $\mathbb{1} \in \mathrm{Diff}$. By duality, it is also sufficient to prove the statement for $N > 0$ and we stick to the case $N = 1$, the general case being handled in a similar fashion. We consider $\psi \in \mathrm{Diff}$ such that $\|\psi - \mathbb{1}\|_{C^0} < \mathrm{inj}(\mathcal{M})$, the injectivity radius of $\mathcal{M} = SM_{g_0}$, seen as a Riemannian manifold endowed with the Sasaki metric. Observe that for $f \in C^\infty(\mathcal{M})$, one has, writing $\gamma(z,t) := \exp_z(t \exp_z^{-1}(\psi(z)))$:

$$\begin{aligned}
\|\psi^* f - f\|_{L^2}^2 &= \int_{\mathcal{M}} |f(\psi(z)) - f(z)|^2 d\mu(z) \\
&\leq \int_{\mathcal{M}} \left| \int_0^1 \langle \nabla f(\gamma(z,t)), \dot{\gamma}(z,t) \rangle dt \right|^2 d\mu(z) \\
&\leq \|\psi - \mathbb{1}\|_{C^0}^2 \int_0^1 \int_{\mathcal{M}} |\nabla f(\gamma(t,z))|^2 d\mu(z) dt \\
&\leq \|\psi - \mathbb{1}\|_{C^0}^2 \int_0^1 \int_{\mathcal{M}} |\nabla f(w)|^2 G(t,w,\psi) d\mu(w) dt \\
&\lesssim \|\psi - \mathbb{1}\|_{C^0}^2 \|f\|_{H^1}^2,
\end{aligned}$$

where $G(t,w,\psi) := |\det d_z\gamma(z,t)|^{-1}, w = \gamma(z,t)$ and G is uniformly bounded in L^∞ for $t \in [0,1], z \in \mathcal{M}$ and ψ in a neighbourhood of the identity in C^∞. This concludes the proof. $\square$

Continuity argument. In this paragraph, we prove Proposition 2.6.2.

Proof of Proposition 2.6.2. We deal with the operator

$$\mathrm{An} \ni g \mapsto \pi_{2*} R_0^+(g) \int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, H^N),$$

the other being similar. We are going to use the perturbative arguments developed in [Gue18, DGRS18]. Let $\mathcal{M} := SM_{g_0}$ and $\mathbb{S}^*\mathcal{M} := T^*\mathcal{M}/\sim$ be the sphere bundle (where $(z,\xi) \sim (z',\xi')$ if and only if $z = z'$ and there exists $\lambda > 0$ such that $\xi = \lambda\xi'$). We denote by $\kappa : T^*\mathcal{M} \to \mathbb{S}^*\mathcal{M}$ the canonical projection.

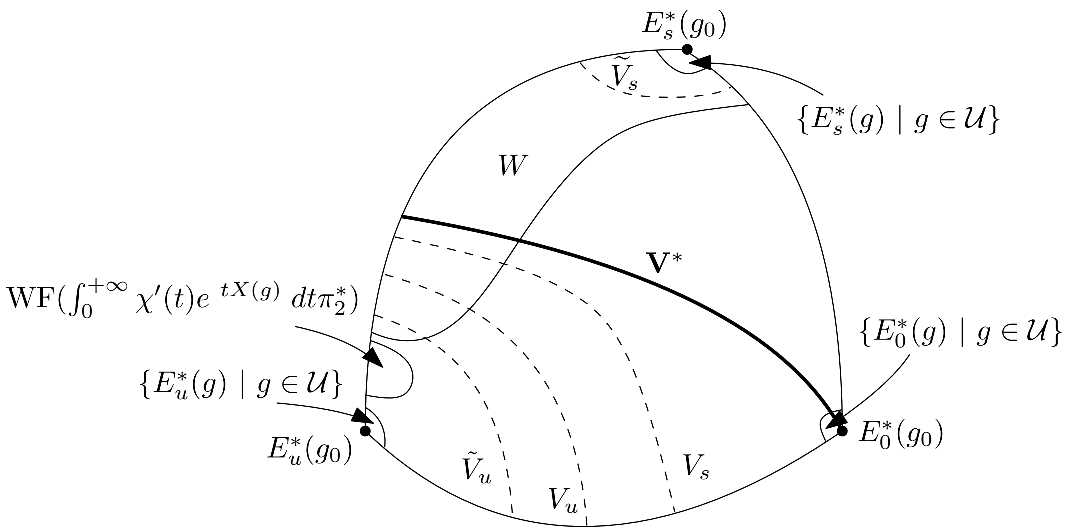

FIGURE 2.6 – A representation of $\mathbb{S}^*\mathcal{M}$

For $\mathcal{U}_{g_0}$ a small C^∞-neighborhood of g_0, we can thus introduce $\left\{\kappa(E_{0,s,u}^*(g)) \mid g \in \mathcal{U}_{g_0}\right\}$ which contain (and are close to) $\kappa(E_{0,s,u}^*(g_0))$. We choose :

- $\widetilde{V}_u \subset V_u$ are relatively compact neighborhoods of $\kappa(E_u^*(g_0))$ containing the set $\{\kappa(E_u^*(g)) \mid g \in \mathcal{U}_{g_0}\}$,
- $\widetilde{V}_s \subset V_s$ are relatively compact neighborhoods of $\kappa(E_s^*(g_0))$ containing the set $\{\kappa(E_s^*(g)) \mid g \in \mathcal{U}_{g_0}\}$,
- V_s contains $\mathbb{V}^* \cap \cup_{g \in \mathcal{U}_{g_0}} E_s^*(g) \oplus E_u^*(g)$,
- and W is a relatively compact neighborhood of $\cup_{g \in \mathcal{U}_{g_0}} \kappa(E_u^*(g) \oplus E_s^*(g)) \cap V_s$.

We refer to Figure 2.6 for a picture of these different sets. By abuse of notations, we will sometimes confuse a set in $\mathbb{S}^*\mathcal{M}$ with its conic lift to $T^*\mathcal{M}$. If $T > 0$ is chosen large enough, one can ensure that *uniformly in* $g \in \mathcal{U}_{g_0}$, one has

$$\mathrm{WF}\left(\int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* f\right) \subset \widetilde{V}_u,$$

where $f \in C^{-\infty}(M, \otimes_S^2 T^*M) := \cup_{s \in \mathbb{R}} H^s(M, \otimes_S^2 T^*M)$. This will be made more precise in a few lines, although not exactly stated this way (we will take advantage of propagation in the other direction). The key fact is the following

Lemma 2.6.2. *For all metrics g in a C^∞ neighborhood $\mathcal{U}_{g_0}$ of g_0, for all conic open set W such that $W \cap \cup_{g \in \mathcal{U}_{g_0}} (E_0^*(g) \oplus E_u^*(g)) = \{0\}$, there exists a time $T > 0$ such that $\Phi_{-t}^g(W) \subset \widetilde{V}_s$ for all $t \geq T$.*

Proof. This is a rather standard lemma in hyperbolic dynamics and we refer to [DGRS18, Section 3.1] for elements of proof. ☐

From the construction techniques of escape function in [Gue18, DGRS18], one can show that there exists a uniform escape function $m \in C^\infty(\mathbb{S}^*\mathcal{M})$ such that $m \equiv 1$ on V_s, $m \equiv -1$ on V_u, and $\mathbf{X}(g) \cdot m \leq 0$ for all g in a C^∞-neighborhood of g_0, and a function $G_m \in S^0_{\rho,1-\rho}(T^*\mathcal{M})$ (for all $\rho < 1$), constructed from m, such that there exist constants $C_1, R > 0$ (independent of g) such that on $|\xi| \geq R$ intersected with a conic neighborhood of $\Sigma := \cup_{g \in \mathcal{U}_{g_0}} E_s^*(g) \oplus E_u^*(g)$, one has $\mathbf{X}(g) \cdot G_m \leq -C_1 < 0$ (where $\mathbf{X}(g)$ is the symplectic lift to $T^*\mathcal{M}$ of $X(g)$) and for $|\xi| \geq R$, $\mathbf{X}(g) \cdot G_m \leq 0$. For $N > 0$, we introduce $A_N := \mathrm{Op}(e^{NG})$. Up to a lower order modification, we can assume that A_N is invertible. We then define the following scale of anisotropic Sobolev spaces for $N \gg N' > 0$:

$$\mathcal{H}^{N,N'} := A_N^{-1}\left(H^{N'}(\mathcal{M})\right)$$

As it was proved in [FS11, DZ16], the spectrum of $X(g)$ is discrete on $\mathcal{H}^{N,N'}$.

Lemma 2.6.3. *The map*

$$\mathrm{An} \ni g \mapsto \int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, \mathcal{H}^{N,-1})$$

is continuous for $N \in \mathbb{N}$ large enough.

Proof. The proof mainly relies on a version of Egorov Theorem with parameter (see §2.6.2). Let $a \in C^\infty(T^*\mathcal{M})$ by a symbol of order 0 that is 0-homogeneous for $|\xi| > 1$ and such that :

- $a \equiv 1$ on $\kappa^{-1}(\widetilde{V}_u) \cap \{|\xi| > 1\}$,
- and $a \equiv 0$ on $(T^*\mathcal{M} \setminus \kappa^{-1}(V_u)) \cap \{|\xi| > 1\}$.

We define $A := \mathrm{Op}(a)$. The lemma boils down to proving that the maps

$$\mathrm{Met} \ni g \mapsto (\mathbb{1} - A) \int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, H^{N-1})$$

and

$$\mathrm{Met} \ni g \mapsto A \int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, H^{-N-1})$$

are continuous. The second one is rather obvious using Lemma 2.6.1, that is the continuity of $\mathrm{Met} \times \mathbb{R} \ni (g,t) \mapsto e^{-tX(g)} \in \mathcal{L}(H^{-N}, H^{-N-1})$. We deal with the first one. We write $\mathbb{1} - A = (\mathbb{1} - A)B + (\mathbb{1} - A)(\mathbb{1} - B) = C_1 + C_2$, where $\mathrm{ell}(B) \subset \mathrm{WF}(B) \subset W$ (see Figure 2.6). We have that $\mathrm{WF}(C_1)$ is localized near W, $\mathrm{WF}(C_2)$ is localized near the complementary $\mathbb{S}^*\mathcal{M} \setminus (W \cup \widetilde{V}_u)$.

The operators $X(g)$ are all uniformly elliptic on $\mathrm{WF}(C_2)$ and thus $X(g)^{2N}$ are all uniformly elliptic too. As a consequence, we can find a microlocal parametrix $Q(g) \in \Psi^{-2N}(\mathcal{M})$ such that $Q(g)X(g)^{2N} = C_2 + R(g)$ where $R(g) \in \Psi^{-\infty}(\mathcal{M})$. The construction of the parametrix is continuous that is

$$\mathrm{Met} \ni g \mapsto (Q(g), R(g)) \in \Psi^{-2N}(\mathcal{M}) \times \Psi^{-\infty}(\mathcal{M})$$

is continuous. Then

$$\begin{aligned} C_2 \int_0^{+\infty} \chi'(t) e^{-tX(g)} \mathrm{d}t \pi_2^* &= \left(Q(g) X(g)^{2N} - R(g)\right) \int_0^{+\infty} \chi'(t) e^{-tX(g)} \mathrm{d}t \pi_2^* \\ &= Q(g) \int_0^{+\infty} \chi^{(2N+1)}(t) e^{-tX(g)} \mathrm{d}t \pi_2^* - R(g) \int_0^{+\infty} \chi'(t) e^{-tX(g)} \mathrm{d}t \pi_2^* \end{aligned}$$

Now, $\int_0^{+\infty} \chi^{(2N+1)}(t) e^{-tX(g)} dt \pi_2^* \in \mathcal{L}(H^{-N}, H^{-N-1})$ and $Q(g) \in \mathcal{L}(H^{-N-1}, H^{N-1})$ and both operators depend continuously on $g \in \mathrm{Met}$ by the previous arguments. Thus

$$\mathrm{Met} \ni g \mapsto Q(g) \int_0^{+\infty} \chi^{(2N+1)}(t) e^{-tX(g)} \mathrm{d}t \pi_2^* \ \in \mathcal{L}(H^{-N}, H^{N-1})$$

is continuous. The second integral with $R(g)$ is dealt in the same fashion.

We now need to deal with the part containing C_1. Using the continuous version of Egorov's Theorem (see §2.6.2) :

$$\begin{aligned} \int_0^{+\infty} \chi'(t) C_1 e^{-tX(g)} dt \pi_2^* &= \int_0^{+\infty} \chi'(t) e^{-tX(g)} e^{tX(g)} C_1 e^{-tX(g)} \mathrm{d}t \pi_2^* \\ &= \int_0^{+\infty} \chi'(t) e^{-tX(g)} C_1(t,g) \mathrm{d}t \pi_2^*, \end{aligned}$$

The principal symbol of $C_1(t,g) = \mathrm{Op}(c_1(t,g))$ is $\sigma(C_1) \circ \Phi_t$. What is more is that the wavefront set satisfies $\mathrm{WF}(C_1(t,g)) \subset \Phi_{-t}^g(\mathrm{WF}(C_1))$, so that roughly speaking, it is "moved" towards $E_s^*(g)$. In particular, by Lemma 2.6.2, there exists $T > 0$ large enough so that uniformly in g in a C^∞ neighborhood of g_0, for all $t \geq T$,

$$\mathrm{WF}(C_1(t,g)) \subset \Phi_{-t}^g(\mathrm{WF}(C_1)) \subset \Phi_{-t}^g(W) \subset \widetilde{V}_s$$

and thus $\mathrm{WF}(C_1(t,g)) \cap \mathbb{V}^* = \{0\}$. Since $\mathrm{WF}(\pi_2^* f) \subset \mathbb{V}^*$, this implies that $C_1(t,g)\pi_2^* f \in C^\infty(\mathcal{M})$ for all $t \geq T$ and more precisely,

$$\mathrm{Met} \ni g \mapsto C_1(t,g)\pi_2^* \in \Psi^{-\infty}(M, \otimes_S^2 T^*M, \mathcal{M})$$

is continuous. From now on $T > 0$ is chosen large enough so that Lemma 2.6.2 is satisfied. In other words,

$$\mathrm{Met} \ni g \mapsto \int_0^{+\infty} \chi'(t) C_1 e^{-tX(g)} \mathrm{d}t \pi_2^* \in \mathcal{L}(H^{-N}, H^{N-1})$$

is continuous. □

Lemma 2.6.4. *The map*

$$\mathrm{An} \ni g \mapsto R_0^+(g) \in \mathcal{L}(\mathcal{H}^{N,-1}, \mathcal{H}^{N,-2})$$

is continuous for $N \in \mathbb{N}$ large enough.

Proof. First, there exists a uniform $r > 0$ such that for all $g \in \mathcal{U}_{g_0}$, $(X(g) - \lambda)^{-1}$ has no pole in $D(0,r) \subset \mathbb{C}$ except at 0. Indeed, by [FT13], for X_{g_0}, there exists a spectral gap of size at least $-(n+1)\nu(g_0)/2$, where $\nu(g_0)$ is the expansion rate in (2.1.2). For g in a C^2-neighborhood of g_0, this expansion rate is uniform in g. In other words, for g close enough to g_0 in the C^2-topology, $X(g)$ has a *finite number of resonances* in the strip

$\{-(n+1)\nu(g_0)/10 \leq \Re(z) \leq 0\}$. By [Gue18], these resonances depend continuously on $g \in C^\infty(M, \otimes^2_{S,+} T^*M)$.

Let Γ be the circle of radius $r/2$. Since $R_0^+(g) = \int_\Gamma (X(g)-\lambda)^{-1}\lambda^{-1} d\lambda$, in order to prove the continuity of $g \mapsto R_0^+(g)$, it is thus sufficient to prove that the map

$$\mathrm{Met} \times \Omega \ni (g,\lambda) \mapsto (X(g)-\lambda)^{-1} \ \in \mathcal{L}(\mathcal{H}^{N,-1}, \mathcal{H}^{N,-2})$$

is continuous, where $\Omega := D(0,r) \setminus D(0,r/4)$. But this follows directly from [DGRS18, Proposition 6.1] .

□

The combination of the previous Lemmas 2.6.3 and 2.6.4 imply Proposition 2.6.2. Indeed,

$$\underbrace{\pi_{2*}}_{\in \mathcal{L}(\mathcal{H}^{N,-2}, H^{N-2})} \underbrace{R_0^+(g)}_{\in \mathcal{L}(\mathcal{H}^{N,-1}, \mathcal{H}^{N,-2})} \underbrace{\int_0^{+\infty} \chi'(t) e^{-tX(g)} dt \pi_2^*}_{\in \mathcal{L}(H^{-N}, \mathcal{H}^{N,-1})} \in \mathcal{L}(H^{-N}, H^{N-2})$$

with continuous dependance on $g \in \mathrm{An}$. The fact that $\pi_{2*} \in \mathcal{L}(\mathcal{H}^{N,-2}, H^{N-2})$ is due to the careful choice of the escape function m.

□

Chapitre 3

The marked length spectrum of Anosov manifolds

« *Le clair de lune à travers les hautes branches,*
Tous les poètes disent, sans exception, qu'il est beaucoup plus
Que le clair de lune à travers les hautes branches. »

Le Gardeur de troupeaux, Fernando Pessoa (Alberto Caeiro)

This chapter is a compilation of the two articles :

- *The marked length spectrum of Anosov manifolds*, written in collaboration with Colin Guillarmou and published in **Annals of Mathematics**,
- *Geodesic stretch and marked length spectrum rigidity*, written in collaboration with Colin Guillarmou and Gerhard Knieper.

Sommaire

In this chapter, we prove that the marked length spectrum of a Riemannian manifold (M, g_0) with Anosov geodesic flow and non-positive curvature locally determines the metric in the sense that two close enough metrics with the same marked length spectrum are isometric. This result is valid in any dimensions. In addition, we provide a new stability estimate quantifying how the marked length spectrum controls the distance between the isometry classes of metrics. In dimension two, we obtain similar results for general metrics with Anosov geodesic flows. We also locally solve a rigidity conjecture of Croke [Cro04] relating volume and marked length spectrum for the same category of metrics. By a compactness argument, we show that the set of negatively curved metrics (up to isometry) with the same marked length spectrum, the same volume and with curvature in a bounded set of C^∞ is finite.

In a second part, we investigate the link between the geodesic stretch introduced by Knieper [Kni95], the generalized X-ray transforms of Guillarmou [Gui17a] and the marked length spectrum of Anosov Riemannian manifolds. In particular, we prove that in a neighborhood of a fixed metric g_0, the marked length spectrum of a metric g "at infinity" (more precisely, the ratio of the two marked length spectra of g and g_0 as lengths tend to $+\infty$) is sufficient to determine the metric which weakens the conjecture of Burns-Katok [BK85]. Moreover, we prove that the geodesic stretch with respect to the Liouville measure and a certain thermodynamic pressure quantify the distance between the isometry classes of g and g_0. We also introduce a natural semidefinite metric G on the space of smooth metrics on the manifold M which is defined as the Hessian of the geodesic stretch of infinitesimal variations. We prove that this metric is continuous and provide a locally uniform lower bound $G_g(u, u) \geq C\|u\|^2$, where u is a symmetric solenoidal 2-tensor. When $M = S$ is a surface, we show that this metric induces a canonical metric on the Teichmüller space $\mathcal{T}(S)$ which is called the pressure metric and is a multiple of the Weil-Petersson metric. Eventually, as another consequence of the continuity of the metric G on Met, we prove a uniform version of the local rigidity of the marked length spectrum.

3.1 The Burns-Katok conjecture

Let (M, g_0) be a smooth closed Riemannian manifold. As mentioned in the previous chapter, if the metric g_0 admits an Anosov geodesic flow in the sense of (2.1.1), the set of lengths of closed geodesics is discrete and is called the *length spectrum of* g_0. The closed geodesics are parametrized by the set $\mathcal{C}$ of free-homotopy classes, or equivalently the set of conjugacy classes of the fundamental group $\pi_1(M)$ (see [Kli74]). It is an old problem in Riemannian geometry to understand if the length spectrum determines the metric g_0 up to isometry. In 1980, Vigneras [Vig80] found counterexamples in constant negative curvature. The first examples of manifolds with Anosov geodesic flow are the negatively curved closed manifolds. We can thus define a map, called the *marked length spectrum*, by

$$L_{g_0} : \mathcal{C} \to \mathbb{R}^+, \quad L_{g_0}(c) := \ell_{g_0}(\gamma(c)) \tag{3.1.1}$$

where, if γ is a C^1-curve, $\ell_{g_0}(\gamma)$ denotes its length with respect to g_0. We recall the following long-standing conjecture stated in Burns-Katok [BK85] (and probably considered even before) :

Conjecture 3.1.1. *[BK85, Problem 3.1] If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have same marked length spectrum, i.e $L_g = L_{g_0}$,* *then they are isometric, i.e. there exists a smooth diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Note that if $\phi : M \to M$ is a diffeomorphism isotopic to the identity, then $L_{\phi^* g_0} = L_{g_0}$. The analysis of the linearised operator at a given metric g_0 is now well-understood, starting from the pionnering work of Guillemin-Kazhdan [GK80a], and pursued by the works of Croke-Sharafutdinov [CS98], Dairbekov-Sharafutdinov [DS03] and more recently by Paternain-Salo-Uhlmann [PSU14a, PSU15] and Guillarmou [Gui17a]. It is known that the linearized operator, the so-called X-ray transform (see Lemma 3.2.1), is injective for non-positively curved manifolds with Anosov geodesic flows in all dimensions, and for all Anosov geodesic flows in dimension 2. These works imply the deformation rigidity result : there is no 1-parameter family of such metrics (more precisely, of isometry classes) with the same marked length spectrum.

Concerning the non-linear problem (Conjecture 3.1.1), there are only very few results : in dimension 2 and non-positive curvature, a breakthrough was done by Otal [Ota90] and Croke [Cro90] who solved that problem [1]. It was extended by Croke-Fathi-Feldman [CFF92] to surfaces when one of the metrics has non positive curvature and the other has no conjugate points. Katok [Kat88] previously had a short proof for metrics in a fixed conformal class, in dimension 2, and his proof can be easily extended to higher dimensions. Beside the conformal case, for higher dimension the only known rigidity result is due to Hamenstädt [Ham99], based on the celebrated entropy rigidity work of Besson-Courtois-Gallot [BCG95] : she showed that if two negatively curved metrics g and g_0 on M have the same marked length spectrum and if the Anosov foliation of g_0 is C^1, then $\mathrm{vol}(g) = \mathrm{vol}(g_0)$, thus, as L_g determines the topological entropy, the results of [BCG95] imply Conjecture 3.1.1 when g_0 is a locally symmetric space. For general metrics the problem is largely open. We refer to the surveys/lectures of Croke and Wilkinson [Cro90, Wil14] for an overview of the subject. The main difficulty to obtain a local rigidity result is that the linearised operator takes values on functions on a discrete set and is not a tractable operator to obtain non-linear results. The Conjecture 3.1.1 actually also makes sense for Anosov geodesic flows without the negative curvature assumption.

3.2 Local rigidity of Anosov manifolds

From now on, we denote by Met the Fréchet manifold consisting of smooth metrics on M. We denote by $\mathrm{Met}^{k,\alpha}$ the set of metrics with regularity $C^{k,\alpha}$, $k \in \mathbb{N}, \alpha \in (0,1)$.

3.2.1 Statement of the results

Our first result is a local rigidity statement asserting that the marked length spectrum parametrizes locally the isometry classes of metrics. As far as we know, this is the first (non-linear) progress towards Conjecture 3.1.1 in dimension $n+1 \geq 3$ for general metrics.

Theorem 3.2.1. *Let (M, g_0) be :*

- *either a closed smooth Riemannian surface with Anosov geodesic flow,*
- *or a closed smooth Riemannian manifold of dimension $n+1 \geq 3$ with Anosov geodesic flow and non-positive sectional curvature,*

1. Otal's work was in negative curvature and Croke's paper in non-positive curvature.

and let $N > 3(n+1)/2 + 8$. There exists $\varepsilon > 0$ such that for any smooth metric $g \in \mathrm{Met}^N$ with same marked length spectrum as g_0 and such that $\|g - g_0\|_{C^N} < \varepsilon$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.

We actually prove a slightly stronger result in the sense that g can be chosen to be in the Hölder space $C^{N,\alpha}$ with $(N, \alpha) \in \mathbb{N} \times (0, 1)$ satisfying $N + \alpha > 3(n+1)/2 + 8$. Note also that $\varepsilon > 0$ is chosen small enough so that the metrics g have Anosov geodesic flow too. This result is new even if $\dim(M) = 2$, as we make no assumption on the curvature. If $\dim(M) > 2$ and g is Anosov, the same result holds outside a finite dimensional manifold of metrics. This implies the following result supporting Conjecture 3.1.1 :

Corollary 3.2.1. *Let (M, g_0) be an $(n+1)$-dimensional compact Riemannian manifold with negative curvature and let $N > 3(n+1)/2 + 8$. Then there exists $\varepsilon > 0$ such that for any smooth metric $g \in \mathrm{Met}^N$ with same marked length spectrum as g_0 and such that $\|g - g_0\|_{C^N(M)} < \varepsilon$, there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Since two C^0-conjugate Anosov geodesic flows that are close enough have the same marked length spectrum, we also deduce that for g_0 fixed as above, each metric g which is close enough to g_0 and has geodesic flow conjugate to that of g_0 is isometric to g. This also leads us to ask the natural question : for g_0 a fixed negatively curved metric, is there a neighborhood of g_0 such that each metric in this neighbourhood with the same *length spectrum* as g_0 is isometric to g_0 ? This question is closely related to the question of finiteness of isospectral metrics asked by Sarnak in [Sar90], and at the moment we are unable to answer it.

To prove these results, a natural strategy would be to apply an implicit function theorem. The linearized operator is I_2, the X-ray transform on symmetric 2-tensors studied in the previous chapter : it consists in integrating symmetric 2-tensors along closed geodesics of the metric g_0 (see Lemma 3.2.1). It is known to be injective under the assumptions of Theorem 3.2.1 by [GK80a, CS98, PSU14a, PSU15, Gui17a], but as mentioned before, the main difficulty to apply this to the non-linear problem is that I_2 maps to functions on the discrete set $\mathcal{C}$ and it seems unlikely that its range is closed. To circumvent this problem, we use some completely new approach from [Gui17a] that replaces the operator I_2 by the operator Π_2 introduced in the previous chapter. This new operator plays the same role as the normal operator $I_2^* I_2$ that is strongly used in the context of manifolds with boundary. On the other hand, Π_2 is not constructed from I_2 and the additional crucial ingredient that allows us to relate the operators I_2 and Π_2 is the positive Livsic theorem due to Lopes-Thieullen [LT05] (see Theorem 2.1.2).

Combining these methods with some ideas developed by [CDS00, Lef18b] in the case with boundary, we are able to prove a new rigidity result which has similarities with the minimal filling volume problem appearing for manifolds with boundary and is a problem asked by Croke in [Cro04, Question 6.8].

Theorem 3.2.2. *Let (M, g_0) be as in Theorem 3.2.1 and let $N > \frac{n+1}{2} + 2$. There exists $\varepsilon > 0$ such that for any smooth metric g satisfying $\|g - g_0\|_{C^N} < \varepsilon$, the following holds true : if $L_g(c) \geq L_{g_0}(c)$ for all conjugacy class $c \in \mathcal{C}$ of $\pi_1(M)$, then $\mathrm{vol}(g) \geq \mathrm{vol}(g_0)$. If in addition $\mathrm{vol}(g) = \mathrm{vol}(g_0)$, then there exists a diffeomorphism $\phi : M \to M$ such that $\phi^* g = g_0$.*

Again, in the proof, we actually just need $g \in C^{N,\alpha}$ with $(N, \alpha) \in \mathbb{N} \times (0, 1)$ satisfying $N + \alpha > (n+1)/2 + 2$. This result (but without the assumption that g is close to g_0)

was proved by Croke-Dairbekov [CD04] for negatively curved *surfaces* and for metrics in a conformal class in higher dimension (by applying the method of [Kat88]). Theorem 3.2.2 is the first general result in dimension $n > 2$ and is new even when $n = 2$ as we do not assume negative curvature.

Next, we get Hölder stability estimates quantifying how close are metrics with close marked length spectrum. In that aim we fix a metric g_0 with Anosov geodesic flow and define for g close to g_0 in some C^N norm, the quantity $\mathcal{L}(g) := L_g/L_{g_0} \in \ell^\infty(\mathcal{C})$. We are able to show the

Theorem 3.2.3. *Let (M, g_0) satisfy the assumptions of Theorem 3.2.1 and let $N > 3(n+1)/2 + 8$. For all $s > 0$ small there is a positive $\nu = \mathcal{O}(s)$ and a constant $C > 0$ such that the following holds : there exists $\varepsilon > 0$ small such that for any C^N metric g satisfying $\|g - g_0\|_{C^N} < \varepsilon$, there is a diffeomorphism ϕ close to the identity such that*

$$\|\phi^* g - g_0\|_{C^s} \leq C\|g - g_0\|_{C^N}^{(1+\nu)/2} \|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2}$$

where $\mathbf{1}(c) := 1$ for each $c \in \mathcal{C}$.

We note that this Hölder stability estimate is the first quantitative estimate on the marked length rigidity problem. It is even new for negatively curved surfaces where the injectivity of $g \mapsto L_g$ (modulo isometry) is known by [Cro90, Ota90].

We conclude by some finiteness results. On a closed manifold M, we consider for $\nu_1 \geq \nu_0 > 0$, $\theta_0 > 0$ and $C_0 > 0$ the set of smooth metrics g with Anosov geodesic flow satisfying the estimates (2.1.1) where the constants C, ν verify $C \leq C_0$, $\nu \in [\nu_0, \nu_1]$ and $d_G(E_s, E_u) \geq \theta_0$ if d_G denotes the distance in the Grassmanian of the unit tangent bundle SM induced by the Sasaki metric. We write $\mathrm{An}(\nu_0, \nu_1, C_0, \theta_0)$ for the set of such metrics. This is a closed set that consists of uniform Anosov geodesic flows. For example, metrics with curvatures contained in $[-a^2, -b^2]$ with $a > b > 0$ satisfy such property [Kli95, Theorem 3.2.17]. In what follows, we denote by $\mathcal{R}_g$ the curvature tensor of g.

Theorem 3.2.4. *Let M be a smooth closed manifold and let $\nu_1 \geq \nu_0 > 0$, $C_0 > 0$ and $\theta_0 > 0$. If $\dim M = 2$, for each sequence of positive numbers $B := (B_k)_{k\in\mathbb{N}}$, there is at most finitely many isometry classes of metrics g in $\mathrm{An}(\nu_0, \nu_1, C_0, \theta_0)$ satisfying the curvature bounds $|\nabla_g^k \mathcal{R}_g|_g \leq B_k$ and with the same marked length spectrum. If $\dim M > 2$ the same holds true if in addition g have non-positive curvature and uniformly bounded volume.*

Restricting to negatively curved metrics we get the finiteness results which is new if $\dim M > 2$:

Corollary 3.2.2. *Let M be a compact manifold. Then, for each $a, b > 0$ and each sequence $B = (B_k)_{k\in\mathbb{N}}$ of positive numbers, there is at most finitely many smooth isometry classes of metrics with sectional curvature bounded above by $-a^2 < 0$, curvature tensor bounded by B (in the sense of Theorem 3.2.4), volume bounded above by b and same marked length spectrum.*

We remark that the C^∞ assumptions on the background metric g_0 in all our results and the boundedness assumptions on the C^∞ norms of the curvatures in Theorem 3.2.4 can be relaxed to C^k for some fixed k depending on the dimension. [2]

2. The smoothness assumptions come from the fact we are using certain results based on microlocal analysis ; it is a standard fact that only finitely many derivatives are sufficient for microlocal methods.

3.2.2 The marked length spectrum and its linearisation

We consider a smooth manifold M equipped with a smooth Riemannian metric g. We assume that the geodesic flow φ_t of g on the unit tangent bundle SM is Anosov. We will call *Anosov manifolds* such Riemannian manifolds and let

$$\mathrm{An} := \{g \in C^\infty(M, \otimes^2_{S,+} T^*M) \mid g \text{ has Anosov geodesic flow}\}.$$

Here, φ_t with generating vector field X is called Anosov in the sense of (2.1.1), where the norm is given in terms of the Sasaki metric of g. By Anosov structural stability [Ano67, dlLMM86], An is an open set. In particular, a metric $g \in \mathrm{An}$ has no conjugate points (see [Kli74]) and there is a unique geodesic $\gamma(c)$ in each free-homotopy class $c \in \mathcal{C}$. We can thus define the *marked length spectrum* of g by (3.1.1).

It will also be important for us to consider the mapping $g \mapsto L_g$ from the space of metrics to the set of sequences. In order to be in a good functional setting and since we shall work locally, we fix a smooth metric $g_0 \in \mathrm{An}$ and consider the metrics g in a neighborhood $\mathcal{U}_{g_0}$ of g_0 in $C^N(M, \otimes^2_{S,+} T^*M)$ for some N large enough which will be chosen later. We can consider the map

$$\mathcal{L} : \mathcal{U}_{g_0} \to \ell^\infty(\mathcal{C}), \quad \mathcal{L}(g)(c) := L_g(c)/L_{g_0}(c). \tag{3.2.1}$$

which we call the g_0*-normalized marked length spectrum*. We notice from the definition of the length that $\mathcal{L}(g) \in [0, 2]$ if $g \leq 2g_0$, justifying that $\mathcal{L}$ maps to $\ell^\infty(\mathcal{C})$.

Proposition 3.2.1. *The functional (3.2.1) is C^2 near g_0 if we choose the topology $C^3(M, \otimes^2_{S,+} T^*M)$. In particular, there is a neighborhood $\mathcal{U}_{g_0} \subset C^3(M, \otimes^2_{S,+} T^*M)$ of g_0 and $C = C(g_0) > 0$ such that for all $g \in \mathcal{U}_{g_0}$*

$$\|\mathcal{L}(g) - \mathbf{1} - d\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty} \leq C\|g - g_0\|^2_{C^3}. \tag{3.2.2}$$

Proof. Let $\mathcal{M} := SM_{g_0}$ be the unit tangent bundle for g_0 and X_0 the geodesic vector field. We use the stability result in the work of De la Llave-Marco-Moryion [dlLMM86, Appendix A] which says that there is a neighborhood $\mathcal{V}_{X_0}$ in $C^2(\mathcal{M}, T\mathcal{M})$ of X_0 and a C^2 map $X \in \mathcal{V}_{X_0} \mapsto \theta_X \in C^0(\mathcal{M})$ such that for each $X \in \mathcal{V}_{X_0}$ and each fixed periodic orbit γ_{X_0} of X_0, there is a closed orbit γ_X freely-homotopic to γ_{X_0} and the period $\ell(\gamma_X)$ is C^2 as a map $X \in \mathcal{V}_{X_0} \mapsto \ell(\gamma_X) \in \mathbb{R}^+$ given by

$$\ell(\gamma_X) = \int_{\gamma_{X_0}} \theta_X.$$

In particular, we see that $X \in \mathcal{V}_{X_0} \mapsto \ell(\gamma_X)/\ell(\gamma_{X_0})$ is C^2 and its derivatives of order $j = 1, 2$ are bounded :

$$\|d^j \ell(\gamma_X)/\ell(\gamma_{X_0})\|_{C^2 \to \mathbb{R}} \leq \sup_{X \in \mathcal{V}_{X_0}} \|d^j \theta_X\|_{C^2 \to C^0} \leq C$$

for some C depending on $\mathcal{V}_{X_0}$ but uniform in γ_{X_0}. Now we fix $c \in \mathcal{C}$ and choose the geodesic $\gamma_{g_0}(c)$ for g_0 as being the element γ_{X_0} above, and we take $\mathcal{U}_{g_0}$ a small neighborhood of g_0 in the C^3 topology. The map $X : g \in \mathcal{U}_{g_0} \mapsto X_g \in C^2(\mathcal{M}, T\mathcal{M})$ is defined so that X_g is the geodesic vector field of g, where we used the natural diffeomorphism between $\mathcal{M} = SM_{g_0}$ and $SM_g := \{(x, v) \in TM, g_x(v, v) = 1\}$ obtained by scaling the fibers to pull-back the field on $\mathcal{M}$. It is a C^∞ map between the Banach space $C^3(M, \otimes^2_{S,+} T^*M)$ and $C^2(\mathcal{M}, T\mathcal{M})$. Thus the composition $g \mapsto \ell(\gamma_{X_g})$, which is simply the map $g \mapsto L_g(c)$, is C^2 on $\mathcal{U}_{g_0}$ and the second derivative is uniformly bounded in $\mathcal{U}_{g_0}$. The inequality (3.2.2) follows directly. □

The central object on which stands our proof is the X-ray transform over symmetric 2-tensors studied in the previous chapter. It is nothing more than the linearization $d\mathcal{L}$ that appeared in Proposition 3.2.1. It is a direct computation, which appeared already in [GK80a] that

Lemma 3.2.1. *For $h \in C^3(M, \otimes_S^2 T^*M)$,*

$$d\mathcal{L}_{g_0} h(c) = \frac{1}{2L_{g_0}(c)} \int_0^{L_{g_0}(c)} h_{\gamma_c(t)}(\dot{\gamma}_c(t), \dot{\gamma}_c(t))dt = 1/2 \times I_2^{g_0} h(c),$$

where $t \mapsto \gamma_c(t)$ is the arc-length parametrization of the g_0-geodesic homotopic to c and $\dot{\gamma}_c(t)$ its time derivative.

Proof. The proof is immediate, using the fact that the g_0-geodesic $\gamma_{g_0}(c)$ is a critical point of the length functional. □

When the background metric is fixed, we will remove the g_0 index and just write I_2, I instead of $I_2^{g_0}, I^{g_0}$. When (M, g_0) is Anosov, we recall that I_2 is solenoidal injective that is injective on $C^\infty(M, \otimes_S^m T^*M) \cap \ker D^*$ when g_0 has non-positive curvature [CS98, Theorem 1.3] (see also Appendix B) or $\dim(M) = 2$ [PSU14a, Gui17a]. We notice that similar results have also been obtained in the case of domains with strictly convex boundary in $\mathbb{R}^n$ in relation with spectral rigidity : for example, De Simoi-Kaloshin-Wei [dSKW17, Theorem 4.9] prove a similar injectivity result for domains with $\mathbb{Z}^2$ symmetry close to the circle ; the billiard dynamic is of course very different from our case. More generally, me refer to the books of Petkov-Stoyanov [PS92, PS17] where a variety of topics relating the length spectrum and the Laplace spectrum of a billiard problem in a domain of Euclidean space is discussed.

3.2.3 Preliminary results

As before, we fix a smooth Riemannian manifold (M, g_0) with Anosov flow and we shall consider metrics g with regularity $C^{N,\alpha}$ for some $N \geq 3, \alpha > 0$ to be determined later and such that $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$, for some $\varepsilon > 0$ small enough so that g also has Anosov flow.

Reduction of the problem. The metric g_0 is divergence-free with respect to itself : $D^* g_0 = -\mathrm{Tr}(\nabla g_0) = 0$, where the Levi-Civita connection ∇ and trace Tr are defined with respect to g_0. By a standard argument developed in Lemma B.1.7, there is a slice consisting of solenoidal tensors transverse to the diffeomorphism action $(\phi, g) \mapsto \phi^* g$ at the metric g_0 ; here ϕ varies in the group of $C^{N+1,\alpha}$-diffeomorphisms on M homotopic to the identity. We shall write $\mathrm{Diff}_0^{N,\alpha}(M)$ for the group of $C^{N,\alpha}(M)$ diffeomorphisms homotopic to the identity, with $N \geq 2, \alpha \in (0,1)$. Since $\mathcal{L}(\phi^* g_0) = \mathcal{L}(g_0) = \mathbf{1}$ for all $\phi \in \mathrm{Diff}_0^{N+1,\alpha}(M)$, it suffices to work on that transverse slice to study the marked length spectrum. This fact is classical and probably goes back to [Ebi68] but we still provide a proof for the sake of completeness (see Lemma B.1.7). We introduce $f := \phi^* g - g_0 \in C^{N,\alpha}(M, \otimes_S^2 T^*M)$, which is, by construction, divergence-free and satisfies $\|f\|_{C^{N,\alpha}} \lesssim \|g - g_0\|_{C^{N,\alpha}} \lesssim \varepsilon$. Our goal will be to prove that $f \equiv 0$, if ε is chosen small enough and $L_g = L_{g_0}$.

Geometric estimates. We let g be in a neighborhood of g_0.

Lemma 3.2.2. *Assume that $L_g(c) \geq L_{g_0}(c)$ for each $c \in \mathcal{C}$. If $\gamma_{g_0}(c)$ denotes the unique geodesic freely homotopic to c for g_0, then*

$$I_2^{g_0} f(c) = \int_{\gamma_{g_0}(c)} \pi_2^* f \geq 0.$$

Proof. We denote by $\gamma_g(c)$ the g-geodesic in the free-homotopy class c. One has :

$$\int_{\gamma_{g_0}(c)} \pi_2^* f = \int_{\gamma_{g_0}(c)} \pi_2^* g - \int_{\gamma_{g_0}(c)} \pi_2^* g_0 = E_g(\gamma_{g_0}(c)) - L_{g_0}(c),$$

where $E_g(\gamma_{g_0}(c)) = \int_0^{\ell_{g_0}(\gamma_{g_0}(c))} g_{\gamma_{g_0}(c)(t)}(\dot\gamma_{g_0}(c)(t), \dot\gamma_{g_0}(c)(t))dt$ is the energy functional for g. By using Cauchy-Schwartz,

$$E_g(\gamma_{g_0}(c)) \geq \ell_g(\gamma_{g_0}(c))^2 / \ell_{g_0}(\gamma_{g_0}(c)),$$

and since $\gamma_{g_0}(c)$ is freely-homotopic to c, we get $\ell_g(\gamma_{g_0}(c)) \geq \ell_g(\gamma_g(c))$. Since $\ell_g(\gamma_g(c)) = L_g(c) \geq L_{g_0}(c) = \ell_{g_0}(\gamma_{g_0}(c))$ by assumption, we obtain the desired inequality. ☐

Next, we can use the following result :

Lemma 3.2.3. *There exists $\varepsilon > 0$ small enough, $C > 0$, such that if $\|g - g_0\|_{C^0} \leq \varepsilon$ and $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$, then with $f := g - g_0$, one has*

$$\int_{SM} \pi_2^* f \, d\mu \leq C \|f\|_{L^2}^2.$$

Here μ is the Liouville measure of the metric g_0.

Proof. Let $g_\tau := g_0 + \tau f$ with $f \in C^3(M, \otimes_S^2 T^*M)$. A direct computation gives that $\int_M \mathrm{Tr}_{g_0}(f) \mathrm{dvol}_{g_0} = \int_{SM} \pi_2^* f \, d\mu$. Then the argument of [CDS00, Proposition 4.1] by Taylor expanding $\mathrm{vol}(g_\tau)$ in τ directly provides the result. ☐

Finally, we conclude this section with the following :

Lemma 3.2.4. *Assume that $I_2^{g_0} f(c) \geq 0$ for all $c \in \mathcal{C}$. Then, there exists a constant $C = C(g_0) > 0$, such that :*

$$0 \leq \int_{SM} \pi_2^* f \, d\mu \leq C \left(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + \|f\|_{C^3(M)}^2 \right). \tag{3.2.3}$$

Proof. For the Anosov geodesic flow of g_0, the Liouville measure is the unique equilibrium state associated to the potential given by $J^u(z) := -\partial_t \left(\det d\varphi_t(z)|_{E_u(z)}\right)|_{t=0}$ (the unstable Jacobian). By Parry's formula (see [Par88, Paragraph 3]), we have :

$$\forall F \in C^0(SM), \quad \lim_{T \to \infty} \frac{1}{N(T)} \sum_{c \in \mathcal{C}, L_{g_0}(c) \leq T} \frac{e^{\int_{\gamma(c)} J^u}}{L_{g_0}(c)} \int_{\gamma(c)} F = \frac{1}{\mathrm{vol}(SM)} \int_{SM} F \, d\mu, \tag{3.2.4}$$

where, as before, $\gamma(c)$ is the g_0-geodesic in c and $N(T)$ is the constant of normalisation corresponding to the sum when $F = 1$. The first inequality in (3.2.3) then follows from that formula and the assumption $I_2^{g_0} f \geq 0$. For the second inequality in (3.2.3) we use Proposition 3.2.1 with the fact that $d\mathcal{L}_{g_0} f = \frac{1}{2} I_2^{g_0} f$ to deduce that there exists $C(g_0) > 0$ such that

$$\|I_2^{g_0} f\|_{\ell^\infty(\mathcal{C})} \leq 2\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + C(g_0)\|f\|_{C^3}^2. \tag{3.2.5}$$

Thus, we get for any $T > 0$

$$\frac{1}{N(T)} \sum_{c \in \mathcal{C}, L_{g_0}(c) \leq T} e^{\int_{\gamma(c)} J^u} I_2^{g_0} f(c) \leq \|I_2^{g_0} f\|_{\ell^\infty(\mathcal{C})} \leq 2\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty(\mathcal{C})} + C(g_0)\|f\|_{C^3}^2 \tag{3.2.6}$$

and the left-hand side converges to $\frac{1}{\mathrm{vol}(SM)} \int_{SM} \pi_2^* f \, d\mu$ by Parry's formula (3.2.4), which is the sought result by letting $T \to \infty$. □

We note that in the previous proof, the approximation of $\int_{SM} \pi_2^* f$ by $I_2^{g_0} f(c)$ could also be done using the Birkhoff ergodic theorem and the Anosov closing lemma to approximate $\int_{SM} \pi_2^* f$ by some $I_2^{g_0} f(c)$ for some $c \in \mathcal{C}$ so that $L_{g_0}(c)$ is large.

3.2.4 Proofs of the main results

Proof of Theorem 3.2.1 and Theorem 3.2.2. We fix g_0 with Anosov geodesic flow on M and assume that either M is a surface or that g_0 has non-positive curvature in order to have that $I_2^{g_0}$ is solenoidal injective. Fix $N \geq 3$ to be chosen later and $\alpha > 0$ small. As explained in Lemma B.1.7, if $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$, then there is $\phi \in \mathrm{Diff}_0^{N+1,\alpha}(M)$ with $\|\phi^* g - g_0\|_{C^{N,\alpha}} \lesssim \varepsilon$ and $D^*(\phi^* g - g_0) = 0$.

We write $f := \phi^* g - g_0$ and assume that $L_g \geq L_{g_0}$: this implies $L_{\phi^* g} \geq L_{g_0}$ thus $I_2^{g_0} f(c) \geq 0$ for all $c \in \mathcal{C}$ by Lemma 3.2.2. By Theorem 2.1.2, we know that there exists $h \in C^\beta(SM)$ and $F \in C^\beta(SM)$ for some $0 < \beta < \alpha$ (depending on g_0 and linearly on α) such that $\pi_2^* f + Xh = F \geq 0$, with

$$\|\pi_2^* f + Xh\|_{C^\beta} \leq C\|\pi_2^* f\|_{C^\alpha} \leq C\|f\|_{C^\alpha}, \tag{3.2.7}$$

where $C = C(g_0)$. Take $0 < s \ll \beta$ very small (it will be fixed later) and let $\beta' < \beta$ be very close to β. Thus we obtain

$$\begin{aligned}
\|f\|_{H^{-1+s}} &\lesssim \|\Pi_2 f\|_{H^s}, && \text{by Lemma 2.5.5} \\
&\lesssim \|\pi_{2*}\Pi(\pi_2^* f + Xh)\|_{H^s} && \text{by Theorem 2.4.2} \\
&\lesssim \|\pi_2^* f + Xh\|_{H^s}, && \text{by Lemma 2.5.9} \\
&\lesssim \|\pi_2^* f + Xh\|_{L^2}^{1-\nu} \|\pi_2^* f + Xh\|_{H^{\beta'}}^{\nu}, && \text{by interpolation with } \nu = \tfrac{s}{\beta'}.
\end{aligned} \tag{3.2.8}$$

Note that by (7.5.3) we have a control :

$$\|\pi_2^* f + Xh\|_{H^{\beta'}} \lesssim \|\pi_2^* f + Xh\|_{C^\beta} \lesssim C\|f\|_{C^\alpha}. \tag{3.2.9}$$

And we can once more interpolate between Lebesgue spaces so that :

$$\|\pi_2^* f + Xh\|_{L^2} \lesssim \|\pi_2^* f + Xh\|_{L^1}^{1/2} \|\pi_2^* f + Xh\|_{L^\infty}^{1/2} \lesssim \|\pi_2^* f + Xh\|_{L^1}^{1/2} \|f\|_{C^\alpha}^{1/2}. \tag{3.2.10}$$

Next, using that $\pi_2^* f + Xh \geq 0$, we have

$$\|\pi_2^* f + Xh\|_{L^1} = \int_{SM} (\pi_2^* f + Xh) d\mu = \int_{SM} \pi_2^* f \, d\mu. \tag{3.2.11}$$

We will now consider two cases : in case (1) we assume that $L_g = L_{g_0}$, while in case (2) we assume that $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ (recall we have also assumed $L_g \geq L_{g_0}$). Combining Lemma 3.2.2 and Lemma 3.2.4, we deduce that in case (1),

$$\|\pi_2^* f + Xh\|_{L^1} \lesssim \|f\|_{C^3}^2,$$

while in case (2), we get by Lemma 3.2.3 that if $\varepsilon > 0$ is small enough,

$$\|\pi_2^* f + Xh\|_{L^1} \lesssim \|f\|_{L^2}^2.$$

These facts combined with (3.2.10) yield

$$\|\pi_2^* f + Xh\|_{L^2} \lesssim \begin{cases} \|f\|_{C^3}.\|f\|_{C^\alpha}^{1/2}, & \text{case (1)} \\ \|f\|_{L^2}.\|f\|_{C^\alpha}^{1/2}, & \text{case (2)} \end{cases}.$$

Thus we have shown

$$\|f\|_{H^{-1+s}} \lesssim \begin{cases} \|f\|_{C^3}^{1-\nu}\|f\|_{C^\alpha}^{\frac{1+\nu}{2}}, & \text{case (1)} \\ \|f\|_{L^2}^{1-\nu}.\|f\|_{C^\alpha}^{\frac{1+\nu}{2}}, & \text{case (2)} \end{cases} \tag{3.2.12}$$

We choose α very small and $0 < s \ll \beta < \alpha$, $j \in \{\alpha, 3\}$ and $N_0 > n/2 + j + s$: by interpolation and Sobolev embedding we have

$$\|f\|_{C^j} \lesssim \|f\|_{H^{n/2+j+s}} \lesssim \|f\|_{H^{-1+s}}^{1-\theta_j}\|f\|_{H^{N_0}}^{\theta_j} \tag{3.2.13}$$

with $\theta_j = \frac{n/2+j+1+2s}{N_0+s+1}$. If $N_0 > \frac{3}{2}n + 8$, we see that $\gamma := \frac{1}{2}(1-\theta_\alpha)(1+\nu) + (1-\theta_3)(1-\nu) > 1$ if $s > 0$ and α are chosen small enough, thus in case (1) we get with $\gamma' := (1+\nu)\theta_\alpha/2 + (1-\nu)\theta_3$

$$\|f\|_{H^{-1+s}} \lesssim \|f\|_{H^{-1+s}}^{\gamma}\|f\|_{H^{N_0}}^{\gamma'}.$$

Thus if $f \neq 0$ we obtain, if $\|f\|_{H^{N_0}} \leq \varepsilon$

$$1 \lesssim \|f\|_{H^{-1+s}}^{\gamma-1}\|f\|_{H^{N_0}}^{\gamma'} \lesssim \|f\|_{H^{N_0}}^{\gamma-1+\gamma'} \lesssim \varepsilon^{\gamma-1+\gamma'}.$$

Since $\gamma-1+\gamma' > 0$, we see that by taking $\varepsilon > 0$ small enough we obtain a contradiction, thus $f = 0$. This proves Theorem 3.2.1 by choosing $N \geq N_0$. In case (2) (corresponding to Theorem 3.2.2), this is the same argument except that we get a slightly better result due to the L^2 norm in (3.2.12) : N_0 can be chosen to be any number $N_0 > n/2 + 2$. To conclude, we have shown that if $\|g - g_0\|_{C^{N,\alpha}} < \varepsilon$ for $N \in \mathbb{N}$ with $N + \alpha > n/2 + 2$, then $L_g \geq L_{g_0}$ implies that either $\mathrm{vol}(g) \leq \mathrm{vol}(g_0)$ and $\phi^* g = g_0$ for some $C^{N+1,\alpha}$ diffeomorphism (thus actually $\mathrm{vol}(g) = \mathrm{vol}(g_0)$), or $\mathrm{vol}(g) \geq \mathrm{vol}(g_0)$. Note that in both cases, if g is smooth then ϕ is smooth.

Stability estimates for the marked length spectrum. Proof of Theorem 3.2.3. We will apply the same reasoning as before to get a stability estimate for the non-linear problem (the marked length spectrum). We proceed as before and reduce to considering $f = \phi^* g - g_0$ where $\phi \in \mathrm{Diff}_0^{N+1,\alpha}(M)$ and $\|f\|_{C^{N,\alpha}} \lesssim \varepsilon$ with $\varepsilon = \|g - g_0\|_{C^{N,\alpha}}$. By Theorem 2.1.4, and using (3.2.5) we have for some fixed exponents $\alpha > 0$ small, $0 < s \ll \alpha$ and ν (in particular ν, α, s can be made arbitrarily small) :

$$\begin{aligned} \|f\|_{C^s} &\lesssim \|I_2^{g_0} f\|_{\ell^\infty}^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} \\ &\lesssim (\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty} + \|f\|_{C^3}^2)^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} \\ &\lesssim \left(\|\mathcal{L}(g) - \mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2} + \|f\|_{C^3}^{1-\nu}\|f\|_{C^\alpha}^{(1+\nu)/2}\right). \end{aligned}$$

Since $1-\nu+(1+\nu/2)>1$, we can interpolate like in the previous proofs and obtain for $N_0>0$ large enough :

$$\|f\|_{C^s} \lesssim \|\mathcal{L}(g)-\mathbf{1}\|_{\ell^\infty}^{(1-\nu)/2}\|f\|_{C^\alpha}^{(1+\nu)/2}+\|f\|_{C^s}\|f\|_{C^{N_0}}^{\gamma}, \tag{3.2.14}$$

for some $\gamma>0$. Taking $\varepsilon>0$ small enough and $\|f\|_{C^{N_0}}\lesssim\varepsilon$, we can swallow the second term on the right-hand side in the left hand side. This provides the sought inequality.

Compactness theorems and proof of Theorem 3.2.4. We let M be a closed smooth manifold equipped with an Anosov geodesic flow. By the proof of [Kni12, Theorem 4.8], the universal cover $\widetilde{M}$ and the fundamental group $\pi_1(M):=\pi_1(M,x_0)$ (for some arbitrary $x_0\in M$) are hyperbolic in the sense of Gromov [Gro87]. We shall denote by $\mathcal{R}_g$ the curvature tensor associated to the metric g and by $\mathrm{inj}(g)$ the injectivity radius of g. We proceed by contradiction : let $(g_n)_{n\geq 0}$ be a sequence of smooth metrics on M in the class $\mathrm{An}(\nu_0,\nu_1,C_0,\theta_0)$ such that $L_{g_n}=L_{g_0}$ and such that for each $k\in\mathbb{N}$ there is $B_k>0$ such that $|\nabla^k_{g_n}\mathcal{R}_{g_n}|_{g_n}\leq B_k$ for all n, and we assume that for each $n\neq n'$, g_n is not isometric to $g_{n'}$. Since the metrics have Anosov flow, they have no conjugate points and thus

$$\mathrm{inj}(g_n)=\tfrac{1}{2}\min_{c\in\mathcal{C}}L_{g_n}(c)=\tfrac{1}{2}\min_{c\in\mathcal{C}}L_{g_0}(c).$$

By Hamilton's compactness result [Ham95, Theorem 2.3], if $\mathrm{vol}(g_n)$ is uniformly bounded, there is a family of smooth diffeomorphisms ϕ_n on M such that $g'_n:=\phi_n^*g_n$ converges to $g\in\mathrm{An}(\nu_0,\nu_1,C_0,\theta_0)$ in the C^∞ topology (note that $\mathrm{An}(\nu_0,\nu_1,C_0,\theta_0)$ is invariant by pull-back through smooth diffeomorphisms). Denote by $\phi_{n*}\in\mathrm{Out}(\pi_1(M))$ the action of ϕ_n on the set of conjugacy classes $\mathcal{C}$. The universal cover $\widetilde{M}$ of M is a ball since M has no conjugate points, and $\pi_1(M)$ is a hyperbolic group thus we can apply the result of Gromov [Gro87, Theorem 5.4.1] saying that the outer automorphism group $\mathrm{Out}(\pi_1(M))$ is finite if $\dim M\geq 3$. This implies in particular that there is a subsequence $(\phi_{n_j})_{j\in\mathbb{N}}$ such that ${\phi_{n_j}}_*(c)=\phi_{n_0*}(c)$ for all $c\in\mathcal{C}$ and all $j\in\mathbb{N}$ where as before $\mathcal{C}$ is the set of conjugacy classes of $\pi_1(M)$. But $\phi_{n_0}^*g_{n_j}$ have the same marked length spectrum as $\phi_{n_0}^*g_0$ for all j, thus $L_{g'_{n_j}}=L_{\phi_{n_0}^*g_0}$ for all j. Since $g'_{n_j}\to g$ in C^∞, we have $L_g=L_{g'_{n_j}}$ for all j and by Theorem 3.2.1, we deduce that there is j_0 such that for all $j\geq j_0$, g'_{n_j} is isometric to g. This gives a contradiction.

Now, if $\dim M=2$, $\mathrm{Out}(\pi_1(M))$ is a discrete infinite group. We first show that for each $c\in\mathcal{C}$, the set of classes $(\phi_n^{-1})_*(c)\in\mathcal{C}$ is finite as n ranges over $\mathbb{N}$. Assume the contrary, then consider γ_n the geodesic for g_n in the class c, one has $L_{g_n}(c)=\ell_{g_n}(\gamma_n)=\ell_{g_0}(\gamma_0)$, by assumption. Now $\phi_n^{-1}(\gamma_n)$ is a g'_n geodesic in the class $(\phi_n^{-1})_*(c)$ with length $\ell_{g'_n}(\phi_n^{-1}(\gamma_n))=\ell_{g_n}(\gamma_n)=\ell_{g_0}(\gamma_0)$. We know that there are finitely many g-geodesics with length less than $\ell_{g_0}(\gamma_0)$, but we also have

$$L_g((\phi_n^{-1})_*(c))\leq\ell_g(\phi_n^{-1}(\gamma_n))\leq\ell_{g'_n}(\phi_n^{-1}(\gamma_n))(1+\varepsilon)\leq\ell_{g_0}(\gamma_0)(1+\varepsilon),$$

if $\|g'_n-g\|_{C^3}\leq\varepsilon$. Thus we obtain a contradiction for n large. The extended mapping class group[3] $\mathrm{Mod}(M)$ is isomorphic to $\mathrm{Out}(\pi_1(M))$ (see [FM12, Theorem 8.1]). By [FM12, Proposition 2.8][4], if M has genus at least 3, there is a finite set $\mathcal{C}_0\subset\mathcal{C}$ such that if $\phi_*\in\mathrm{Mod}(M)$ is the identity on $\mathcal{C}_0$ then ϕ is homotopic to the identity, while if

3. Extended in the sense that it includes orientation reversing elements.

4. See also the proof of Theorem 3.10 in [FM12].

M has genus 2, the same condition implies that ϕ is either homotopic to the identity or to an hyperelliptic involution h. In both cases, we can extract a subsequence ϕ_{n_j} such that $\phi_{n_{j*}} = \phi_{n_{0*}}$ for all $j \geq 0$ and we conclude like in the higher dimensional case. By [CK94, FO87], L_g determines $\mathrm{vol}(g)$ if $\dim(M) = 2$, thus the volume bound is satisfied.

3.3 Asymptotic behavior of the marked length spectrum

One of the aims of this section is to further investigate the previous local rigidity result from different perspectives : new stability estimates and a refined characterization of the condition under which the isometry may hold. More precisely, our first result is that we can locally relax the assumption that the two marked length spectra of g and g_0 exactly coincide to the weaker assumption that they "coincide at infinity" (in some sense that is made precise below) and still obtain the isometry. From now on, it will be important to distinguish between primitive and non primitive closed geodesics. As a consequence, *$\mathcal{C}$ now denotes the set of primitive free homotopy classes.*

3.3.1 Statement of the results

In the following result, given $c \in \mathcal{C}$, we also write $\delta_{g_0}(c)$ to denote the probability Dirac measure carried by the unique g_0-geodesic $\gamma_{g_0}(c) \in c$. We will say that $L_g/L_{g_0} \to 1$ when

$$\lim_{j\to+\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} = 1,$$

for any sequence $(c_j)_{j\in\mathbb{N}}$ of primitive free homotopy classes such that $L_{g_0}(c_j) \to_{j\to+\infty} +\infty$, or equivalently $\lim_{j\to\infty} L_g(c_j)/L_{g_0}(c_j) = 1$, if $\mathcal{C} = (c_j)_{j\in\mathbb{N}}$ is ordered by the increasing lengths $L_{g_0}(c_j)$. We notice that it is important to consider only the set of *primitive* closed geodesics for $\lim_{j\to+\infty} L_g(c_j)/L_{g_0}(c_j) = 1$ to be (a priori) not equivalent to $L_g = L_{g_0}$. Indeed, assuming that $\lim_{j\to\infty} L_g(c_j)/L_{g_0}(c_j) = 1$ for *every* free homotopy classes such that $L_{g_0}(c_j) \to_{j\to+\infty} +\infty$ would immediately imply that $L_g = L_{g_0}$ by simply considering iterates of a given free homotopy class c_0.

Theorem 3.3.1. *Let (M, g_0) be a smooth Anosov Riemannian $(n+1)$-dimensional manifold and further assume that its curvature is nonpositive if $n+1 \geq 3$. There exists $k \in \mathbb{N}$ depending only on n, $\varepsilon > 0$ depending on g_0 such that for each $\alpha \in (0,1)$ the following holds : let $g \in \mathrm{Met}^{k,\alpha}$ such that $\|g - g_0\|_{C^{k,\alpha}} \leq \varepsilon$ and assume that $L_g/L_{g_0} \to 1$. Then g is isometric to g_0.*

We develop a new strategy of proof, different from the previous section, which relies on the introduction of the *geodesic stretch* between two metrics. This quantity was introduced by Croke-Fathi [CF90] and was further studied by Knieper [Kni95]. If g is close enough to g_0, then by Anosov structural stability, the geodesic flows φ^{g_0} and φ^g are orbit conjugate via a homeomorphism ψ, i.e. they are conjugate up to a time reparametrization. The *infinitesimal stretch* is the infinitesimal function of time reparametrization a_g and satisfies $d\psi_g(z)X_{g_0}(z) = a_g(z)X_g(\psi_g(z))$ where $z \in SM_{g_0}$ and X_{g_0} (resp. X_g) denotes the geodesic vector field of g_0 (resp. g). The *geodesic stretch between g and g_0 with respect to the Liouville measure $\mu^{\mathrm{L}}_{g_0}$ of g_0* is then defined by

$$I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) := \int_{SM_{g_0}} a_g \, d\mu^{\mathrm{L}}_{g_0}.$$

It turns out to be equal to

$$I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) = \lim_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

if $(c_j)_{j\in\mathbb{N}} \subset \mathcal{C}$ is a sequence so that the uniform probability measures $(\delta_{g_0}(c_j))_{j\in\mathbb{N}}$ supported on the closed geodesics of g_0 in the class c_j converge to $\mu^{\mathrm{L}}_{g_0}$ in the weak sense of measures. While it has an interest on its own, it also turns out that this method involving the geodesic stretch provides a new estimate which quantifies locally the distance between isometry classes in terms of this geodesic stretch functional

Theorem 3.3.2. *Let (M, g_0) be a smooth Riemannian $(n+1)$-dimensional manifold with Anosov geodesic flow and further assume that its curvature is nonpositive if $n+1 \geq 3$. There exist $k \in \mathbb{N}$ large enough depending only on n, some constants $C, C', \varepsilon > 0$ depending on g_0 such that for all $\alpha \in (0,1)$, the following holds : for each $g \in \mathrm{Met}^{k,\alpha}$ with $\|g - g_0\|_{C^{k,\alpha}(M)} \leq \varepsilon$, there exists a $C^{k+1,\alpha}$-diffeomorphism $\phi : M \to M$ such that*

$$\begin{aligned}\|\phi^* g - g_0\|_{H^{-\frac{1}{2}}(M)} &\leq C\left(|1 - I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g)|^{\frac{1}{2}} + |\mathbf{P}(-J^u_{g_0} + a_g - 1)|^{\frac{1}{2}}\right) \\ &\leq C'\left(|\mathcal{L}_+(g)|^{\frac{1}{2}} + |\mathcal{L}_-(g)|^{\frac{1}{2}}\right)\end{aligned}$$

where $J^u_{g_0}$ is the unstable Jacobian of φ^{g_0}, $\mathbf{P}$ denotes the topological pressure for the φ^{g_0} flow, a_g is the reparameterization coefficient relating φ^{g_0} and φ^g defined above, and

$$\mathcal{L}_+(g) := \limsup_{j\to\infty} L_g(c_j)/L_{g_0}(c_j) - 1, \quad \mathcal{L}_-(g) := \liminf_{j\to\infty} L_g(c_j)/L_{g_0}(c_j) - 1.$$

We remark that $\mathbf{P}(-J^u_{g_0} + a_g - 1) = 0$ if $L_g/L_{g_0} \to 1$. This result is an improvement of the Hölder stability result (see Theorem 3.2.3) as it only involves the asymptotic behaviour of L_g/L_{g_0}. We will show that the combination of the Hessians of the geodesic stretch at g_0 and of the pressure functional can be expressed in terms of $\Pi^{g_0}_2$, interpreted as a variance operator, which enjoys uniform lower bounds at least once we have factored out the gauge (the diffeomorphism action by pull-back on metrics).

Using the continuity of the normal operator $g \mapsto \Pi^g_2 \in \Psi^{-1}$ (see Theorem 2.6.1), we will prove the

Theorem 3.3.3. *Let (M, g_0) be a smooth Riemannian $(n+1)$-dimensional manifold with Anosov geodesic flow and further assume that its curvature is nonpositive if $n+1 \geq 3$. Then there exists $k \in \mathbb{N}$, $\varepsilon > 0$ and $C_{g_0} > 0$ depending on g_0 such that for all $g_1, g_2 \in \mathrm{Met}$ such that $\|g_1 - g_0\|_{C^k} \leq \varepsilon$, $\|g_2 - g_0\|_{C^k} \leq \varepsilon$, there is a C^k- diffeomorphism $\phi : M \to M$ such that*

$$\|\phi^* g_2 - g_1\|_{H^{-\frac{1}{2}}} \leq C_{g_0}(|\mathcal{L}_+(g_1, g_2)|^{\frac{1}{2}} + |\mathcal{L}_-(g_1, g_2)|^{\frac{1}{2}})$$

with

$$\mathcal{L}_+(g_1, g_2) := \limsup_{j\to\infty} L_{g_2}(c_j)/L_{g_1}(c_j) - 1, \quad \mathcal{L}_-(g_1, g_2) := \liminf_{j\to\infty} L_{g_2}(c_j)/L_{g_1}(c_j) - 1.$$

In particular if $L_{g_1} = L_{g_2}$, then g_2 is isometric to g_1.

The coercive estimate of Lemma 2.5.6 allows also to define a *pressure metric* on the open set consisting of isometry classes of Anosov non-positively curved metric (contained in $\mathrm{Met}/\mathrm{Diff}_0$ if Diff_0 is the group of smooth diffeomorphisms isotopic to the identity) by setting for $h_1, h_2 \in T_{g_0}(\mathrm{Met}/\mathrm{Diff}_0) \subset C^\infty(M, \otimes_S^2 T^*M)$

$$G_{g_0}(h_1, h_2) := \langle \Pi_2^{g_0} h_1, h_2 \rangle_{L^2(M, d\,\mathrm{vol}_{g_0})}.$$

We show in Section 3.3.4 that this metric is well-defined and restricts to (a multiple of) the Weil-Petersson metric on Teichmüller space if $\dim M = 2$: it is related to the construction of Bridgeman-Canary-Labourie-Sambarino [BCLS15, BCS18] and Mc Mullen [MM08], but with the difference that we work here in the setting of variable negative curvature and the space of metrics considered here is infinite dimensional.

We conclude by the following remark : the results above suggest that $|\mathcal{L}_+(g_1, g_2)|^{1/2} + |\mathcal{L}_-(g_1, g_2)|^{1/2}$ is a kind of distance between the isometry classes of g_1 and g_2, which is related to the Riemannian pressure metric G. It would be interesting the understand this link further.

3.3.2 Definition of the geodesic stretch

The space of Riemannian metrics. The group $\mathrm{Diff}_0(M)$ of smooth diffeomorphisms on M that are isotopic to the identity is a Fréchet Lie group in the sense of [Ham82, Section 4.6]. The right action

$$\mathrm{Met} \times \mathrm{Diff}_0 \to \mathrm{Met}, \quad (g, \phi) \mapsto \phi^* g$$

is smooth and proper [Ebi68, Ebi70]. Moreover, if g is a metric with Anosov geodesic flow, it is direct to see from ergodicity that there are no Killing vector fields thus the isotropy subgroup $\{\phi \in \mathrm{Diff}_0 \mid \phi^* g = g\}$ of g is finite. For negatively curved metrics it is shown in [Fra66] that the action is free, i.e. the isotropy group is trivial. One cannot apply the usual quotient theorem [Tro92, p.20] in the setting of Banach or Hilbert manifolds but rather smooth Fréchet manifolds instead (using Nash-Moser theorem). Thus, in the setting of the space Met^- of negatively curved smooth metrics, which is a Frechet manifold, the slice theorem says that there is a neighborhood $\mathcal{U}$ of a fixed Anosov metric g_0, a neighborhood $\mathcal{V}$ of Id in Diff_0 and a Frechet submanifold $\mathcal{S}$ containing g_0 so that

$$\mathcal{S} \times \mathcal{V} \to \mathcal{U}, \quad (g, \phi) \mapsto \phi^* g \tag{3.3.1}$$

is a diffeomorphism of Frechet manifolds, and $T_{g_0}\mathcal{S} = \{h \in T_{g_0}\mathrm{Met} \mid D_{g_0}^* h = 0\}$. Moreover $\mathcal{S}$ parametrizes the set of orbits $g \cdot \mathrm{Diff}_0$ for g near g_0 and $T_g\mathcal{S} \cap T(g \cdot \mathrm{Diff}_0) = 0$.

On the other hand, if one considers $\mathrm{Met}^{k,\alpha}$, the space of metrics with $C^{k,\alpha}$ regularity and $\mathrm{Diff}_0^{k+1,\alpha} := \mathrm{Diff}_0^{k+1,\alpha}(M)$, the group of diffeomorphisms with $C^{k+1,\alpha}$ regularity, then both spaces are smooth Banach manifolds. However, the action of $\mathrm{Diff}_0^{k+1,\alpha}$ on $\mathrm{Met}^{k,\alpha}$ is no longer smooth but only topological which also prevents us from applying the quotient theorem.

Nevertheless, recalling g_0 is smooth, if we consider $\mathcal{O}^{k,\alpha}(g_0) := g_0 \cdot \mathrm{Diff}_0^{k+1,\alpha} \subset \mathrm{Met}^{k,\alpha}$, then this is a smooth submanifold of $\mathrm{Met}^{k,\alpha}$ and

$$T_g \mathcal{O}^{k,\alpha}(g_0) = \left\{ D_g p \mid p \in C^{k+1,\alpha}(M, T^*M) \right\}.$$

Notice that the decomposition of tensors in potential/solenoidal parts (see Theorem B.1.1) in $C^{k,\alpha}$ regularity exactly says that given $g \in \mathcal{O}^{k,\alpha}(g_0)$, one has the decomposition :

$$T_g\mathrm{Met} = T_g\mathcal{O}^{k,\alpha}(g_0) \oplus \ker D_g^*|_{C^{k,\alpha}(M, S^2T^*M)}. \tag{3.3.2}$$

Thus, an infinitesimal perturbation of a metric $g \in \mathcal{O}^{k,\alpha}(g_0)$ by a symmetric 2-tensor that is solenoidal with respect to g is actually an infinitesimal displacement *transversally to the orbit* $\mathcal{O}^{k,\alpha}(g_0)$.

Thermodynamic formalism. Let f be a Hölder-continuous function on SM_{g_0}. We recall that its *pressure* [Wal82, Theorem 9.10] is defined by :

$$\mathbf{P}(f) := \sup_{\mu \in \mathfrak{M}_{\text{inv}}} \left(\mathbf{h}_\mu(\varphi_1^{g_0}) + \int_{SM_{g_0}} f \, d\mu \right), \tag{3.3.3}$$

where $\mathfrak{M}_{\text{inv}}$ denotes the set of invariant (by the flow φ^{g_0}) Borel probability measures and $\mathbf{h}_\mu(\varphi_1^{g_0})$ is the metric entropy of the flow $\varphi_1^{g_0}$ at time 1. It is actually sufficient to restrict the sup to ergodic measures $\mathfrak{M}_{\text{inv,erg}}$ [Wal82, Corollary 9.10.1]. Since the flow is Anosov, the supremum is always achieved for a unique invariant ergodic measure μ_f [HF, Theorem 9.3.4] called the *equilibrium state* of f. The measure μ_f is also mixing and positive on open sets which rule out the possibility of a finite combination of Dirac measures supported on a finite number of closed orbits. Moreover μ_f can be written as an infinite weighted sum of Dirac masses $\delta_{g_0}(c_j)$ supported over the geodesics $\gamma_{g_0}(c_j)$, where $c_j \in \mathcal{C}$ are the primitive classes (see [Par88] for the case $\mathbf{P}(f) \geq 0$ or [PPS15, Theorem 9.17] for the general case). For example when $\mathbf{P}(f) \geq 0$,

$$\int u \, d\mu_f = \lim_{T \to \infty} \frac{1}{N(T,f)} \sum_{\{j | L_{g_0}(c_j) \in [T, T+1]\}} e^{\int_{\gamma_{g_0}(c_j)} f} \int_{\gamma_{g_0}(c_j)} u, \tag{3.3.4}$$

where $N(T,f) := \sum_{j, L_{g_0}(c_j) \in [T,T+1]} L_{g_0}(c_j) e^{\int_{\gamma_{g_0}(c_j)} f}$. When $f \equiv 0$, this is the measure of maximal entropy, also called the *Bowen-Margulis measure* $\mu_{g_0}^{\text{BM}}$. In that case $\mathbf{P}(0) = \mathbf{h}_{\text{top}}(\varphi_1^{g_0})$ is the topological entropy of the flow. When $f = -J_{g_0}^u$, where $J_{g_0}^u : x \mapsto \partial_t |\det d\varphi_t^g(x)|_{E_u(x)}|_{t=0}$ is the *geometric potential*, one obtains the Liouville measure $\mu_{g_0}^{\text{L}}$ induced by the metric g_0. In that case, $\mathbf{P}(-J_{g_0}^u) = 0$. If we fix an exponent of Hölder regularity $\nu > 0$, then the map $C^\nu(SM_{g_0}) \ni f \mapsto \mathbf{P}(f)$ is real analytic [Rue04, Corollary 7.10].

Geodesic stretch. We fix a smooth metric $g_0 \in \text{Met}$ with Anosov geodesic flow and we view the geodesic flow and vector fields of any metric g close to g as living on the unit tangent bundle SM_{g_0} for g_0 by simply pulling them back by the diffeomorphism

$$(x,v) \in SM_{g_0} \to (x, v/|v|_g) \in SM_g.$$

We denote by 2ν the exponent of Hölder regularity of the stable/unstable bundles of g_0. We fix some constant $k \geq 2$ and $\alpha \in (0,1)$. There exists a neighborhood $\mathcal{U} \subset \text{Met}^{k,\alpha}$ of g_0 such that, by the structural stability theorem [dlLMM86], for any $g \in \mathcal{U}$, there exists a Hölder homeomorphism $\psi_g : SM_{g_0} \to SM_{g_0}$, differentiable in the flow direction, which is an orbit conjugacy i.e. such that

$$d\psi_g(z) X_{g_0}(z) = a_g(z) X_g(\psi_g(z)), \qquad \forall z \in SM_{g_0}, \tag{3.3.5}$$

where a_g is a Hölder-continuous function on SM_{g_0}. Moreover, the map

$$g \mapsto (a_g, \psi_g) \in C^\nu(SM_{g_0}) \times C^\nu(SM_{g_0}, SM_{g_0})$$

is C^{k-2} [Con92, Proposition 1.1] and ψ_g is homotopic to Id. Note that neither a_g nor ψ_g are unique but a_g is unique up to a coboundary and in all the following paragraphs, adding a coboundary to a_g will not affect the results. The structural stability theorem can be proved via a local inverse theorem (see [dlLMM86] for instance) and in order to do so, one has to choose a hyperplane distribution that is transverse to the flow. A canonical choice here is $E_s(g_0) \oplus E_u(g_0) = \ker \alpha$ since we have a (contact) geodesic flow and that this distribution is smooth. Once this distribution is chosen, the local inverse theorem provides a canonical function a_g.

From (3.3.5), we obtain that for $t \in \mathbb{R}$, $z \in SM_{g_0}$, $\varphi^g_{\kappa_{a_g}(z,t)}(\psi_g(z)) = \psi_g(\varphi^{g_0}_t(z))$ with :

$$\kappa_{a_g}(z,t) = \int_0^t a_g(\varphi^{g_0}_s(z))\, ds, \qquad (3.3.6)$$

If $c \in \mathcal{C}$ be a free homotopy class, then one has :

$$L_g(c) = \int_0^{L_{g_0}(c)} a_g(\varphi^{g_0}_s(z))\, ds, \qquad (3.3.7)$$

for any $z \in \gamma_{g_0}(c)$, the unique g_0-closed geodesic in c.

Boundary at infinity. We denote by $\widetilde{M}$ the universal cover of M. Given a metric $g \in \mathrm{Met}$ on M, we denote by $\widetilde{g}$ its lift to the universal cover. Given two metrics g_1 and g_2 on M, there exists a constant $c > 0$ such that $c^{-1}g_1 \leq g_2 \leq cg_1$. This implies that any $\widetilde{g}_1$-geodesic is a quasi-geodesic for $\widetilde{g}_2$. In particular, this implies that the *ideal* (or *visual*) boundary $\partial_\infty \widetilde{M}$ is independent of the choice of g and is naturally endowed with the structure of topological manifold. Note that when restricting to metrics with pinched sectional curvatures $-a^2 \leq \kappa \leq -b^2$, the regularity of the ideal boundary becomes Hölder (the Hölder regularity depending on a and b). We refer to [BH99, Chapter H.3] for further details. We denote by $\mathcal{G}_g := S_{\widetilde{g}}\widetilde{M}/\sim$ (where $z \sim z'$ if and only if there exists a time $t \in \mathbb{R}$ such that $\varphi_t(z) = z'$) the set of g-geodesics on $\widetilde{M}$: this is smooth $2n$-dimensional manifold. Moreover, there exists a Hölder continuous homeomorphism $\Phi_g : \mathcal{G}_g \to \partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$, where Δ is the diagonal in $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M}$.

We now consider a fixed metric g_0 on M and a metric g in a neighborhood of g_0. If ψ_g denotes an orbit-conjugacy between the two geodesic flows, then ψ_g induces a homeomorphism $\Psi_g : \mathcal{G}_{g_0} \to \mathcal{G}_g$. The map

$$\Phi_g \circ \Psi_g \circ \Phi_{g_0}^{-1} : \partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta \to \partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$$

is nothing but the identity.

Given $z = (x, v) \in SM_{g_0}$, we denote by $c_{g_0}(z) : t \mapsto c_{g_0}(z,t) \in M$ the unique geodesic[5] such that $c_{g_0}(z,0) = x, \dot{c}_{g_0}(z,0) = v$. We consider $\widetilde{c}_{g_0}(z)$, a lift of $c_{g_0}(z)$ to the universal cover $\widetilde{M}$ and introduce the function

$$b : SM_{g_0} \times \mathbb{R} \to \mathbb{R}, \qquad b(z,t) := d_{\widetilde{g}}(\widetilde{c}_{g_0}(z,0), \widetilde{c}_{g_0}(z,t)),$$

which computes the $\widetilde{g}$-distance between the endpoints of the $\widetilde{g}_0$-geodesic joing $\widetilde{c}_{g_0}(z,0)$ to $\widetilde{c}_{g_0}(z,t)$. It is an immediate consequence of the triangular inequality that $(z,t) \mapsto b(z,t)$ is a subadditive cocycle for the geodesic flow φ^{g_0}, that is :

$$b(z, t+s) \leq b(z,t) + b(\varphi^t_{g_0}(z), s), \quad \forall z \in SM_{g_0}, t, s \in \mathbb{R}$$

5. For the sake of simplicity, we identify the geodesic and its arc-length parametrization.

As a consequence, by the subadditive ergodic theorem (see [Wal82, Theorem 10.1] for instance), we obtain the following

Lemma 3.3.1. *Let μ be an invariant probability measure for the flow $\varphi_t^{g_0}$. Then, the quantity*

$$I_\mu(g_0, g, z) := \lim_{t\to+\infty} b(z,t)/t$$

exists for μ-almost every $z \in SM_{g_0}$, $I_\mu(g_0, g, \cdot) \in L^1(SM_{g_0}, d\mu)$ and this function is invariant by the flow $\varphi_t^{g_0}$.

We define the *geodesic stretch of the metric g, relative to the metric g_0, with respect to the measure μ* by :

$$I_\mu(g_0, g) := \int_{SM_{g_0}} I_\mu(g_0, g, z)\, d\mu(z)$$

When the measure μ in the previous definition is ergodic, the function $I_\mu(g_0, g, \cdot)$ is thus (μ-almost everywhere) equal to the constant $I_\mu(g_0, g)$. We denote by $\delta_{g_0}(c)$ the normalized measure supported on $\gamma_{g_0}(c)$, that is :

$$\delta_{g_0}(c) : f \mapsto \frac{1}{L_{g_0}(c)} \int_0^{L_{g_0}(c)} f(\varphi_t^{g_0}(z))\, dt.$$

Since the geodesic flow $\varphi_t^{g_0}$ on SM_{g_0} satisfies the closing lemma property, by [CS10, Lemma 2.2], we know that the set $\{\delta_{g_0}(c) \mid c \in \mathcal{C}\}$ is dense in the set of all ergodic invariant Borel probability measures. Given $\mu \in \mathfrak{M}_{\text{inv,erg}}$, we can thus find a sequence $(\gamma_{g_0}(c_j))_{j\geq 0}$ of closed g_0-geodesics such that $\lim_{j\to\infty} \delta_{g_0}(c_j) = \mu$ in the weak-sense. Moreover, if μ is chosen to be an equilibrium state, then it is positive on any open sets and this implies that $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu$ for a sequence $(\gamma_{g_0}(c_j))_{j\geq 0}$ such that $L_{g_0}(c_j) \to +\infty$.

We can actually describe the stretch using the time reparametrization a_g.

Lemma 3.3.2. *Let μ be an invariant ergodic Borel probability measure for the flow $\varphi_t^{g_0}$. Then :*

$$I_\mu(g_0, g) = \int_{SM_{g_0}} a_g\, d\mu = \lim_{j\to+\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)},$$

where $(c_j)_{j\geq 0} \in \mathcal{C}^{\mathbb{N}}$ is such that $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu$.

Proof. We first prove the left equality. Let $z \in SM_{g_0}$ and $\widetilde{c}_{g_0}(z)$ be a lift of $c_{g_0}(z)$ to the universal cover. Let $\widetilde{c}_g(\psi_g(z))$ be the unique lift of $c_g(\psi_g(z))$ with the same endpoints on the ideal boundary as $\widetilde{c}_{g_0}(z)$. Then, there exists a constant $C > 0$ such that :

$$|d_{\widetilde{g}}(\widetilde{c}_{g_0}(z,0), \widetilde{c}_{g_0}(z,t)) - \underbrace{d_{\widetilde{g}}\left(\widetilde{c}_g(\psi_g(z),0), \widetilde{c}_g(\psi_g(z), \kappa_{a_g}(t,z))\right)}_{=\kappa_{a_g}(t,z)}| \leq C.$$

This implies, using (3.3.6) that :

$$\lim_{t\to+\infty} b(z,t)/t = \lim_{t\to+\infty} \kappa_{a_g}(z,t)/t = \lim_{t\to+\infty} \frac{1}{t}\int_0^t a_g(\varphi_s^{g_0}(z))\, ds = \int_{SM_{g_0}} a\, d\mu,$$

for μ-almost every $z \in SM_{g_0}$, by the Birkhoff ergodic Theorem [Wal82, Theorem 1.14]. By (3.3.7) we also have

$$\int_{SM_{g_0}} a_g\, d\mu = \lim_{j\to\infty} \langle \delta_{g_0}(c_j), a_g\rangle = \lim_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)}$$

thus the proof is complete. ∎

As a consequence, we immediately obtain the

Corollary 3.3.1. *Let $g \in \mathcal{U}$, a fixed neighborhood of g_0 in $\mathrm{Met}^{k,\alpha}$, and assume that for any sequence of primitive free homotopy classes $(c_j)_{j\geq 0} \in \mathcal{C}^{\mathbb{N}}$ such that $L_{g_0}(c_j) \to \infty$, the ratio $L_g(c_j)/L_{g_0}(c_j) \to_{j\to+\infty} 1$. Then, for any ergodic probability measure μ with respect to φ^{g_0} that is an equilibrium state for some Hölder potential, we have $I_\mu(g_0, g) = 1$.*

Combining with Theorem 3.2.1, we also easily obtain :

Theorem 3.3.4. *Let (M, g_0) be a smooth Riemannian $(n+1)$-dimensional manifold with Anosov geodesic flow, topological entropy $\mathbf{h}_{\mathrm{top}}(g_0) = 1$ and assume that its curvature is nonpositive if $n+1 \geq 3$. Then there exists $k \in \mathbb{N}$ large enough depending only on n, $\varepsilon > 0$ small enough such that the following holds : there is $C > 0$ depending on g_0 so that for each $g \in C^k(M, \otimes_S^2 T^*M)$ with $\|g - g_0\|_{C^k} \leq \varepsilon$, if*

$$\mathbf{h}_{\mathrm{top}}(g) = 1, \qquad \lim_{j\to+\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} = 1,$$

for some sequence $(c_j)_{j\in\mathbb{N}}$ of primitive free homotopy classes such that $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu_{g_0}^{BM}$, then g is isometric to g_0.

Proof. Given a metric g, one has by [Kni95, Theorem 1.2] that

$$\mathbf{h}_{\mathrm{top}}(g) \geq \frac{\mathbf{h}_{\mathrm{top}}(g_0)}{I_{\mu_{g_0}^{\mathrm{BM}}}(g_0, g)}, \tag{3.3.8}$$

with equality if and only if φ_{g_0} and φ_g are, up to a scaling, time-preserving conjugate, that is there exists homeomorphism ψ such that $\psi \circ \varphi_{g_0}^{ct} = \varphi_g^t \circ \psi$ with $c := \mathbf{h}_{\mathrm{top}}(g)/\mathbf{h}_{\mathrm{top}}(g_0)$. In particular, restricting to mtrics with entropy 1 one obtains that $I_{\mu_{g_0}^{\mathrm{BM}}}(g_0, g) \geq 1$ with equality if and only if the geodesic flows are conjugate, that is if and only if $L_g = L_{g_0}$. As a consequence, given g_0, g with entropy 1 such that $L_g(c_j)/L_{g_0}(c_j) \to_{j\to+\infty} 1$ for some sequence $\delta_{g_0}(c_j) \rightharpoonup_{j\to+\infty} \mu_{g_0}^{\mathrm{BM}}$, we obtain that $I_{\mu_{g_0}^{\mathrm{BM}}}(g_0, g) = 1$, hence $L_g = L_{g_0}$. If $k \in \mathbb{N}$ was chosen large enough at the beginning, we can then conclude by the local rigidity of the marked length spectrum (see Theorem 3.2.1). $\square$

It is of no harm to assume that g_0 has entropy 1 : indeed, considering λg_0 for some constant $\lambda > 0$, the entropy scales as $\mathbf{h}_{\mathrm{top}}(\lambda g) = \mathbf{h}_{\mathrm{top}}(g)/\sqrt{\lambda}$ [Pat99, Lemma 3.23]. In particular, this also implies the local rigidity of the marked length spectrum because the topological entropy is determined by the marked length spectrum since $s = \mathbf{h}_{\mathrm{top}}(g)$ is the first pole of the Ruelle zeta function

$$\zeta_g(s) := \prod_{c\in\mathcal{C}} (1 - e^{-sL_g(c)}).$$

We will provide a more direct alternate proof of this fact in a next paragraph using the convexity of the geodesic stretch.

Variance of Anosov flows. As already mentioned in the previous chapter, we can make a link between the operator Π and the variance in the central limit theorem for Anosov geodesic flows. This link will be crucial in the following paragraphs. The *variance* of φ_t with respect to the Liouville measure μ^{L} is defined for $u \in C^\nu(SM), \nu \in (0,1)$ real-valued by :

$$\mathrm{Var}_{\mu^{\mathrm{L}}}(u) := \lim_{T\to\infty} \frac{1}{T} \int_{SM} \Big(\int_0^T u(\varphi_t(z))\, dt\Big)^2 d\mu^{\mathrm{L}}(z), \tag{3.3.9}$$

under the condition that $\int_{SM} u\, d\mu^{\mathrm{L}} = 0$. We observe, using the fact that φ_t preserves μ^{L}, that

$$\begin{aligned}\mathrm{Var}_{\mu^{\mathrm{L}}}(u) &= \lim_{T\to\infty} \frac{1}{T} \int_{SM_{g_0}} \int_0^T \int_0^T u(\varphi_{t-s}(z))u(z)\, dtds d\mu^{\mathrm{L}}(z) \\ &= \lim_{T\to\infty} \int_0^1 \int_{\mathbb{R}} \mathbf{1}_{[(t-1)T,tT]}(r) \langle u\circ\varphi_r, u\rangle_{L^2}\, drdt.\end{aligned}$$

where the L^2 pairing is with respect to μ^{L}. By exponential decay of correlations [Liv04], we have for $|r|$ large

$$|\langle u\circ\varphi_r, u\rangle_{L^2}| \leq Ce^{-\nu|r|}\|u\|^2_{C^\alpha}$$

for some $\nu > 0$, $C > 0$ independent of u. Thus by Lebesgue theorem, we obtain the :

Lemma 3.3.3. *Let $u \in C^\nu(SM), \nu \in (0,1)$. Then :*

$$\mathrm{Var}_{\mu^{\mathrm{L}}}(u) = \langle \Pi u, u\rangle_{L^2}$$

3.3.3 A functional on the space of metrics

A submanifold of the space of metrics. Recall that 2ν is the exponent of Hölder continuity of the stable/unstable vector bundles for the fixed metric g_0. Given a metric g in a $C^{k,\alpha}$-neighborhood of g_0, we define the potential

$$V_g := J^u_{g_0} + a_g - 1 \in C^\nu(SM_{g_0}). \tag{3.3.10}$$

Remark that $g \mapsto V_g \in C^\nu(SM_{g_0})$ is C^{k-2} and for $g = g_0$, $V_{g_0} = J^u_{g_0}$. We introduce the spaces

$$\mathcal{N}^{k,\alpha} := \left\{ g \in \mathrm{Met}^{k,\alpha} \mid \mathbf{P}(-V_g) = 0 \right\}, \tag{3.3.11}$$

and $\mathcal{N}^{k,\alpha}_{\mathrm{sol}} := \mathcal{N}^{k,\alpha} \cap \ker D^*_{g_0}$. In particular, $g_0 \in \mathcal{N}^{k,\alpha}_{\mathrm{sol}}$. Given $g \in \mathcal{N}^{k,\alpha}$, we denote by m_g the unique equilibrium state for the potential V_g. We will also denote $\mathcal{N}$ for the case where $k = \infty$.

Lemma 3.3.4. *There exists a neighborhood $\mathcal{U}_{g_0} \subset \mathrm{Met}^{k,\alpha}$ of g_0 such that $\mathcal{N}^{k,\alpha} \cap \mathcal{U}_{g_0}$ is a codimension one C^{k-2}-submanifold of $\mathcal{U}_{g_0}$ and $\mathcal{N}^{k,\alpha}_{\mathrm{sol}} \cap \mathcal{U}_{g_0}$ is a C^{k-2}-submanifold of $\mathcal{U}_{g_0}$. Similarly, there is $\mathcal{U}_{g_0} \subset \mathrm{Met}$ an open neighborhood so that $\mathcal{N} \cap \mathcal{U}_{g_0}$ is a Frechet submanifold of* Met.

Proof. To prove this lemma, we will use the notion of differential calculus on Banach manifolds as it is stated in [Zei88, Chapter 73]. Note that $\mathrm{Met}^{k,\alpha}$ is a smooth Banach manifold and $\mathcal{N}^{k,\alpha} \subset \mathrm{Met}^{k,\alpha}$ is defined by the implicit equation $F(g) = 0$ for

$$F : g \mapsto \mathbf{P}(-V_g) \in \mathbb{R}. \tag{3.3.12}$$

The map F being C^{k-2} (see §3.3.2, the pressure is real analytic so F inherits the regularity of $g \mapsto a_g$), we only need to prove that dF_{g_0} does not vanish by [Zei88, Theorem 73.C]. This will immediately give that $T_{g_0}\mathcal{N}^{k,\alpha} = \ker dF_{g_0}$.

We first need a deformation lemma. For the sake of simplicity, we write the objects $\cdot_\lambda$ instead of $\cdot_{g_\lambda}$.

Lemma 3.3.5. *Consider a smooth deformation $(g_\lambda)_{\lambda\in(-1,1)}$ of g_0 inside $\mathrm{Met}^{k,\alpha}$. Then, there exists a Hölder-continuous function $f : SM_{g_0} \to \mathbb{R}$ such that*

$$\pi_2^* \left(\partial_\lambda g_\lambda|_{\lambda=0}\right) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0} f.$$

Proof. Let c be a fixed free homotopy class, $\gamma_0 \in c$ be the unique closed g_0-geodesic in the class c, which we parametrize by unit-speed $z_0 : [0, \ell_{g_0}(\gamma_0)] \to SM_{g_0}$. We define $z_\lambda(s) = \psi_\lambda(z_0(s)) = (\alpha_\lambda(s), \dot\alpha_\lambda(s))$ (the dot is the derivative with respect to s) where ψ_λ is the conjugation between g_λ and g_0 : this gives a non-unit-speed parametrization of γ_λ, the unique closed g_λ-geodesic in c. We recall that $\pi : TM \to M$ is the projection. We obtain using (3.3.5)

$$\begin{aligned}
\int_0^{\ell_{g_0}(\gamma_0)} g_\lambda(\dot\alpha_\lambda(s), \dot\alpha_\lambda(s)) ds &= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda\left(\partial_s(\pi\circ z_\lambda(s)), \partial_s(\pi\circ z_\lambda(s))\right) ds \\
&= \int_0^{\ell_{g_0}(\gamma_0)} g_\lambda\left(\partial_s(\pi\circ\psi_\lambda\circ z_0(s)), \partial_s(\pi\circ\psi_\lambda\circ z_0(s))\right) ds \\
&= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s)) \underbrace{g_\lambda\left(d\pi(X_{g_\lambda}(z_\lambda(s))), d\pi(X_{g_\lambda}(z_\lambda(s)))\right)}_{=1} ds \\
&= \int_0^{\ell_{g_0}(\gamma_0)} a_\lambda^2(z_0(s)) ds.
\end{aligned}$$

Since $s \mapsto \alpha_0(s)$ is a unit-speed geodesic for g_0, it is a critical point of the energy functional (with respect to g_0). Thus, by differentiating the previous identity with respect to λ and evaluating at $\lambda = 0$, one obtains :

$$\int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda g_\lambda|_{\lambda=0}(\dot\alpha_0(s), \dot\alpha_0(s)) ds = 2\int_0^{\ell_{g_0}(\gamma_0)} \partial_\lambda a_\lambda|_{\lambda=0}(z_0(s)) ds.$$

As a consequence, $\pi_2^*\left(\partial_\lambda g_\lambda|_{\lambda=0}\right) - 2\partial_\lambda a_\lambda|_{\lambda=0}$ is a Hölder-continuous function in the kernel of the X-ray transform : by the usual Livsic theorem, there exists a function f (with the same Hölder regularity), differentiable in the flow direction, such that $\pi_2^*\left(\partial_\lambda g_\lambda|_{\lambda=0}\right) - 2\partial_\lambda a_\lambda|_{\lambda=0} = X_{g_0} f$. □

We can now complete the proof of Lemma 3.3.4. We first prove the first part concerning $\mathcal{N}^{k,\alpha}$. By [PP90, Proposition 4.10], we have :

$$dF_g.u = -\int_{SM_{g_0}} da_g.u \, dm_g$$

where m_g is the equilibrium measure of V_g. In particular, observe that for $g = g_0$, one has :

$$dF_{g_0}.u = -\int_{SM_{g_0}} da_{g_0}.u \, d\mu^{\mathrm{L}}_{g_0},$$

since $m_{g_0} = \mu^{\mathrm{L}}_{g_0}$. Then, using Lemma 3.3.5, one obtains

$$dF_{g_0}.u = -\int_{SM_{g_0}} da_{g_0}.u \, d\mu^{\mathrm{L}}_{g_0} = -\frac{1}{2}\int_{SM_{g_0}} \pi_2^* u \, d\mu^{\mathrm{L}}_{g_0} = -c_2\langle u, g_0\rangle_{L^2}, \qquad (3.3.13)$$

for some constant $c_2 > 0$. This is obviously surjective and we also obtain :

$$T_{g_0}\mathcal{N}^{k,\alpha} = \ker dF_{g_0} = \left\{u \in C^{k,\alpha}(M, \otimes_S^2 T^*M) \mid \langle u, g_0\rangle = 0\right\} = (\mathbb{R}g_0)^\perp,$$

where the orthogonal is understood with respect to the L^2-scalar product.

We now deal with $\mathcal{N}^{k,\alpha}_{\text{sol}}$. First observe that $\ker D^*_{g_0}$ is a closed linear subspace of $\text{Met}^{k,\alpha}$ and thus a smooth submanifold of $\text{Met}^{k,\alpha}$. By [Zei88, Corollary 73.50], it is sufficient to prove that $\ker D^*_{g_0}$ and $\mathcal{N}^{k,\alpha}$ are transverse at g_0. But observe that $g_0 \in \ker D^*_{g_0} \simeq T_{g_0} \ker D^*_{g_0}$ and thus

$$T_{g_0} \ker D^*_{g_0} + T_{g_0}\mathcal{N}^{k,\alpha} = T_{g_0}\text{Met}^{k,\alpha}$$

showing transversality.

The case of $\mathcal{N}^\infty$ follows directly from Nash-Moser theorem since F is a smooth tame map from a Frechet space to $\mathbb{R}$, with a right inverse H_g for dF_g that is continuous in g : just take $H_g 1 := g$. □

We next show that metrics with the same marked length spectrum at infinity belong to $\mathcal{N}^{k,\alpha}$.

Lemma 3.3.6. *Let $g \in \mathcal{U}$. If for any sequence of primitive free homotopy classes $(c_j)_{j\geq 0} \in \mathcal{C}^{\mathbb{N}}$ such that $L_{g_0}(c_j) \to \infty$, the ratio $L_g(c_j)/L_{g_0}(c_j) \to_{j\to+\infty} 1$, then $g \in \mathcal{N}^{k,\alpha}$.*

Proof. By §3.3.2, one has :

$$\mathbf{P}(-V_g) = \sup_{\mu \in \mathfrak{M}_{\text{inv,erg}}} \left(\mathbf{h}_\mu(\varphi^{g_0}) - \int_{SM_{g_0}} (J^u_{g_0} + a_g - 1)\, d\mu \right).$$

Note that by Corollary 3.3.1, for the equilibrium state m_g of $-V_g$, one has $\int_{SM_{g_0}} (a_g - 1)\, dm_g = I_\mu(g_0, g) - 1 = 0$. Thus :

$$\mathbf{P}(-V_g) = \mathbf{P}(-J^u_{g_0}) = \mathbf{P}(-V_{g_0}) = 0$$

proving the claim. □

By Lemma 3.3.2, we know that

$$I_{\mu^{\text{L}}_{g_0}}(g_0, g) = \int_{SM_{g_0}} a_g \, d\mu^{\text{L}}_{g_0}.$$

We introduce the functional

$$\Phi : \mathcal{N}^{k,\alpha}_{\text{sol}} \to \mathbb{R}, \qquad \Phi(g) := I_{\mu^{\text{L}}_{g_0}}(g_0, g). \tag{3.3.14}$$

Note that Φ is C^{k-2}, its regularity being limited by that of $\mathcal{N}^{k,\alpha}_{\text{sol}}$. Given $h \in T_{g_0}\mathcal{N}^{k,\alpha}_{\text{sol}}$, we thus obtain

$$d\Phi_{g_0}.h = \int_{SM_{g_0}} da_{g_0}.h \, d\mu^{\text{L}}_{g_0} = 0,$$

that is, g_0 is a critical point of the functional Φ on $\mathcal{N}^{k,\alpha}_{\text{sol}}$. We can extend Φ to

$$\widetilde{\Phi} : \text{Met}^{k,\alpha} \to \mathbb{R}, \quad \widetilde{\Phi}(g) = I_{\mu^{\text{L}}_{g_0}}(g_0, g),$$

and we note that

$$d\widetilde{\Phi}_{g_0}.h = \int_{SM_{g_0}} da_{g_0}.h \, d\mu^{\text{L}}_{g_0} = -dF_{g_0}.h = -C_n \langle h, g_0 \rangle \tag{3.3.15}$$

for some constant $C_n > 0$ depending only on $n+1 = \dim(M)$ and $\langle h, g_0 \rangle = \int_M \text{Tr}_{g_0}(h) d\,\text{vol}_{g_0}$.

Lemma 3.3.7. *The map $\Phi : \mathcal{N}^{k,\alpha}_{\text{sol}} \to \mathbb{R}$ is strictly convex at g_0 and there is $C > 0$ such that*

$$d^2\Phi_{g_0}(h,h) = \frac{1}{4}\langle \Pi_2^{g_0} h, h\rangle \geq C\|h\|^2_{H^{-\frac{1}{2}}(M)}$$

for all $h \in T_{g_0}\mathcal{N}^{k,\alpha}_{\text{sol}}$.

Proof. Since g_0 is a critical point of Φ, we have $d^2\Phi_{g_0}(h,h) = \partial^2_\lambda \Phi(g_\lambda)|_{\lambda=0}$ where $g_\lambda := g_0 + \lambda h + \mathcal{O}(\lambda^2)$ is a smooth curve of metrics in $\mathcal{N}^{k,\alpha}$, and we write $a_\lambda := a_{g_\lambda}$, $V_\lambda := V_{g_\lambda}$ and denote by $\dot{x}$ and $\ddot{x}$ the derivatives with respect to λ. By Lemma 3.3.2, we have

$$\partial^2_\lambda \Phi(g_\lambda)|_{\lambda=0} = \int_{SM_{g_0}} \ddot{a}_0 \, d\mu^{\mathrm{L}}_{g_0}. \tag{3.3.16}$$

But we also know that $\mathbf{P}(-V_\lambda) = 0$, thus if we differentiate twice, we obtain

$$d^2\mathbf{P}_{-V_0}(\dot{V}_0, \dot{V}_0) - d\mathbf{P}_{-V_0}(\ddot{V}_0) = 0. \tag{3.3.17}$$

By (3.3.13), we have

$$\int_{SM_{g_0}} \dot{V}_0 \, d\mu^{\mathrm{L}}_{g_0} = \int_{SM_{g_0}} \dot{a}_0 \, d\mu^{\mathrm{L}}_{g_0} = -dF_{g_0}.h = 0,$$

thus we obtain by [PP90, Proposition 4.11] that

$$\begin{aligned} d^2\mathbf{P}_{-V_0}(\dot{V}_0, \dot{V}_0) &= \mathrm{Var}_{\mu^{\mathrm{L}}_{g_0}}(\dot{V}_0) = \langle \Pi^{g_0}\dot{V}_0, \dot{V}_0\rangle, \\ d\mathbf{P}_{-V_0}(\ddot{V}_0) &= \int_{SM_{g_0}} \ddot{V}_0(z)\, d\mu^{\mathrm{L}}_{g_0} \end{aligned} \tag{3.3.18}$$

where $\mathrm{Var}_{\mu^{\mathrm{L}}_{g_0}}(h)$ is the variance defined in (3.3.9), equal to $\langle \Pi^{g_0} h, h\rangle$ by (3.3.3). Also note that $\dot{V}_0 = \dot{a}_0 = \frac{1}{2}\pi_2^*\dot{g}_0 + X_{g_0} f$ for some $f \in C^\nu(SM_{g_0})$, $\nu > 0$, by Lemma 3.3.5. We also have $\ddot{V}_0 = \ddot{a}_0$. As a consequence, we get from (3.3.16), (3.3.17) and (3.3.18)

$$\begin{aligned} d^2\Phi_{g_0}(h,h) &= \int_{SM_{g_0}} \ddot{a}_0 \, d\mu^{\mathrm{L}}_{g_0} \\ &= \langle \Pi^{g_0}\dot{V}_0, \dot{V}_0\rangle \\ &= \frac{1}{4}\langle \Pi^{g_0}\pi_2^*\dot{g}_0, \pi_2^*\dot{g}_0\rangle = \frac{1}{4}\left(\langle \Pi_2^{g_0}\dot{g}_0, \dot{g}_0\rangle - \langle \pi_2^*\dot{g}_0, \mathbf{1}\rangle^2\right) = \frac{1}{4}\langle \Pi_2^{g_0}\dot{g}_0, \dot{g}_0\rangle, \end{aligned}$$

where we used in the third identity that $\Pi^{g_0} X_{g_0} f = 0 = X_{g_0}\Pi^{g_0} f$ if both f and $X_{g_0} f$ are in $C^\nu(SM_{g_0})$ for some $\nu > 0$. In the last equality, $\langle \pi_2^*\dot{g}_0, \mathbf{1}\rangle^2 = c_2\langle \dot{g}_0, g_0\rangle = 0$, since $T_{g_0}\mathcal{N}^{k,\alpha} = (\mathbb{R}g_0)^\perp$. Since $h \in T_{g_0}\mathcal{N}^{k,\alpha}_{\text{sol}}$, h is divergence-free with respect to g_0. The result then follows from Lemma 2.5.6. □

Isometry classes and geodesic stretch. We now prove Theorems 3.3.1 and 3.3.2. Of course, the first one being implied by the second one, we focus on the latter.

Proof of Theorem 3.3.2. From now, C_{g_0}, C'_{g_0} will denote positive constants depending only on g_0, and whose value may change from line to line. Let us pick $g \in \mathrm{Met}^{k,\alpha}$ with $k \geq 5, \alpha \in (0,1)$ as before. We denote by $g' = \phi^* g \in \ker D^*_{g_0}$, with $\phi \in \mathrm{Diff}_0^{k+1,\alpha}$, the

g_0-solenoidal metric obtained by applying Lemma B.1.7. We write the Taylor expansion of both $F(g) = \mathbf{P}(-V_g)$ and $\widetilde{\Phi}(g) = I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g)$ at g_0 using (3.3.15)

$$\begin{aligned}\widetilde{\Phi}(g') &= 1 + d\widetilde{\Phi}_{g_0}(g' - g_0) + \frac{1}{2}d^2\widetilde{\Phi}_{g_0}(g' - g_0, g' - g_0) + \mathcal{O}(\|g' - g_0\|^3_{C^{5,\alpha}}) \\ F(g') &= -d\widetilde{\Phi}_{g_0}(g' - g_0) + \frac{1}{2}d^2F_{g_0}(g' - g_0, g' - g_0) + \mathcal{O}(\|g' - g_0\|^3_{C^{5,\alpha}}).\end{aligned}$$

As in (3.3.17) and using (3.3.18), we have for $h \in C^{k,\alpha}(M, \otimes^2_S T^*M)$

$$\begin{aligned}d^2F_{g_0}(h, h) =& d^2\mathbf{P}_{-V_0}(da_{g_0}h, da_{g_0}h) - d\mathbf{P}_{-V_0}.d^2a_{g_0}(h, h) \\ =& d^2\mathbf{P}_{-V_0}(da_{g_0}h, da_{g_0}h) - d^2\widetilde{\Phi}_{g_0}(h, h)\end{aligned},$$

thus we get

$$\widetilde{\Phi}(g') - 1 + F(g') = \frac{1}{2}d^2\mathbf{P}_{-V_0}\big(da_{g_0}(g' - g_0), da_{g_0}(g' - g_0)\big) + \mathcal{O}(\|g' - g_0\|^3_{C^{5,\alpha}}). \quad (3.3.19)$$

Using [PP90, Proposition 4.11], we get for each $u \in C^\nu(SM_{g_0})$

$$d^2\mathbf{P}_{-V_0}(u, u) = \big\langle \Pi^{g_0}(u - \langle u, \mathbf{1}\rangle), (u - \langle u, \mathbf{1}\rangle)\big\rangle = \langle \Pi^{g_0}u, u\rangle,$$

because $\Pi^{g_0}\mathbf{1} = 0$ and where $\langle u, \mathbf{1}\rangle = \int_{SM_{g_0}} u \, d\mu^{\mathrm{L}}_{g_0}$. By Lemma 3.3.5, $da_{g_0}(g' - g_0) = \frac{1}{2}\pi_2^*(g' - g_0) + X_{g_0}f$ for some $f \in C^\nu(SM_{g_0})$, which then yields for $h := g' - g_0$

$$d^2\mathbf{P}_{-V_0}(da_{g_0}h, da_{g_0}h) = \frac{1}{4}\left(\langle \Pi_2^{g_0}h, h\rangle - \langle h, g_0\rangle^2\right) - d^2\widetilde{\Phi}_{g_0}(h, h),$$

where by our normalization convention $\langle g_0, g_0\rangle = 1$. Combining with (3.3.19), we obtain

$$|\widetilde{\Phi}(g') - 1| + |F(g')| \geq \frac{1}{8}\left(\langle \Pi_2^{g_0}h, h\rangle - \langle h, g_0\rangle^2\right) - C_{g_0}\|h\|^3_{C^{5,\alpha}}.$$

By Lemma 2.5.6 and the fact that $D^*_{g_0}h = 0$, we deduce that

$$|\widetilde{\Phi}(g') - 1| + |F(g')| \geq C_{g_0}\|h\|^2_{H^{-\frac{1}{2}}(M)} - \frac{1}{8}\langle h, g_0\rangle^2 - C'_{g_0}\|h\|^3_{C^{5,\alpha}}.$$

But since by (3.3.15), there is $C_n > 0$ depending only on n so that for $\|g - g_0\|_{C^{5,\alpha}}$ small enough

$$|\langle h, g_0\rangle|^2 = C_n^{-2}|d\widetilde{\Phi}_{g_0}h|^2 \leq 2C_n^{-2}|\widetilde{\Phi}(g') - 1|^2 + C_{g_0}\|h\|^4_{C^{5,\alpha}}$$

we conclude that

$$|\widetilde{\Phi}(g) - 1| + |F(g)| \geq C_{g_0}\|h\|^2_{H^{-\frac{1}{2}}(M)} - C'_{g_0}\|h\|^3_{C^{5,\alpha}}.$$

Using Sobolev embedding and interpolation estimates, we get

$$\|g' - g_0\|^3_{C^{5,\alpha}} \leq C_{g_0}\|g' - g_0\|^3_{H^{\frac{n+1}{2}+5+\alpha'}} \leq C'_{g_0}\|g' - g_0\|^2_{H^{-1/2}}\|g' - g_0\|_{H^k},$$

with $k > \frac{3}{2}(n + 1) + 16 + 3\alpha$ and $\alpha' > \alpha$. Thus assuming that $\|g - g_0\|_{C^{k,\alpha}}$ is small enough depending on C_{g_0}, we obtain

$$\|g' - g_0\|^2_{H^{-1/2}} \leq C_{g_0}\left(|\widetilde{\Phi}(g) - 1| + |F(g)|\right), \quad (3.3.20)$$

We also recall that

$$\mathbf{P}(-V_g) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{c\in\mathcal{C}, L_{g_0}(c)\le T} e^{-\int_{\gamma_{g_0}(c)} V_g} = \lim_{T\to\infty} \frac{1}{T} \log \sum_{c\in\mathcal{C}, L_{g_0}(c)\le T} e^{-\int_{\gamma_{g_0}(c)} J^u_{g_0}} e^{L_{g_0}(c)-L_g(c)}.$$

Thus, if we order $\mathcal{C} = (c_j)_{j\in\mathbb{N}}$ by the lengths (i.e. $L_{g_0}(c_j) \ge L_{g_0}(c_{j-1})$), and we define

$$\mathcal{L}_+(g) := \limsup_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1, \quad \mathcal{L}_-(g) := \liminf_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - 1,$$

we see that for all $\delta > 0$ small, there is $T_0 > 0$ large so that for all j with $T \ge L_{g_0}(c_j) \ge T_0$

$$e^{\min(T(-\mathcal{L}_+(g)-\delta), T_0(-\mathcal{L}_+(g)-\delta))} \le e^{L_{g_0}(c)-L_g(c)} \le e^{\max(T(-\mathcal{L}_-(g)+\delta), T_0(-\mathcal{L}_-(g)+\delta))}$$

thus we deduce that, using $\mathbf{P}(-V_{g_0}) = 0$,

$$-\mathcal{L}_+(g) - \delta \le \mathbf{P}(-V_g) \le -\mathcal{L}_-(g) + \delta.$$

Since $\delta > 0$ is arbitrarily small, we obtain $|\mathbf{P}(-V_g)| \le \max(|\mathcal{L}_+(g)|, |\mathcal{L}_-(g)|)$ and combining with (3.3.20) and Lemma 3.3.2, we get the announced result. $\square$

We remark that the proof above (using (3.3.19)) also shows that if we work on the slice of metrics

$$\{g \in \mathrm{Met}^{k,\alpha} \mid D^*_{g_0} g = 0, \int_M \mathrm{Tr}_{g_0}(h)\, d\mathrm{vol}_{g_0} = 0\}$$

then there is $C_{g_0} > 0, \varepsilon > 0$ such that if $\|g - g_0\|_{C^{k,\alpha}} < \varepsilon$ with ε small enough

$$\|g - g_0\|^2_{H^{-\frac{1}{2}}(M)} \le C_{g_0}\Big(\limsup_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)} - \liminf_{j\to\infty} \frac{L_g(c_j)}{L_{g_0}(c_j)}\Big).$$

We also prove Theorem 3.3.3.

Proof of Theorem 3.3.3. Let $g_0 \in \mathcal{M}$ be Anosov and assume g_0 has non-positive curvature if $n+1 \ge 3$. Using Lemma B.1.7, for $g_1, g_2 \in \mathrm{Met}$ close enough to g_0 in $C^{k,\alpha}$ norm, we can find $\phi \in \mathrm{Diff}_0^{k+1,\alpha}$ (with $k \ge 5$ to be chosen later) such that $D^*_{g_1}(\phi^* g_2) = 0$. Moreover $g_2' = \phi^* g_2$ satisfies

$$\|g_2' - g_1\|_{C^{k,\alpha}} \le C(\|g_1 - g_0\|_{C^{k,\alpha}} + \|g_2 - g_0\|_{C^{k,\alpha}})$$

for some C depending only on g_0. We can then rewrite the proof of Theorem 3.3.2 but by replacing g_0 by g_1. This gives that for g_1, g_2 close enough to g_0 in $\mathrm{Met}^{5,\alpha}$

$$|\widetilde{\Phi}_{g_1}(g_2) - 1| + |F_{g_1}(g_2)| \ge C_n \rangle \Pi_2^{g_1}(g_2 - g_1'), (g_2' - g_1)\rangle - C'_{g_1}\|g_2 - g_1\|^3_{C^{5,\alpha}}$$

where C_n depends only on $n+1 = \dim(M)$ and C_{g_1} depends on $\|g_1\|_{C^{5,\alpha}}$, and $F_{g_1}(g_2) := \mathbf{P}(-J^u_{g_1} - a_{g_1,g_2} + 1)$ while $\widetilde{\Phi}_{g_1}(g_2) = I_{\mu^{\mathrm{L}}_{g_1}}(g_1, g_2)$, where a_{g_1,g_2} is the time reparameterization coefficient in the conjugation between the flows φ^{g_1} and φ^{g_2} and the pressure and the stretch are taken with respect to the flow φ^{g_1}. Combining Theorem 2.6.1 and Lemma 2.5.7, we deduce that there is $C_{g_0}, C'_{g_0} > 0$ depending only on g_0 so that for $g_1, g_2 \in \mathrm{Met}$ in a small enough neighborhood of g_0 in the C^∞ topology,

$$|\widetilde{\Phi}_{g_1}(g_2) - 1| + |F_{g_1}(g_2)| \ge C_{g_0}\|g_2' - g_1\|_{H^{-\frac{1}{2}}(M)} - C'_{g_0}\|g_2 - g_1\|^3_{C^{5,\alpha}}.$$

This means that there is $\varepsilon > 0$ depending on g_0 and k large enough so that for all $g_1, g_2 \in \mathrm{Met}$ smooth satisfying $\|g_j - g_0\|_{C^{k,\alpha}(M)} \le \varepsilon$ the estimate above hold. Reasoning like in the proof of Theorem 3.3.2, we obtain the result. $\square$

3.3.4 The pressure metric on the space of negatively curved metrics

Definition of the pressure metric using the variance. On Met^-, the cone of smooth negatively-curved metrics, we introduce the non-negative symmetric bilinear form

$$G_g(h_1, h_2) := \langle \Pi_2^g h_1, h_2 \rangle_{L^2(M, d\,\mathrm{vol}_g)}, \tag{3.3.21}$$

defined for $g \in \mathrm{Met}$, $h_j \in T_g\mathrm{Met} \simeq C^\infty(M, \otimes_S^2 T^*M)$. It is nondegenerate on $T_g\mathrm{Met} \cap \ker D_g^*$, namely $G_g(h,h) \geq C_g \|h\|^2_{H^{-1/2}}$ by Lemma 2.5.6. To get a uniform bound on the coercivity of Π_2^g for g near a given metric g_0, we need to apply Lemma 2.5.7 and the continuity of $g \mapsto \Pi_2^g \in \Psi^{-1}(M)$ proved in Theorem 2.6.1. Combining these facts, we obtain

Proposition 3.3.1. *Let $g_0 \in \mathrm{Met}^-$, then the bilinear form G defined in (3.3.21) produces a Riemannian metric on the quotient space $\mathrm{Met}^-/\mathrm{Diff}_0$ near the class $[g_0]$, where $\mathrm{Met}^-/\mathrm{Diff}_0$ is identified with the slice $\mathcal{S}$ passing through g_0 as in (3.3.1).*

Proof. It suffices to show that G is non-degenerate on $T\mathcal{S}$. Let $h \in T_g\mathcal{S}$ and assume that $G_g(h,h) = 0$. We can write $h = L_V g + h'$ where $D_g^* h' = 0$ and V is a smooth vector field. By Lemma 2.5.6 we obtain $0 = G_g(h,h) \geq C\|h'\|_{H^{-1/2}}$. Thus $h = \mathcal{L}_V g$, but we also know that $T_g\mathcal{S} \cap \{\mathcal{L}_V g \mid V \in C^\infty(M, T^*M)\} = \{0\}$ since $\mathcal{S}$ is a slice. Therefore $h = 0$. □

Definition using the intersection number. Let us assume that g is in a fixed C^2-neighborhood of g_0. Since $J_{g_0}^u > 0$, we obtain that $V_g = J_{g_0}^u + a_g - 1 > 0$ if g is close enough to g_0. By [Sam14, Lemma 2.4], there exists a unique constant $\mathbf{h}_{V_g} \in \mathbb{R}$ such that $\mathbf{P}(-\mathbf{h}_{V_g} V_g) = 0$. In particular, $\mathcal{N}$ coincides in a neighborhood of g_0 with the set $\{g \in \mathrm{Met} \mid \mathbf{h}_{V_g} = 1\}$. One can express the constant $\mathbf{h}_{V_g}$ as $\mathbf{h}_{V_g} = \mathbf{h}_{\mathrm{top}}(\varphi_t^{g_0, V_g})$, where $\varphi_t^{g_0,V_g}$ is a time-reparametrization of the geodesic flow of g_0 (see [BCLS15, Section 3.1.1]). More precisely, given $f \in C^\nu(SM_{g_0})$ a Hölder-continuous positive function on SM_{g_0}, we introduce $\mathbf{h}_f$ to be the unique real number such that $\mathbf{P}(-\mathbf{h}_f f) = 0$ and we set :

$$SM_{g_0} \times \mathbb{R} \ni (z,t) \mapsto \kappa_f(z,t) := \int_0^t f(\varphi_s^{g_0}(z))\, ds.$$

For a fixed $z \in SM_{g_0}$, this is a homeomorphism on $\mathbb{R}$ and thus allows to define :

$$\varphi^{g_0,f}_{\kappa_f(z,t)}(z) := \varphi_t^{g_0}(z). \tag{3.3.22}$$

We now follow the approach of [BCLS15, Section 3.4.1]. Given two Hölder-continous functions $f, f' \in C^\nu(SM_{g_0})$ such that $f > 0$, one can define an *intersection number* [BCLS15, Eq. (13)]

$$\mathbf{I}_{g_0}(f, f') := \frac{\int_{SM_{g_0}} f'\, d\mu_{-\mathbf{h}_f f}}{\int_{SM_{g_0}} f\, d\mu_{-\mathbf{h}_f f}},$$

where $d\mu_{-\mathbf{h}_f f}$ is the equilibrium measure for the potential $-\mathbf{h}_f f$. We have the following result, which follows from [BCLS15, Proposition 3.8] stated for Anosov flows on compact metric spaces :

Proposition 3.3.2 (Bridgeman-Canary-Labourie-Sambarino [BCLS15]). *Let $f, f' : SM_{g_0} \to \mathbb{R}_+$ be two Hölder-continuous positive functions. Then :*

$$\mathbf{J}_{g_0}(f, f') := \frac{\mathbf{h}_{f'}}{\mathbf{h}_f}\mathbf{I}_{g_0}(f, f') \geq 1$$

with equality if and only if $\mathbf{h_f} f$ and $\mathbf{h_{f'}} f'$ are cohomologous for the geodesic flow $\varphi_t^{g_0}$ of g_0. The quantity $\mathbf{J}_{g_0}(f, f')$ is called the renormalized intersection number.

We apply the previous proposition with $f := J^u_{g_0}$ (then $\mathbf{h}_{J^u_{g_0}} = 1$) and $f' := V_g$. Without assuming that $g \in \mathcal{N}$ (that is we do not necessarily assume that $\mathbf{h}_{V_g} = 1$), we have

$$\begin{aligned}\mathbf{J}_{g_0}(J^u_{g_0}, V_g) = \mathbf{h}_{V_g}\mathbf{I}_{g_0}(J^u_{g_0}, V_g) &= \mathbf{h}_{V_g}\frac{\int_{SM_{g_0}}(J^u_{g_0} + a_g - \mathbf{1})\, d\mu^{\mathrm{L}}_{g_0}}{\int_{SM_{g_0}} J^u_{g_0} d\mu^{\mathrm{L}}_{g_0}} \\ &= \mathbf{h}_{V_g}\frac{\mathbf{h}_L(g_0) + I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) - 1}{\mathbf{h}_L(g_0)} \geq 1\end{aligned}$$

where $\mathbf{h}_L(g_0)$ is the entropy of Liouville measure for g_0. In the specific case where $g \in \mathcal{N}$, $\mathbf{h}_{V_g} = 1$ and we find that $I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) \geq 1$ with equality if and only if a_g is cohomologous to 1, that is if and only if $L_g = L_{g_0}$, or alternatively if and only if φ^g and φ^{g_0} are time-preserving conjugate. This computation holds as long as $J^u_{g_0} + a_g - 1 > 0$ (which is true in a C^2-neighborhood of g_0).

In particular, on $\mathcal{N}$, we have the linear relation

$$\mathbf{J}_{g_0}(J^u_{g_0}, V_g) = 1 + \frac{I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g) - 1}{\mathbf{h}_L(g_0)}.$$

In the notations of [BCLS15, Proposition 3.11], the second derivative computed for the family $(g_\lambda)_{\lambda \in (-1,1)} \in \mathcal{N}^\infty$ is

$$\partial^2_\lambda \mathbf{J}_{g_0}(J^u_{g_0}, V_{g_\lambda})|_{\lambda=0} = \frac{1}{\mathbf{h}_L(g_0)}\partial^2_\lambda I_{\mu^{\mathrm{L}}_{g_0}}(g_0, g_\lambda)|_{\lambda=0} = \frac{\langle \Pi_2^{g_0}\dot{g}_0, \dot{g}_0\rangle}{4\mathbf{h}_L(g_0)} \tag{3.3.23}$$

and is called the *pressure form*. When considering a slice transverse to the Diff_0 action on $\mathcal{N}$, it induces a metric called the *pressure metric* by Lemma 2.5.6. To summarize :

Lemma 3.3.8. *Given a smooth metric g_0, the metric G_{g_0} restricted to $\mathcal{N}$ can be obtained from the renormalized intersection number by*

$$G_{g_0}(h, h) = 4\mathbf{h}_L(g_0)\partial^2_\lambda \mathbf{J}_{g_0}(J^u_{g_0}, V_{g_\lambda})|_{\lambda=0}$$

where $(g_\lambda)_{\lambda \in (-1,1)}$ is any family of metrics such that $g_\lambda \in \mathcal{N}$ and $\dot{g}_0 = h \in T_{g_0}\mathcal{N}$.

Link with the Weil-Petersson metric. We now assume that $M = S$ is an orientable surface of genus ≥ 2 and let $\mathcal{T}(S)$ be the Teichmüller space of S. We fix a hyperbolic metric g_0. Given $\eta, \rho \in \mathcal{T}(S)$, the intersection number is defined as

$$\mathbf{I}(\eta, \rho) := \mathbf{I}_{g_0}(a_{g_\eta}, a_{g_\rho}) = \frac{\int_{SM_{g_0}} a_{g_\rho} d\mu_\eta}{\int_{SM_{g_0}} a_{g_\eta} d\mu_\eta}$$

where $[g_\eta] = \eta, [g_\rho] = \rho$ and μ_η is the equilibrium state of $-h_{a_{g_\eta}} a_{g_\eta}$. Note that $\mathbf{h}_{a_{g_\eta}} = \mathbf{h}_{\mathrm{top}}(\varphi_t^{g_0,a_\eta}) = 1$ since φ^{g_0,a_η} is conjugate to the geodesic flow of g_η, which in turn has constant curvature and that [Sam14, Lemma 2.4], $a_{g_\eta} d\mu_\eta / \int_{SM_{g_0}} a_{g_\eta} d\mu_\eta$ is the measure of maximal entropy of the flow $\varphi_t^{g_0,a_\eta}$, thus also the normalized Liouville measure of g_η (viewed on SM_{g_0}). This number $\mathbf{I}(\eta,\rho)$ is in fact *independent of* g_0 as it can alternatively be written

$$\mathbf{I}(\eta,\rho) = \lim_{T\to\infty} \frac{1}{N_T(\eta)} \sum_{c\in\mathcal{C}, L_{g_\eta}(c)\le T} \frac{L_{g_\rho}(c)}{L_{g_\eta}(c)},$$

where $N_T = \sharp\{c \in \mathcal{C} \mid L_{g_\eta}(c) \le T\}$ (see [BCS18, Proof of Th. 4.3]). In particular, taking $g_0 = g_\eta$, one has

$$\mathbf{I}(\eta,\rho) = I_{\mu^{\mathrm{L}}_{g_\eta}}(g_\eta, g_\rho).$$

As explained in [BCS18, Theorem 4.3], up to a normalization constant c_0 depending on the genus only, the Weil-Petersson metric on $\mathcal{T}(S)$ is equal to

$$\|u\|^2_{\mathrm{WP}} = c_0 \partial_\lambda^2\, \mathbf{I}(\eta,\eta_\lambda)|_{\lambda=0} = c_0 \partial_\lambda^2 I_{\mu^{\mathrm{L}}_{g_\eta}}(g_\eta, g_{\eta_\lambda})|_{\lambda=0}, \tag{3.3.24}$$

where $\dot\eta_0 = u$ and $(g_{\eta_\lambda})_{\lambda\in(-1,1)}$ is a family of hyperbolic metrics such that $[g_{\eta_\lambda}] = \eta_\lambda$, $\eta = \eta_0 = [g_0]$. This fact follows from combined works of Thurston, Wolpert [Wol86] and Mc Mullen [MM08] : the length of a random geodesic γ on (S, g_0) with respect to g_{η_λ} has a local minimum at $\lambda = 0$ and the Hessian is positive definite (Thurston), equals to the Weil-Petersson norm squared of $\dot g$ (Wolpert [Wol86]) and given by a variance (Mc Mullen [MM08]). Here random means equidistributed with respect to the Liouville measure of g_0. We can check that the metric G also corresponds to this metric

Proposition 3.3.3. *The metric G on $\mathcal{T}(S)$ is a multiple of the Weil-Petersson metric.*

Proof. This follows directly from (3.3.23), (3.3.24) and the fact that $\mathbf{h}_L(g_\eta) = 1$ if g_η has curvature -1. □

Remark 3.3.1. We notice that the positivity of the metric in the case of Teichmüller space follows only from some convexity argument in finite dimension. In the case of general metrics with negative curvature, the coercive estimate of Lemma 2.5.6 on the variance is much less obvious due to the infinite dimensionality of the space. As it turns out, this is the key for the local rigidity in Theorem 3.3.1.

3.3.5 Distances from the marked length spectrum

In this paragraph, we discuss different notions of distances involving the marked length spectrum on the space of isometry classes of negatively-curved metrics.

Length distance. We define the following map :

Definition 3.3.1. Let k be as in Theorem 3.3.3. We define the marked length distance map $d_L : \mathrm{Met}^{k,\alpha} \times \mathrm{Met}^{k,\alpha} \to \mathbb{R}^+$ by

$$d_L(g_1,g_2) := \limsup_{j\to\infty} \left| \log \frac{L_{g_1}(c_j)}{L_{g_2}(c_j)} \right|^{\frac{1}{2}} + \limsup_{j\to\infty} \left| \log \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} \right|^{\frac{1}{2}}.$$

We get as a Corollary of Theorem 3.3.3 :

Corollary 3.3.2. *The map d_L descends to the set of isometry classes near g_0 and defines a distance in a small $C^{k,\alpha}$-neighborhood of the isometry class of g_0.*

Proof. It is clear that d_L is invariant by action of diffeomorphisms homotopic to the identity since $L_g = L_{\psi^* g}$ for such diffeomorphisms ψ. Now let g_1, g_2, g_3 three metrics. We have

$$\begin{aligned}\limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_2}(c_j)}\right|^{\frac{1}{2}} &= \limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_3}(c_j)}\frac{L_{g_3}(c_j)}{L_{g_2}(c_j)}\right|^{\frac{1}{2}} \\ &\leq \limsup_{j\to\infty}\left|\log\frac{L_{g_1}(c_j)}{L_{g_3}(c_j)}\right|^{\frac{1}{2}} + \limsup_{j\to\infty}\left|\log\frac{L_{g_3}(c_j)}{L_{g_2}(c_j)}\right|^{\frac{1}{2}}.\end{aligned}$$

thus d_L satisfies the triangular inequality. Finally, By Theorem 3.3.3, if $d_L(g_1, g_2) = 0$ with g_1, g_2 in the $C^{k,\alpha}$ neighborhood $\mathcal{U}_{g_0}$ of Theorem 3.3.3, we have g_1 isometric to g_2, showing that d_L produces a distance on the quotient of $\mathcal{U}_{g_0}$ by diffeomorphisms. □

We also note that Theorem 3.3.3 states that there is $C_{g_0} > 0$ such that for each $g_1, g_2 \in C^{k,\alpha}(M; S^2T^*M)$ close to g_0 there is a diffeomorphism such that

$$d_L(g_1, g_2) \geq C_{g_0}\|\psi^* g_1 - g_2\|_{H^{-1/2}}$$

showing that the pressure norm is controlled by the d_L distance.

Thurston distance. We also introduce the Thurston distance on metrics with topological entropy 1, generalizing the distance introduced by Thurston in [Thu98] for surfaces on Teichmüller space (all hyperbolic metrics on surface have topological entropy equal to 1). We denote by $\mathcal{E}$ (resp. $\mathcal{E}^{k,\alpha}$) the space of metrics in Met (resp. in $\mathrm{Met}^{k,\alpha}$) with topological entropy $\mathbf{h}_{\mathrm{top}} = 1$. With the same arguments than in Lemma 3.3.4, this is a codimension 1 submanifold of Met and if $g_0 \in \mathcal{E}^{k,\alpha}$, one has :

$$T_{g_0}\mathcal{E}^{k,\alpha} := \left\{ h \in C^{k,\alpha}(M; S^2T^*M) \mid \int_{S_{g_0}M} \pi_2^* h \, d\mu_{g_0}^{\mathrm{BM}} = 0 \right\} \tag{3.3.25}$$

Definition 3.3.2. We define the Thurston non-symmetric distance map $d_T : \mathcal{E}^{k,\alpha} \times \mathcal{E}^{k,\alpha} \to \mathbb{R}^+$ by

$$d_T(g_1, g_2) := \limsup_{j\to\infty} \log\frac{L_{g_2}(c_j)}{L_{g_1}(c_j)}.$$

We will prove the

Proposition 3.3.4. *The map d_T descends to the set of isometry classes of metrics in $\mathcal{E}^{k,\alpha}$ (for $k \in \mathbb{N}$ large enough, $\alpha \in (0,1)$) with topological entropy equal to 1 and defines a non-symmetric distance in a small $C^{k,\alpha}$-neighborhood of the diagonal.*

Moreover, this distance is non-symmetric in the pair (g_1, g_2) which is also the case of the original distance introduced by Thurston [Thu98] but this is just an artificial limitation [6] : *"It would be easy to replace L [7] by its symmetrization $1/2(L(g,h)+L(h,g))$, but it seems that, because of its direct geometric interpretations, L is more useful just as it is."* In order to justify that this is a distance, we start with the

6. Thurston, [Thu98].
7. In the notations of Thurston, $L(g,h) = \limsup_{j\to\infty} \log\frac{L_g(c_j)}{L_h(c_j)}$.

Lemma 3.3.9. *Let $g_1, g_2 \in \mathrm{Met}$. Then :*

$$\limsup_{j\to\infty} \frac{L_{g_2}(c_j)}{L_{g_1}(c_j)} = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2)$$

Here m is seen as an invariant ergodic measure for the flow $\varphi_t^{g_1}$ living on $S_{g_1}M$. However, writing $M = \Gamma\backslash\widetilde{M}$ with $\Gamma \simeq \pi_1(M, x_0)$ for $x_0 \in M$, it can also be identified with a geodesic current on $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$, that is a Γ-invariant Borel measure, also invariant by the flip $(\xi, \eta) \mapsto (\eta, \xi)$ on $\partial_\infty\widetilde{M} \times \partial_\infty\widetilde{M} \setminus \Delta$. This point of view has the advantage of being independent of g_1 (see [ST18]).

Proof. First of all, we claim that

$$\sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2) = \sup_{m\in\mathfrak{M}_{\mathrm{inv}}} I_m(g_1, g_2).$$

Of course, it is clear that $\sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2) \leq \sup_{m\in\mathfrak{M}_{\mathrm{inv}}} I_m(g_1, g_2)$ and thus we are left to prove the reverse inequality. By compactness, we can consider a measure $m_0 \in \mathfrak{M}_{\mathrm{inv}}$ realizing $\sup_{m\in\mathfrak{M}_{\mathrm{inv}}} I_m(g_1, g_2)$. By Choquet representation Theorem (see [Wal82, pp. 153]), there exists a (unique) probability measure τ on $\mathfrak{M}_{\mathrm{inv,erg}}$ such that m_0 admits the ergodic decomposition $m_0 = \int_{\mathfrak{M}_{\mathrm{inv,erg}}} m \, d\tau(m)$. Thus :

$$\begin{aligned}
I_{m_0}(g_1, g_2) &= \int_{S_{g_1}M} a_{g_1,g_2} \, dm_0 \\
&= \int_{\mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_1}M} a_{g_1,g_2} \, dm \, d\tau(m) \\
&\leq \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_1}M} a_{g_1,g_2} \, dm \int_{\mathfrak{M}_{\mathrm{inv,erg}}} d\tau(m) = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2),
\end{aligned}$$

which eventually proves the claim.

Let $(c_j)_{j\in\mathbb{N}}$ be a subsequence such that $\lim_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j)$ realizes the lim sup. Then, by compactness, we can extract a subsequence such that $\delta_{g_1}(c_j) \rightharpoonup m \in \mathfrak{M}_{\mathrm{inv}}$. Thus :

$$L_{g_2}(c_j)/L_{g_1}(c_j) = \langle \delta_{g_1}(c_j), a_{g_1,g_2} \rangle \to_{j\to+\infty} \langle m, a_{g_1,g_2} \rangle = I_m(g_1, g_2),$$

which proves, using our preliminary remark, that

$$\limsup_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2).$$

To prove the reverse inequality, we consider a measure $m_0 \in \mathfrak{M}_{\mathrm{inv,erg}}$ such that $I_{m_0}(g_1, g_2) = \sup_{m\in\mathfrak{M}_{\mathrm{inv,erg}}} I_m(g_1, g_2)$ (which is always possible by compactness). Since m_0 is invariant and ergodic, there exists a sequence of free homotopy classes $(c_j)_{j\in\mathbb{N}}$ such that $\delta_{g_1}(c_j) \rightharpoonup m_0$. Then, like previously, one has

$$I_{m_0}(g_1, g_2) = \lim_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j) \leq \limsup_{j\to+\infty} L_{g_2}(c_j)/L_{g_1}(c_j),$$

which provides the reverse inequality. □

We can now prove Proposition 3.3.4.

Proof of Proposition 3.3.4. By (3.3.8), for $g_1, g_2 \in \mathcal{E}^{k,\alpha}$, we have that $I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) \geq 1$ and thus by Lemma 3.3.9, we obtain that $d_T(g_1, g_2) \geq 0$ (note that g_1 and g_2 do not need to be close for this property to hold). Moreover, triangular inequality is immediate for this distance. Eventually, if $d_T(g_1, g_2) = 0$, then $0 \leq \log I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) \leq d_T(g_1, g_2) = 0$, that is $I_{\mu^{\mathrm{BM}}_{g_1}}(g_1, g_2) = 1$ and by Theorem 3.3.4, it implies that g_1 is isometric to g_2 if g_2 is close enough to g_1 in the $C^{k,\alpha}$-topology (note that this neighborhood depends on g_1). □

We now investigate with more details the structure of the distance d_T. A consequence of Lemma 3.3.9 is the following expression of the Thurston Finsler norm :

Lemma 3.3.10. *Let $g_0 \in \mathcal{E}^{k,\alpha}$ and $(g_t)_{t\in[0,\varepsilon)}$ be a smooth family of metrics and let $f := \partial_t g_t|_{t=0}$. Then :*

$$\|f\|_T := \left.\frac{d}{dt} d_T(g_0, g_t)\right|_{t=0} = \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \pi_2^* f \; dm \tag{3.3.26}$$

The norm $\|\cdot\|_T$ is a Finsler norm *on* $T_{g_0}\mathcal{E}^{k,\alpha} \cap \ker D^*_{g_0}$

Proof. We introduce $u(t) := e^{d_T(g_0,g_t)}$ and write $a_t := a_{g_0,g_t}$ for the time reparametrization (as in (3.3.5)). Then :

$$\begin{aligned} u'(0) = \left.\frac{d}{dt} d_T(g_0, g_t)\right|_{t=0} &= \lim_{t\to 0} \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \frac{a_t - 1}{t} dm \\ &= \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \dot{a}_0 \; dm = \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \pi_2^* f \; dm, \end{aligned}$$

since $\dot{a}_0 = \partial_t a_t|_{t=0}$ and $\pi_2^* f$ are cohomologous by Lemma 3.3.5.

We now prove that this is a Finsler norm in a neighborhood of the diagonal. We fix $g_0 \in \mathcal{E}^{k,\alpha}$. By Lemma B.1.7, isometry classes near g_0 can be represented by solenoidal tensors, namely there exists a $C^{k,\alpha}$-neighborhood $\mathcal{U}$ of g_0 such that for any $g \in \mathcal{U}$, there exists a (unique) $\phi \in \mathrm{Diff}_0^{k+1,\alpha}$ such that $D^*_{g_0}\phi^* g = 0$. Moreover, if $g \in \mathcal{E}^{k,\alpha}$, then $\phi^* g \in \mathcal{E}^{k,\alpha}$. As a consequence, using (3.3.25), the statement now boils down to proving that (3.3.26) is a norm for solenoidal tensors $f \in C^{k,\alpha}(M; S^2T^*M)$ such that $\int_{S_{g_0}M} \pi_2^* f \; d\mu^{\mathrm{BM}}_{g_0} = 0$. Since triangular inequality, $\mathbb{R}_+$-scaling and non-negativity are immediate, we simply need to show that $\|f\|_T = 0$ implies $f = 0$. Now, for such a tensor f, we have

$$\begin{aligned} \mathbf{P}(\pi_2^* f) &= \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \int_{S_{g_0}M} \pi_2^* f dm \\ &\leq \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \mathbf{h}_m(\varphi_1^{g_0}) + \sup_{m \in \mathfrak{M}_{\mathrm{inv,erg}}} \int_{S_{g_0}M} \pi_2^* f dm = \underbrace{\mathbf{h}_{\mathrm{top}}(\varphi_1^{g_0})}_{=1} + 0 \end{aligned}$$

and this supremum is achieved for $m = \mu^{\mathrm{BM}}_{g_0}$ and $\mathbf{P}(\pi_2^* f) = 1$. As a consequence, the equilibrium state associated to the potential $\pi_2^* f$ is the Bowen-Margulis measure $\mu^{\mathrm{BM}}_{g_0}$ (the equilibrium state associated to the potential 0) and thus $\pi_2^* f$ is cohomologous to a constant $c \in \mathbb{R}$ (see [HF, Theorem 9.3.16]) which has to be $c = 0$ since the average of $\pi_2^* f$ with respect to Bowen-Margulis is equal to 0, that is there exists a Hölder-continuous function u such that $\pi_2^* f = Xu$. Since $f \in \ker D^*_{g_0}$, the s-injectivity of the X-ray transform $I_2^{g_0}$ implies that $f \equiv 0$. □

The asymmetric Finsler norm $\|\cdot\|_T$ induces a distance d_F between isometry classes namely

$$d_F(g_1, g_2) = \inf_{\gamma:[0,1]\to\mathcal{E},\gamma(0)=g_1,\gamma(1)=g_2} \int_0^1 \|\dot{\gamma}(t)\|_T \, dt$$

It is easy to prove that $d_T(g_1, g_2) \leq d_F(g_1, g_2)$. Indeed, consider a C^1-path $\gamma : [0,1] \to \mathcal{E}$ such that $\gamma(0) = g_1, \gamma(1) = g_2$. Then, considering $N \in \mathbb{N}, t_i := i/N$, we have by triangular inequality

$$\begin{aligned} d_T(g_1, g_2) \leq \sum_{i=0}^{N-1} d_T(\gamma(t_i), \gamma(t_{i+1})) &= \sum_{i=0}^{N-1} \|\dot{\gamma}(t_i)\|_T (t_{i+1} - t_i) + \mathcal{O}(|t_{i+1} - t_i|^2) \\ &\to_{N\to+\infty} \int_0^1 \|\dot{\gamma}(t)\|_T \, dt, \end{aligned}$$

which proves the claim. In [Thu98], Thurston proves that, in restriction to Teichmüller space, the asymmetric Finsler norm induces the distance d_T, that is $d_T = d_F$. We make the following conjecture, which would imply the marked length spectrum rigidity :

Conjecture 3.3.1. *The distance d_T coincide with d_F for isometry classes of negatively curved metrics with topological entropy equal to* 1.

Deuxième partie

Local rigidity of manifolds with hyperbolic cusps

Chapitre 4

Building parametrices on manifolds with cusps

« *Il est bon qu'il y ait des hérétiques.* »

L'Etrange défaite, Marc Bloch.

This chapter contains most of the article *Local rigidity of manifolds with hyperbolic cusps I. Linear theory and pseudodifferential calculus*, written in collaboration with Yannick Guedes Bonthonneau.

Sommaire

In this chapter, we construct a microlocal framework of inversion of elliptic pseudo-differential operators on manifolds with hyperbolic cusps. This is inspired by Melrose's b-calculus [Mel93] and the cusp calculus developed by Mazzeo-Melrose [MM98]. However, the calculus is somehow different and based on [Bon16, GW17]. This will allow us to treat in further chapters all the operators appearing in the proof of the local rigidity of the marked length spectrum as developed in the previous part in the compact case.

4.1 Introduction

We consider a non-compact smooth manifold N of dimension $d+1$ [1] with a finite number of ends N_ℓ, which take the form

$$Z_{\ell,a} \times F_\ell. \tag{4.1.1}$$

Here, $Z_{\ell,a} = \{z \in Z_\ell \mid y(z) > a\}$, and

$$Z_\ell =]0, +\infty[_y \times \left(\mathbb{R}^d/\Lambda_\ell\right)_\theta.$$

In all generality, $\Lambda_\ell \subset O(d) \ltimes \mathbb{R}^d$ is a crystalographic group. However, according to Bieberbach's Theorem, up to taking a finite cover, we can assume that $\Lambda \subset \mathbb{R}^d$ is a lattice of translations. We will work with that case, and check that the results are stable by taking quotients under free actions of finite groups of isometries.

The slice (F_ℓ, g_{F_ℓ}) is a compact Riemannian manifold. We will use the variables $z = (x, \zeta) \in Z_\ell \times F_\ell$ and $x = (y, \theta) \in [a, +\infty) \times \mathbb{R}^d/\Lambda_\ell$. We assume that N is endowed with a metric g, equal over the cusps to

$$\frac{dy^2 + d\theta^2}{y^2} + g_{F_\ell}.$$

We will also have a vector bundle $L \to N$, and will assume that for each ℓ, there is a vector bundle $L_\ell \to F_\ell$, so that

$$L_{|N_\ell} \simeq Z_\ell \times L_\ell.$$

Whenever L is a hermitian vector bundle with metric g_L, a compatible connection ∇^L is one that satisfies

$$X g_L(Y, Z) = g_L(\nabla^L_X Y, Z) + g_L(Y, \nabla^L_X Z).$$

Taking advantage of the product structure, we impose that when X is tangent to Z,

$$\nabla^L_X Y(x, \zeta) = d_x Y(X) + A_x(X) \cdot Y, \tag{4.1.2}$$

where the connection form $A_x(X)$ is an anti-symmetric endomorphism depending linearly on X, and $A(y\partial_y)$, $A(y\partial_\theta)$ do not depend on y, θ. In particular, we get that the curvature of ∇^L is bounded, as are all its derivatives.

Definition 4.1.1. Such data $(L \to N, g, g_L, \nabla^L)$ will be called an *admissible bundle*.

1. The letter n was tired; we had to resort to its cousin d.

Given a cusp manifold (M,g), namely a manifold whose ends are real hyperbolic cusps, the bundle of differential forms over M is an admissible bundle. Since the tangent bundle of a cusp is trivial, any linearly constructed bundle over M is admissible. For example, the bundle of forms over the Grassmann bundle of M, or over the unit cosphere bundle S^*M. Throughout, the chapter, we will mainly be using Sobolev spaces or Hölder-Zygmund spaces. As usual, when dealing with non-compact manifolds, *weighted spaces* will play an important role. The Sobolev spaces $H^{s,\rho_0,\rho_\perp}(L)$ defined for $s,\rho_0,\rho_\perp \in \mathbb{R}$ are H^s-based Sobolev spaces (see §4.2.1 for the definition of Sobolev norms) with weight y^{ρ_0} on the zero Fourier mode (in the θ variable) and $y^{\rho_\perp}$ for the non-zero Fourier modes. We refer to Definition 4.3.1 for an exact definition.

We are going to prove the following result :

Theorem 4.1.1. *Let L be an admissible bundle in the sense of Definition 4.1.1. Assume that L is endowed with a pseudo-differential operator P. Assume that it is $(\rho_-,\rho_+) - L^2$ (resp. $-L^\infty$)-admissible in the sense of Definitions 4.3.2 (resp. Definition 4.4.3). Also assume that it is uniformly elliptic in the sense of Definition 4.2.2. Then there is a discrete set $S \subset (\rho_-,\rho_+)$ such that for each connected component $I := (\rho_-^I,\rho_+^I) \subset (\rho_-,\rho_+) \setminus S$, there is an operator Q_I that is I-admissible, such that*

$$PQ_I - \mathbb{1} \text{ and } Q_I P - \mathbb{1}$$

are bounded as operators

$$H^{-N,\rho_+^I-\epsilon-d/2,\rho_\perp}(L) \to H^{N,\rho_-^I+\epsilon-d/2,\rho_\perp}(L),$$

(resp. $y^{\rho_+^I-\epsilon}C_*^{-N} \to y^{\rho_-^I+\epsilon}C_*^N$) for all $N>0$ and $\epsilon>0$ small enough. In particular, P is Fredholm with same index on each space $H^{s,\rho_0-d/2,\rho_\perp}$ (resp. $y^{\rho_0}C_*^s$) for $s\in\mathbb{R},\rho_0\in I,\rho_\perp\in\mathbb{R}$.

There is no particular reason for an elliptic pseudo-differential operator to be Fredholm on a non-compact manifolds, even if the ellipticity is uniform at infinity. One has to introduce some kind of ellipticity or boundary condition *at* infinity, which depends on the geometry. However here, the lack of compactness is in some sense only one dimensional, so that many problems can be solved with a one dimensional scattering approach. An important remark is that we will rely on constructions from [GW17], itself based on [Bon16]. In the former paper, the techniques from Melrose [Mel93] had to be adapted to deal with operators that are *not* elliptic. In Section §4.2.4, we will compare our setup to that of Mazzeo and Melrose's *fibred cusp calculus*. In our case, we will require that our operators commute with the generators of local isometries of the cusp, that is ∂_θ and $y\partial_y$ on θ-independent functions. We will be able to allow this to hold modulo compact operators.

Under this assumption, the general strategy goes as follows : first, one inverts P modulo a smoothing remainder that is not compact ; by compact injection of $H^s \hookrightarrow H^{s'}$ for $s>s'$ on the orthogonal of the θ-zeroth Fourier mode (see Lemma 4.3.1), it is sufficient to explicitly invert the operator acting on sections not depending on θ. As in b-calculus, this is done by introducing an indicial operator $I_Z(P)$ (see §4.3.3) which is a convolution operator in the $r=\log y$ variable, defined on "the model at infinity" and acting on sections that are independent of θ. The set S can be computed by hand, as will be explained in Corollary 4.3.1 : it consists of the real parts of the indicial roots of the indicial family $I_Z(P,\lambda)$.

4.2 Pseudo-differential operators

Before we can start the proof of the Theorem, we have to introduce some spaces and some algebras of operators. We want to consider the action of operators on sections of $L \to N$ or more generally from sections of $L_1 \to N$ to sections of $L_2 \to N$ where $L_{1,2}$ are admissible bundles. In the paper [GW17], an algebra of *semi-classical* operators was described using results from [Bon16] ; it consisted of families of operators depending on a small parameter $h > 0$. In this chapter, we will be using *classical* operators, which is equivalent to fixing the value of h to 1.

4.2.1 Functional spaces

Let f be a function on N. We define for an integer $k \geq 0$:

$$\|f\|_{C^k}(z) = \sup_{0 \leq j \leq k} \|\nabla^j f(z)\|,$$

and $C^k(N)$ is the space of functions such that this is uniformly bounded in $z \in N$. We write $f \in C^\infty(N)$ if all the derivatives of f are bounded. If f is infinitely many times differentiable, but its derivatives are not bounded, we simply say that f is *smooth*.

The Christoffel coefficients of the metric in the cusp in the frame

$$X_y := y\partial_y, \ X_\theta := y\partial_\theta, \ X_\zeta := \partial_\zeta$$

are independent of (y, θ). As a consequence, in the cusp, there are uniform constants such that

$$\|f\|_{C^k}(z) \asymp \sup_{|\alpha|} |X_\alpha f(z)|, \tag{4.2.1}$$

(here, α is a multiindex valued in $\{y, \theta, \zeta\}$.) Let $0 < \alpha < 1$. We will write $f \in C^\alpha(N)$ if :

$$\|f\|_{C^\alpha} := \sup_{z \in N} |f(z)| + \sup_{z,z' \in N, z \neq z'} \frac{|f(z) - f(z')|}{d(z,z')^\alpha} = \|f\|_\infty + \|f\|_\alpha < \infty$$

In particular, a function f may be α-Hölder continuous, with a uniform Hölder constant of continuity (i.e. $\|f\|_\alpha < \infty$), but may not be in $C^\alpha(N)$ if $\|f\|_\infty = \infty$ for instance. It also makes sense to define C^α for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ by asking that $f \in C^{[\alpha]}(N)$ and that the $[\alpha]$-th derivatives of f are $\alpha - [\alpha]$ Hölder-continuous.

The Lebesgue spaces $L^p(N)$, for $p \geq 1$, are the usual spaces defined with respect to the measure $d\mu = y^{-d-1} dy d\theta d \operatorname{vol}(\zeta)$ induced by the metric. For $s \in \mathbb{R}$, we define (via the spectral theorem) :

$$\|f\|_{H^s(N)} := \|(-\Delta + 1)^s f\|_{L^2(N)},$$

and $H^s(N)$ is the completion of $C^\infty(N)$ with respect to this norm. We will abuse notations, and denote by y also a smooth extension to N of the coordinates defined in the cusps ; we will assume this extension is positive. For the reader to get familiar with these spaces, let us mention the following embedding lemmas.

Lemma 4.2.1. *Let $0 \leq s < s' < 1$ and $\rho - d/2 < \rho'$. Then $y^\rho C^{s'}(N) \hookrightarrow y^{\rho'} H^s(N)$ is a continuous embedding.*

Lemma 4.2.2. *Let $k \in \mathbb{N}$, $s > \frac{d+1}{2} + k$. Then $y^{-d/2} H^s(N) \hookrightarrow C^k(N)$ is a continuous embedding.*

The shift by $y^{d/2}$ will often appear throughout the chapter and is due to the fact that Sobolev spaces are built from the L^2 space induced by the hyperbolic measure $dyd\theta d\,\mathrm{vol}(\zeta)/y^{d+1}$. We will prove (and even refine) these embedding lemmas in Section §4.4.3.

4.2.2 Pseudo-differential operators on cusps

To describe the class we will be using, it will suffice to say which types of smoothing remainders we will allow, and which quantization we will manipulate. Our class of smoothing operators will be the class $\Psi^{-\infty}_{\mathrm{small}}(L_1, L_2)(= \Psi^{-\infty,L^2}_{\mathrm{small}}(L_1, L_2))$ of operators R that are bounded as

$$R : y^\rho H^{-N}(N, L_1) \to y^\rho H^N(N, L_2),$$

for any $\rho \in \mathbb{R}, N \geq 0$. These are called *$L^2$-small smoothing operators.*

In the compact part, we will use usual pseudo-differential operators with symbols σ in the Kohn-Nirenberg class, satisfying usual estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \sigma| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}.$$

It suffices now to explain what we will be calling a pseudo-differential operator in the *ends*. For this, we consider one end, and we drop the ℓ's. Instead of quantizing Z_a, we work with the full cusp Z.

Let us denote by Op^w the usual Weyl quantization on $\mathbb{R}^{d+1} \times \mathbb{R}^k$. Given $\chi \in C_c^\infty$ equal to 1 around 0, and $a \in \mathcal{S}'(\mathbb{R}^{2d+2k+2})$, we denote by $\mathrm{Op}^w(a)_\chi$ the operator whose kernel is

$$K(y, \theta, x; y', \theta', x') = \chi \left[\frac{y'}{y} - 1 \right] K_{\mathrm{Op}^w(a)}(y, \theta, x; y', \theta', x'). \tag{4.2.2}$$

Next, we can associate $a \in C^\infty(T^*(Z \times \mathbb{R}^k), \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}))$ with its periodic lift

$$\tilde{a} \in C^\infty(T^*(\mathbb{R}_y \times \mathbb{R}^d_\theta \times \mathbb{R}^k_\zeta), \mathcal{L}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})).$$

(supported for $y > 0$). Linear changes of variable have an explicit action on the Weyl quantization on $\mathbb{R}^{d+1+k}$. We deduce that if $f \in C^\infty(Z \times \mathbb{R}^k, \mathbb{R}^{n_1})$, denoting by $\tilde{f}$ the periodic lift to $\mathbb{R}^{d+1} \times \mathbb{R}^k$, $\mathrm{Op}^w(\tilde{a})_\chi \tilde{f}$ is again periodic. In particular, $\mathrm{Op}^w(\tilde{a})_\chi$ defines an operator from compactly supported smooth sections of $\mathbb{R}^{n_1} \to Z \times \mathbb{R}^k$ to distributional sections of $\mathbb{R}^{n_1} \to Z \times \mathbb{R}^k$.

As a consequence, it makes sense to set

$$\mathrm{Op}_{\mathbb{R}^k}(a) f = y^{(d+1)/2} \, \mathrm{Op}^w(a)_\chi [y^{-(d+1)/2} f].$$

Using a partition of unity on F_ℓ, we can globalize this to a Weyl quantization $\mathrm{Op}^w_{N_\ell, L_1 \to L_2}$, and then on the whole manifold $\mathrm{Op}^w_{N, L_1 \to L_2}$ — the arguments in [Zwo12, Section 14.2.3] apply. We will write Op this Weyl quantization on the whole manifold. Since F is compact, one check that the resulting operators are uniformly properly supported above each cusp.

Now, we need to say more about the symbol estimates that we will require. By $\langle \xi \rangle$, we refer to the Japanese bracket of ξ with respect to the natural metric g^* on T^*N, which is equivalent to $g^*_{Z_\ell} + g^*_{F_\ell}$. We denote by Y, J, η the dual variables to y, θ, ζ. In the case F_ℓ is a point, $\langle \xi \rangle = \sqrt{1 + y^2 |\xi|^2}$.

Definition 4.2.1. A *symbol of order* m is a smooth section a of $\mathcal{L}(L_1, L_2) \to T^*N$, that satisfies the usual estimates over N_0, and above each N_ℓ, and in local charts in F_ℓ, for each $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, there is a constant $C > 0$:

$$\left|(y\partial_y)^\alpha (y\partial_\theta)^\beta (\partial_\zeta)^\gamma \ (y^{-1}\partial_Y)^{\alpha'} (y^{-1}\partial_J)^{\beta'} (\partial_\eta)^{\gamma'} a\right|_{\mathcal{L}(L_1,L_2)} \leq C\langle\xi\rangle^{m-\alpha'-|\beta'|-|\gamma'|}.$$

This does not actually depend on the order in which the derivatives were taken. We write $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$.

We denote by $\Psi^m_{\text{small}}(N, L_1 \to L_2)(= \Psi^{m,L^2}_{\text{small}}(N, L_1 \to L_2))$ the class of operators of the form

$$\mathrm{Op}(a) + R,$$

with $R \in \Psi^{-\infty}_{\text{small}}$ and $a \in S^m$.

4.2.3 Microlocal calculus

The following basic results hold

Proposition 4.2.1. *Consider $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$, and $b \in S^{m'}(T^*N, \mathcal{L}(L_2, L_3))$. Then*

1. *$\mathrm{Op}(a)$ is continuous from $y^\rho H^s(N, L_1)$ to $y^\rho H^{s-m}(N, L_2)$ for all $s, \rho \in \mathbb{R}$.*
2. *$\mathrm{Op}(a)\,\mathrm{Op}(b) \in \Psi^{m+m'}_{small}$, and*

$$\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(ab) + \mathcal{O}_{\Psi^{m+m'-1}_{small}}(1).$$

Proof. So far, we can only do the proof of (1) in the case that s, m are integers because we do not know the nature of the operator $(-\Delta + 1)^s$. Using classical results in the compact part, we can restrict our attention to the cusps, and further to the case of $\mathrm{Op}_{\mathbb{R}^k}$. The case when $k = 0$ was dealt with in [Bon16]. As was explained in Appendix A of [GW17], the proofs therein adapt readily to the case $k \geq 1$. We will come back to the case that $s, m \notin \mathbb{Z}$ at the end of this subsection. □

Definition 4.2.2. Let $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$. We will say that a is *uniformly elliptic* in there is a constant $c > 0$ such that for every $(x, \xi) \in T^*N$ with $\|\xi\| > 1/c$ and $u \in L_{1,x}$,

$$\|a(x, \xi)u\|_{L_{2,x}} \geq c\langle\xi\rangle^m_x \|u\|_{L_{1,x}}.$$

Proposition 4.2.2. *Let $a \in S^m(T^*N, \mathcal{L}(L_1, L_2))$ be uniformly elliptic. Then we can find $Q \in \Psi^{-m}(N, \mathcal{L}(L_2, L_1))$ such that*

$$Q\,\mathrm{Op}(a) = \mathbb{1} + R,$$

with $R \in \Psi^{-\infty}_{small}(N, \mathcal{L}(L_1, L_1))$.

Before going on with the proof, observe that the remainder here is not a compact operator, contrary to the case of a compact manifold.

Proof. Here, we can apply the usual parametrix construction. First one can choose a $q_0 \in S^{-m}(T^*N, \mathcal{L}(L_2, L_1))$ such that for $\|\xi\| > 2/c$,

$$q_0 a = \mathbb{1}_{L_1}$$

Then
$$\mathrm{Op}(q_0)\,\mathrm{Op}(a) = \mathbb{1} + \mathrm{Op}(r_1) + R_1.$$
Here, $r_1 \in S^{-1}(T^*N, \mathcal{L}(L_1))$, and R_1 is small smoothing. Then
$$\mathrm{Op}((1 - r_1)q_0)\,\mathrm{Op}(a) = \mathbb{1} + \mathrm{Op}(r_2) + R_2,$$
where $r_2 \in S^{-2}(T^*N, \mathcal{L}(L_1))$ and R_2 is again small smoothing. Now, we can iterate this construction, and find a formal solution $\mathrm{Op}(\tilde{q})$ with
$$\tilde{q} = q_0 - \sum_{i \geq 0} r_i q_0.$$
(the sum is formal, it does *not* converge). Then, by means of a Borel summation, one can find an actual symbol $q \in S^{-m}(T^*N, \mathcal{L}(L_2, L_1))$ such that as $|\xi| \to \infty$, uniformly in x,
$$q \sim q_0 - \sum_{i \geq 0} r_i q_0.$$
As a consequence one gets
$$\mathrm{Op}(q)\,\mathrm{Op}(a) = \mathbb{1} + R,$$
with R small smoothing. □

We can now prove Proposition 4.2.1.

Proof of Proposition 4.2.1. The Laplacian defined by the Friedrichs extension of the quadratic form
$$\int_N g^L(-\Delta^L f, f) d\,\mathrm{vol}_g = \int_N \|\nabla_\bullet f\|^2$$
is uniformly elliptic. Given $N > 0$, by adding a large constant h_N^{-2}, we can obtain a symbol σ such that
$$\mathrm{Op}(\sigma)(-\Delta^L + h_N^{-2}) = h_N^{-2}\mathbb{1}_L + \mathcal{O}_{\Psi^{-N}_{\mathrm{small}}}(1).$$
Following arguments as in the proof of [Zwo12, Theorem 14.8, p.358], one deduces that for each $s \in \mathbb{R}$, there is a uniformly elliptic symbol σ_s of order s such that
$$H^s(N, L) = \mathrm{Op}(\sigma_s) L^2(N, L),$$
with equivalent norms. Together with the product stability of pseudo-differential operators, this finishes the proof of Proposition 4.2.1. □

In the following, we will write $\Lambda_{-s} = \mathrm{Op}(\sigma_s)$.

4.2.4 Fibred cusp calculus

To study Fredholm properties of differential operators on ends of the type (4.1.1), the so-called *fibered-cusp calculus* was introduced by Mazzeo and Melrose in [MM98]. We will explain here why it does not suit our needs entirely, reason for developping our arguments from scratch. The algebra of pseudo-differential operators we have just introduced is an extension of an algebra of *differential* operators. The latter is itself the algebra generated by $\mathcal{V}_0$, the Lie algebra of vector fields of the form
$$ay\partial_y + by\partial_\theta + X(\partial_\zeta),$$

where the coefficients a, b, X are C^∞-bounded on $Z \times F$. A crucial observation is that the Laplacian associated to the metric of $Z \times F$ is in this algebra.

Let us recall on the other hand the setup of the *Fibred-Cusp Calculus* developped by Mazzeo-Melrose [MM98]. We have a manifold N' whose boundary has a finite number of components. Those have a neighbourhood of the form

$$[0, \epsilon[_u \times X,$$

with a bundle map $p : X \to F_\zeta$. The generic coordinate in $p^{-1}(\zeta)$ is denoted θ. The fibred cusp algebra Ψ_{fc}^{diff} is the algebra of differential operators generated by the algebra $\mathcal{V}_{fc}$ of vector fields of the form

$$au^2\partial_u + bu\partial_\zeta + c\partial_\theta,$$

where a, b, c are C^∞ functions of u, ζ, θ (including at $u = 0$). In our case, with $u = 1/y$, we can see that if $V \in \mathcal{V}_0$, $uV \in \mathcal{V}_{fc}$. However, if $V \in \mathcal{V}_{fc}$, $(1/u)V$ is not necessarily in $\mathcal{V}_0$. The purpose of [MM98] was to analyze whether operators in Ψ_{fc}^{diff} have parametrices modulo compact remainders when acting on $L^2(N')$. This involves the inversion of an *indicial operator*, which is a family of operators $\hat{P}(\zeta, \eta)$, parametrized by $(\zeta, \eta) \in T^*F$, acting on the fiber $p^{-1}(\zeta)$ (here $\mathbb{R}^d/\Lambda$). If P is a differential operator of order m in our class, $u^m P \in \Psi_{fc}^{diff}$, so one could apply the results in [MM98]. However, here follows two reasons why this is not satisfying for our purposes.

- In the case that P is not differential, but pseudo-differential of varying order, it is not quite obvious what would replace the correspondence $P \mapsto u^m P$. This is crucial when dealing with anisotropic spaces as in [GW17]. This will intervene when dealing with the non-linear theory of the marked length spectrum.
- We are able to deal with Hölder-Zygmund spaces (instead of $L^2(N')$). As far as we know, this has not been done before with fibred-cusp calculus.

Since we are dealing with a much smaller class than the whole fibred cusp calculus, the criterion for being Fredholm is also simpler. Indeed, we only need to invert a family of operators $I(P, \lambda)$, with $\lambda \in i\mathbb{R}$, each such operator acting on F (the base instead of the fiber).

In the general case of the fibred cusp calculus, one does not require that the fibers $p^{-1}(\zeta)$ are flat manifolds. Let us explain why this is crucial in our case. The central point is to have a space of vector fields that is stable under Lie brackets (a Lie algebra). If yX_1 and yX_2 are two vector fields tangent to the fibers, so that X_1 and X_2 a smooth up to the boundary, we compute

$$[yX_1, yX_2] = y^2[X_1, X_2].$$

In particular, we can only allow vector fields $X_{1,2}$ such that their Lie bracket are $\mathcal{O}(1/y)$ as $y \to +\infty$. If we also require that they do not all vanish themselves as $y \to +\infty$, this is a very strong condition on the fibers. It probably implies that the curvature of the fibers goes to 0 as $y \to +\infty$.

This was the reason for Mazzeo and Melrose to study the algebra $\mathcal{V}_{fc}$. It also suggests that our techniques could be extended to the fibred cusp case, with the assumption that there are family of vector fields in the fibers $p^{-1}(\zeta)$ which are asymptotically parallel. This would be verified if these fibers are almost flat manifolds. For example the case of complex-hyperbolic cusps. We leave this to future investigations, and refer to [Gro78] and [BBC12].

To close this section, let us explain why it should not be surprising that the fibred cusp calculus does not behave very well with propagators. Indeed, consider some propagator e^{itP}. In its microlocal properties, the Hamiltonian flow of the principal symbol of P will appear. It is then important that the class of symbols considered is stable under the action of this flow. In the compact case, to prove such a statement, one relies on the usual statement that if φ is a smooth flow, there is some $\lambda > 0$ such that for $t \in \mathbb{R}$.

$$\|f \circ \varphi_t\|_{C^n} \leq C_n e^{\lambda n|t|} \|f\|_{C^n}.$$

However, the proof of this statement on a manifold uses crucially the fact that the metric has bounded curvature, and bounded covariant derivatives of its curvature tensor. The crux of the problem is then that the curvature of a metric in the form

$$\frac{dy^2 + g_{y,\theta,\zeta}(d\theta)}{y^2} + g_{y,\zeta}(d\zeta)$$

does not even have bounded curvature in general. In particular, there is no reason that propagators of general fibered-cusp operator propagate singularities in a nice fashion at infinity. The examples built in [DPPS15] show even that in the case that the curvature, or its derivatives, are not bounded, new dynamical phenomena appear.

4.3 Parametrices modulo compact operators on weighted Sobolev spaces.

4.3.1 Black-box formalism

Here again, we follow arguments exposed in [GW17]. Associated to each cusp Z, we have extension and restriction operators defined in the following way. Start by letting

$$\Pi_Z f := \int f_{|Z} d\theta.$$

Given $f \in \mathcal{D}'(]a, +\infty[\times F_Z, L_Z)$, we obtain an extended distribution to the manifold $\mathcal{E}_Z f \in \mathcal{D}'(N, L)$ by setting

$$\mathcal{E}_Z f(\phi) = f(\Pi_Z \phi).$$

Conversely, given $f \in \mathcal{D}'(N, L)$, we obtain a restricted distribution to the zero Fourier mode $\mathcal{P}_Z f \in \mathcal{D}'(]a, +\infty[\times F_Z, L_Z)$ by setting

$$\mathcal{P}_Z f(\phi) = f(\mathcal{E}_Z \phi).$$

Given $\chi \in C^\infty([a, +\infty[)$ which is locally constant around a, we define

$$\mathcal{Z}(\chi) f := \sum_Z \chi(a)(\mathbb{1} - \mathcal{E}_Z \mathcal{P}_Z) f + \mathcal{E}_Z(\chi \mathcal{P}_Z f).$$

The operators $\mathcal{E}_Z$, $\mathcal{P}_Z$ and $\mathcal{Z}(\chi)$ together form a black-box formalism, as it was introduced by Sjöstrand and Zworski in [SZ91].

Definition 4.3.1. We pick a function $\tilde{y} \in C^\infty([a, +\infty[)$ such that $\tilde{y}(y) = y$ for $y > 3a$, and $\tilde{y}(y < 2a) = 1$. Then we define for $s, \rho_0, \rho_\perp \in \mathbb{R}$,

$$H^{s,\rho_0,\rho_\perp}(N, L) = \mathcal{Z}(\tilde{y}^{\rho_0 - \rho_\perp})\left(y^{\rho_\perp} H^s\right).$$

These are *weighted* Sobolev spaces, with weight y^{ρ_0} on the zero Fourier mode and weight $y^{\rho_\perp}$ on the non-zero Fourier modes.

Note that we take the same weight on each cusps, this will suffice for our purposes. To obtain compact remainders in parametrices, the following observation going back to [LP76] is essential : for any $\rho_\perp \in \mathbb{R}, s > s'$, the restriction of the injection $y^{\rho_\perp} H^s(N,L) \hookrightarrow y^{\rho_\perp} H^{s'}(N,L)$ to non-constant Fourier modes is compact.

Lemma 4.3.1. *If $\chi \in C^\infty([a,+\infty[)$ is a smooth cutoff function such that $\chi \equiv 1$ for $y > 2a$ and vanishing around $y = a$, then for all $s > s'$:*

$$\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z : H^{s,\rho_0,\rho_\perp}(N,L) \to H^{s',-\infty,\rho_\perp}(N,L)$$

is compact.

By this, we mean that for any $N > 0$, the operator

$$\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z : H^{s,\rho_0,\rho_\perp}(N,L) \to H^{s',-N,\rho_\perp}(N,L)$$

is compact.

Proof. The value of ρ_0 is inessential here, so we take $\rho_0 = \rho_\perp = \rho$. Since $[\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z, y^\rho] = 0$ sufficiently high in the cusp, the lemma boils down to the case $\rho = 0$. For the sake of simplicity, we assume that there is a single cusp and that $L \to N$ is the trivial bundle $N \times \mathbb{R} \to N$, the general case is handled in a similar fashion. Let $\psi_n \in C_c^\infty(N)$ be a smooth cutoff function such that $\psi_n \equiv 1$ on $y < n$ and $\psi_n \equiv 0$ on $y > 2n$. The operators of injection

$$T_n := \psi_n(\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) \in \mathcal{L}(H^s(N), H^{s'}(N))$$

are compact, so it is sufficient to prove that the injection

$$T := \mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z \in \mathcal{L}(H^s(N), H^{s'}(N))$$

is the norm-limit of the operators T_n. In other words, if we can prove that for all $n \in \mathbb{N}$, there exits a constant $C_n > 0$ such that : for all $f \in H^s(N)$ such that $\chi \mathcal{P}_Z f \equiv 0$ (we denote by $H^s_0(N)$ the space of such functions endowed with the norm $\| \cdot \|_{H^s}$), we have

$$\|(\mathbb{1} - \psi_n) f\|_{H^{s'}} \leq C_n \|f\|_{H^s},$$

and that $C_n \to_{n \to +\infty} 0$, then we are done. Using Wirtinger's inequality, one can obtain like in [GW17, Lemma 4.9] that

$$\|\mathbb{1} - \psi_n\|_{\mathcal{L}(H^1_0, L^2_0)} \leq C/n,$$

for some uniform constant $C > 0$ (depending on the lattice Λ). Since we trivially have $\|\mathbb{1} - \psi_n\|_{\mathcal{L}(H^1_0, H^1_0)} \leq 1$, we obtain by interpolation that $\|\mathbb{1} - \psi_n\|_{\mathcal{L}(H^1_0, H^s_0)} \leq (C/n)^{1-s}$ for all $s \in [0,1]$. Since $\|\mathbb{1} - \psi_n\|_{\mathcal{L}(H^k_0, H^k_0)} \leq 1$ for all $k \in \mathbb{Z}$, we can interpolate once again to conclude. □

Lemma 4.3.2. *Consider $\rho_\perp \in \mathbb{R}$, $\rho_0 < \rho_0'$, and $s > s'$. Then $H^{s,\rho_0,\rho_\perp}(N,L) \hookrightarrow H^{s',\rho_0',\rho_\perp}(N,L)$ is a compact injection.*

Proof. One can write $f = (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) f + \mathcal{E}_Z \chi \mathcal{P}_Z f$. The first term is dealt by applying the previous lemma. As to $\mathcal{E}_Z \chi \mathcal{P}_Z f$, this is a classical lemma on $\mathbb{R}$. □

Eventually, we will need this last lemma :

Lemma 4.3.3. *Consider $\rho_\perp, \rho'_\perp \in \mathbb{R}, \rho_0 \in \mathbb{R}, s, s' \in \mathbb{R}$ such that $s > s', \rho_\perp > \rho'_\perp$. Then $H^{s,\rho_0,\rho_\perp} \hookrightarrow H^{s',\rho_0,\rho'_\perp}$ is a continuous embedding.*

Proof. Once again, decomposing in zero and non-zero Fourier modes and using interpolation estimates, it is sufficient to prove that $yH^1 \hookrightarrow L^2$ is a continuous embedding on functions with zero Fourier mode. But :

$$\begin{aligned}\|f\|^2_{yH^1} &= \|y^{-1}f\|^2_{H^1}\\ &\asymp \|y^{-1}f\|^2_{L^2} + \|y\partial_y(y^{-1}f)\|^2_{L^2} + \|y\partial_\theta(y^{-1}f)\|^2_{L^2} + \|\partial_\zeta(y^{-1}f)\|^2_{L^2}\end{aligned}$$

Using Wirtinger's inequality for functions with zero integral, we can control the term $\|y\partial_\theta(y^{-1}f)\|^2_{L^2} = \|\partial_\theta f\|^2_{L^2} \geq \|f\|^2_{L^2}$ and this provides the sought estimate. □

It will be more convenient for zeroth Fourier modes to use the variable $r = \log y$. The following lemma is crucial :

Lemma 4.3.4. *Consider $\chi \in C^\infty([a, +\infty[)$, constant for $y > 2a$, and vanishing around $y = a$. Then the following maps are bounded*

$$H^{s,\rho_0,\rho_\perp}(N, L) \ni f \mapsto \chi\mathcal{P}_Z f \in e^{(\rho_0+d/2)r} H^s(\mathbb{R} \times F_Z, L_Z);$$

$$e^{\rho_0 r} H^s(\mathbb{R} \times F_Z, L_Z) \ni f \mapsto \mathcal{E}_Z(\chi f) \in H^{s,\rho_0-d/2,-\infty}(N, L),$$

where $r = \log y$, and $H^s(\mathbb{R} \times F_Z, L)$ is the usual Sobolev space, built from the L^2 space induced by the measure $dr d\operatorname{vol}_{F_Z}(\zeta)$.

We insist on the fact that there is a shift of $-d/2$ due to the fact that we are considering the usual euclidean measure when working in the r-variable. We will prove this below after Proposition 4.3.1.

4.3.2 Admissible operators

We can now introduce the class of admissible operators.

Definition 4.3.2. Consider $A \in \Psi^m_{\text{small}}(N, \mathcal{L}(L_1, L_2))$ and $I_Z(A) \in \Psi^m(\mathbb{R}_r \times F_Z, L_Z)$ a convolution operator in the r-variable. We will say that A is a *$\mathbb{R}$-L^2-admissible operator* with indicial operator $I_Z(A)$ if the following holds. There exists a cutoff function $\chi \in C^\infty([a, +\infty[)$ (depending on A), such that χ is supported for $y > 2a$, equal to 1 for $y > C$ for some $C > 2a$,

$$\chi[A, \partial_\theta]\chi \text{ and } \mathcal{E}_Z\chi\left[\mathcal{P}_Z A\mathcal{E}_Z - I_Z(A)\right]\chi\mathcal{P}_Z, \tag{4.3.1}$$

are operators bounded from $y^N H^{-N}$ to $y^{-N}H^N$, for all $N \in \mathbb{N}$. The operator $I_Z(A)$ is independent of χ.

When $\rho > \rho'$, the unique convolution operator that is bounded from $e^{\rho r}L^2(dr)$ to $e^{\rho' r}L^2(dr)$ is the null operator. It follows that the indicial operator associated to a L^2 admissible operator is necessarily *unique*. Modulo compact remainders, the first condition in (4.3.1) mean that the operator A preserves the θ-Fourier modes ; the second condition implies that sufficiently high in the cusp, A is a convolution operator in the $r = \log y$ variable when acting on the zeroth Fourier mode. In particular, if B is a compactly supported pseudodifferential operator, B is admissible, and $I_Z(B) = 0$.

Observe that in general, if $P \in \Psi^m$, then in the cusp, $\chi[P, \partial_\theta]\chi$ is in $y^{-\infty}\Psi^m$. Indeed, its symbol can be expressed with derivatives of the symbol of P, that include at least one derivative ∂_θ. However, if $\sigma \in S^m$, $\partial_\theta \sigma \in y^{-\infty} S^m$. What we gain with our assumption is that the order becomes $-\infty$.

An important consequence of the definition is that if A is admissible, then

$$\chi \mathcal{P}_Z A[\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z], \text{ and } \chi[\mathbb{1} - \mathcal{E}_Z \mathcal{P}_Z] A \mathcal{P}_Z \chi \tag{4.3.2}$$

both are continuous from $y^N H^{-N}$ to $y^{-N} H^N$. For the first one, let K be the inverse of ∂_θ in $\{f \in L^2(\mathbb{R}^d/\Lambda), \int f = 0\}$. Abusing notation a little, we consider its action on the cusps ; it is then bounded on every $H^{s,\rho',\rho_\perp}$, for all $s, \rho, \rho_\perp \in \mathbb{R}$. Then

$$\begin{aligned} 0 &= \partial_\theta \chi \mathcal{P}_Z A[\mathbb{1} - \mathcal{E}_Z \mathcal{P}]\chi K \\ &= \chi \mathcal{P}_Z A[\mathbb{1} - \mathcal{E}_Z \mathcal{P}_Z]\chi + \chi \mathcal{P}_Z[\partial_\theta, A][\mathbb{1} - \mathcal{E}_Z \mathcal{P}_Z]\chi K, \end{aligned}$$

which proves the first assertion in (4.3.2) by using the assumption (4.3.1) on $[\partial_\theta, A]$. However the conditions in (4.3.2) are not necessarily stable under products, nor under taking parametrices.

Proposition 4.3.1. *Consider $A = \mathrm{Op}(\sigma)$. Then the first operator in equation (4.3.1) satisfies the required conditions if $\partial_\theta \sigma = 0$. Additionally, the second one also does in each cusp if,*

$$\widetilde{\sigma} : \ (r, z; \lambda, \eta) \ \mapsto \int \sigma_{|Z}(e^r, \theta, \zeta; e^{-r}\lambda, J = 0, \eta) d\theta,$$

does not *depend on r. In that case, the operator $I_Z(A)$ is pseudo-differential, properly supported, and its principal symbol is $\widetilde{\sigma}$. Both these conditions are satisfied when σ is invariant by local isometries of the cusp.*

Finally, an operator A is L^2 admissible if and only if it is of the form $\mathrm{Op}(\sigma) + B + R$, where σ satisfies the conditions above, R is L^2 admissible smoothing, and B is a compactly supported pseudo-differential operator. We deduce that the set of L^2 admissible operators is stable by composition.

From the decomposition (4.1.2), we deduce that ∇^L is a geometric operator. More generally, all the differential operators that can be defined completely locally using only the metric structure are bound to be properly supported geometric operators. For example, the Laplacian or the Levi-Civita connection. In the following, the operators D and D^*D will be local differential operators, so they will be properly supported geometric operators in the sense of the previous definition.

Proof of Proposition 4.3.1. Again, it suffices to work directly with Op_U on $Z \times U$. First, we observe that when $\partial_\theta \sigma = 0$, $\mathrm{Op}_U(\sigma)$ commutes with ∂_θ. Reciprocally, if $[\partial_\theta, \mathrm{Op}(\sigma)]$ is bounded from $y^N H^{-N}$ to $y^{-N} H^N$, it implies that $\partial_\theta \sigma \in y^{-\infty} S^{-\infty}$. In particular, we can replace σ by $\int \sigma d\theta$, and this only adds a negligible correction. For the second condition, one has to do a change of variables. For details, we refer to [GW17, Section 4.1]. □

Now, we can prove Lemma 4.3.4.

Proof of Lemma 4.3.4. Recall that $H^s = \Lambda_{-s} L^2$ with $\Lambda_s = \mathrm{Op}(\sigma_s)$. Actually, since the operators $\mathrm{Op}(\cdot)$ are uniformly properly supported, we can absorb the exponentials y^ρ, in the sense that when $\sigma \in S^m$, $y^\rho \, \mathrm{Op}(\sigma) y^{-\rho} = \mathrm{Op}(\sigma) + \mathcal{O}_{\Psi^{m-1}_{\mathrm{small}}}(1)$. Additionally, the symbol σ_s is built with the metric, so it is invariant under local isometries, and thus it preserves the zeroth Fourier mode. In particular, it suffices to consider the spaces $y^\rho H^s$ instead of $H^{s,\rho_0,\rho_\perp}$.

In particular, it suffices to consider the case $\rho = 0$. We observe that σ_s satisfies the assumptions of Proposition 4.3.1 (because it was built with the symbol of Δ^L which has to commute with the local isometries). The symbol of $I_Z(\mathrm{Op}(\sigma_s))$ is uniformly elliptic in the usual sense on $T^*(\mathbb{R} \times Y)$. From this we deduce that it suffices to consider the case $s = 0$.

Now, it boils down to the observation that the volume measure on the cusp is $y^{-d-1} d\theta dy = e^{-rd} d\theta dr$ with $r = \log y$. □

Lemma 4.3.5. *Let P be an admissible pseudodifferential operator. Then P is bounded as an operator between $H^{s+m,\rho_0,\rho_\perp}$ and $H^{s,\rho_0,\rho_\perp}$.*

Proof. We decompose the operator in four terms :

$$\begin{aligned} P &= (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) P (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) f \\ &\quad + \mathcal{E}_Z \chi \mathcal{P}_Z P (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) f + (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) P \mathcal{E}_Z \chi \mathcal{P}_Z f + \mathcal{E}_Z \chi \mathcal{P}_Z P \mathcal{E}_Z \chi \mathcal{P}_Z f \end{aligned}$$

The first term is bounded as a map

$$\begin{aligned} H^{s+m,\rho_0,\rho_\perp} &\overset{\mathbb{1}-\mathcal{E}_Z\chi\mathcal{P}_Z}{\to} H^{s+m,-\infty,\rho_\perp} \\ &\hookrightarrow y^{\rho_\perp} H^{s+m} \overset{P}{\to} y^{\rho_\perp} H^s \overset{\mathbb{1}-\mathcal{E}_Z\chi\mathcal{P}_Z}{\to} H^{s,-\infty,\rho_\perp} \hookrightarrow H^{s,\rho_0,\rho_\perp}, \end{aligned}$$

where we have used the boundedness of P obtained in Proposition 4.2.1. By (4.3.2), the second and third terms are immediately bounded. As to the last term, it is dealt exactly like the first term. □

4.3.3 Indicial resolvent

Let us consider a $\mathbb{R}$-L^2 admissible operator A of order m, and introduce

$$I_Z(A,\lambda) f(\zeta) = e^{-\lambda r} I_Z(A) \left[e^{\lambda r'} f(\zeta') \right]$$

Since A is small, this defines a *holomorphic* family of operators on F_Z ; it is called the *Indicial family associated to A.*

Lemma 4.3.6. *The Indicial family is a homomorphism in the sense that for all $\mathbb{R}$-L^2 admissible operators P and Q, and for all $\lambda \in \mathbb{C}$,*

$$I_Z(PQ,\lambda) = I_Z(P,\lambda) I_Z(Q,\lambda) \qquad I_Z(P+Q,\lambda) = I_Z(P,\lambda) + I_Z(Q,\lambda)$$

Proof. The only non-trivial part of this statement is that if P, Q are admissible, $I_Z(PQ) = I_Z(P) I_Z(Q)$. To this end, we write (abusing notations for an instant)

$$\begin{aligned} \mathcal{P}_Z P Q \mathcal{E}_Z &= \mathcal{P}_Z P (\mathcal{E}_Z \mathcal{P}_Z + \mathbb{1} - \mathcal{E}_Z \mathcal{P}_Z) Q \mathcal{E}_Z \\ &= \mathcal{P}_Z P \mathcal{E}_Z \mathcal{P}_Z Q \mathcal{E}_Z + \text{ compact} \\ &= I_Z(P) I_Z(Q) + \text{ compact}. \end{aligned}$$

□

Lemma 4.3.7. *Assume A is an* elliptic *$\mathbb{R}$-L^2 admissible operator of order m. Then for each $\lambda \in \mathbb{C}$, $I_Z(A,\lambda)$ is an elliptic pseudo-differential operator of order m, and $I_Z(A,\lambda)^{-1}$ is a meromorphic family of pseudo-differential operators of order $-m$. Its poles are called* indicial roots *of A (at Z). The set*

$$\{\Re s \mid s \text{ is an indicial root}\}$$

is discrete in $\mathbb{R}$.

Proof. The fact that $I_Z(A,\lambda)$ is a pseudo-differential operator follows from a direct computation. One can actually compute the principal symbol of $I_Z(A,\lambda)$. It does not depend on λ :

$$z,\eta \mapsto \sigma(A)(e^r,\theta,\zeta,0,0,\eta).$$

In particular, if A was elliptic, so is $I_Z(A,\lambda)$. However, we will need some uniformity in the ellipticity. We can assume that A decomposes as $\mathrm{Op}(\sigma)+R$ (the compactly supported pseudo-differential operator does not contribute to the indicial family). Let us deal with both parts separately. Let us write

$$I_Z(R)f(r,\zeta) = \int_{\mathbb{R}\times F_Z} K(r-r',\zeta,\zeta')f(r',\zeta')dr'd\zeta',$$

so that the kernel of $I_Z(R,\lambda)$ is

$$\widehat{K}(-i\lambda,\zeta,\zeta'),$$

the Fourier transform being taken in the first variable. Since R is smoothing and $\mathbb{R}$-L^2 admissible, for any $N,k>0$, $\rho\in\mathbb{R}$ and $T>0$, we let $u(r,\zeta)=e^{-\rho T}(-1)^k\delta(r-T)\delta^{(k)}(\zeta,\zeta'')$. Then,

$$e^{\rho r}\mathcal{P}_Z R\mathcal{E}_Z u = e^{\rho r} I_Z(R)u + \mathcal{O}_{H^{N,-N}}(1)$$

The left hand side is valued in all $H^{N,-d/2}$, $N>0$, with bounds uniform in ζ''. According to Lemma 4.4.8, it is thus contained in C^k, $k\geq 0$. However, the first term in the RHS is $e^{\rho(r-T)}\partial^k_{\zeta''}K(r-T,\zeta,\zeta'')$. With $r=r_0+T$, r_0 fixed, and $T\to+\infty$, we deduce that for all $\rho\in\mathbb{R}$ $e^{\rho r}K(r,\zeta,\zeta')$ is C^k (in the Banach sense).

Estimating thus the Fourier transform, we deduce that $I_Z(R,\lambda)$ is a $\mathcal{O}((1+|\Im\lambda|)^{-\infty})$ Sobolev-smoothing operator on $L_Z\to F_Z$, locally uniformly in $\Re\lambda$, in the sense that for all $N\in\mathbb{N}$, for all $s,s'\in\mathbb{R}$, for all $a<b$ there exists a constant $C_{N,s,s',a,b}>0$ such that $\|I_Z(R,\lambda)\|_{\mathcal{L}(H^s,H^{s'})}\leq \frac{C_{N,s,s',a,b}}{(1+|\Im\lambda|)^N}$ for all $a<\Re\lambda<b$.

We now consider a general L^2 admissible operator A, like in Proposition 4.3.1. Let Q be a parametrix for A i.e. such that $QA=\mathbb{1}+R$, where R is a small smoothing operator. We can always choose Q so that it is L^2-admissible. Then, by Lemma 4.3.6, $I_Z(A,\lambda)I_Z(Q,\lambda)=\mathbb{1}+I_Z(R,\lambda)$. From the discussion before, $\mathbb{1}+I_Z(R,\lambda)$ is an analytic Fredholm family, which is eventually invertible when $|\Im\lambda|$ becomes large. It satisfies the assumptions of the Fredholm Analytic theorem. As a consequence, $I_Z(Q,\lambda)(\mathbb{1}+I_Z(R,\lambda))^{-1}$ is a meromorphic family of bounded operators, and where it is bounded, it is equal to $I_Z(A,\lambda)^{-1}$. □

Now, we want to invert $I_Z(A)$ from the knowledge of $I_Z(A,\lambda)^{-1}$. Pick a $\rho\in\mathbb{R}$ such that $I_Z(A,\lambda)^{-1}$ has no pole on $\{\Re\lambda=\rho\}$, and consider the operator S_ρ whose kernel is

$$\int_{\Re\lambda=\rho} e^{\lambda(r-r')}I_Z(A,\lambda)^{-1}d\lambda.$$

Then one finds that S_ρ is bounded from $e^{\rho r}H^s(\mathbb{R}\times F_Z)$ to $e^{\rho r}H^{s+m}(\mathbb{R}\times F_Z)$, and

$$I_Z(A)S_\rho=\mathbb{1}.$$

Since $I_Z(A,\lambda)^{-1}$ is holomorphic, one also get by contour deformation that S_ρ does not change when ρ varies continuously without crossing the real part of an indicial root, so that given a connected component I of $\mathbb{R}\setminus\{\Re\lambda \mid I_Z(A,\lambda) \text{ is not invertible}\}$, we denote S_I the inverse.

4.3.4 General Sobolev admissible operators

When dealing with differential operators, whose kernel is supported exactly on the diagonal, the assumption that one can work with spaces $H^{s,\rho_0,\rho_\perp}$ for any $\rho_0, \rho_\perp \in \mathbb{R}$ is not very important. However, we will be dealing with pseudo-differential operators that are not properly supported. We will also be dealing with parametrices, which cannot be $\mathbb{R}$-admissible since some poles appear.

Definition 4.3.3. Let $\rho_+ > \rho_-$. We say that an operator A is $(\rho_-, \rho_+) - L^2$*-admissible* of order m if it can be decomposed as $A = A_{comp} + A_{cusp} + R$, where A_{comp} is a compactly supported pseudo-differential operator of order m, $A_{cusp} = \mathrm{Op}(\sigma)$ with σ satisfying the conditions of Proposition 4.3.1. Finally, R is $(\rho_-, \rho_+) - L^2$*-smoothing admissible* :

1. For all $\rho_0 \in]\rho_-, \rho_+[$, $\rho_\perp \in \mathbb{R}$ and $N > 0$, R is bounded from $H^{-N,\rho_0-d/2,\rho_\perp}$ to $H^{N,\rho_0-d/2,\rho_\perp}$,
2. For all $\rho_\perp \in \mathbb{R}$ and $N, \epsilon > 0$, $[\partial_\theta, R]$ is bounded as a map
$$H^{-N,\rho_+-d/2-\epsilon,\rho_\perp} \to H^{N,\rho_--d/2+\epsilon,\rho_\perp},$$
3. There is a convolution operator $I_Z(R)$ and $C > a$ such that
$$\chi_C \mathcal{P}_Z R \chi_C \mathcal{E}_Z - \chi_C I_Z(R) \chi_C$$
is an operator bounded from
$$e^{(\rho_+-\epsilon)r} H^{-N}(\mathbb{R} \times F_Z) \to e^{(\rho_-+\epsilon)r} H^N(\mathbb{R} \times F_Z),$$
for all $N, \epsilon > 0$.

The difference between being $\mathbb{R}$-admissible and (ρ_-, ρ_+)-admissible lies only in the behaviour on the zeroth Fourier mode in the cusps, where certain asympototic behaviour is allowed. In the other Fourier modes in θ, all exponential behaviours are allowed.

Each $(\rho_-, \rho_+) - L^2$ admissible operator A is associated with a convolution operator $I_Z(A)$ in each cusp. We can also define the indicial family $I_Z(A, \lambda)$, which is holomorphic in the strip
$$\mathbb{C}_{\rho_-,\rho_+} := \{\lambda \in \mathbb{C},\ \Re\lambda \in (\rho_-, \rho_+)\}.$$

Proposition 4.3.2. *The set of (ρ_-, ρ_+)-L^2 admissible operators is an algebra of operators, and the indicial family is also an algebra homomorphism.*

The proof is the same as that of Proposition 4.3.1 and Lemma 4.3.6. The proof of Lemma 4.3.7 still applies, albeit in $\mathbb{C}_{\rho_-,\rho_+}$ instead of $\mathbb{C}$, so we can still define the set of indicial roots, and the indicial inverses S_I.

4.3.5 Improving Sobolev parametrices

In this section, we will prove Theorem 4.1.1 in the case that the operator is (ρ_-, ρ_+)-L^2-admissible (except the part about the Fredholm index that we will deal with in the next section). Recall that in Proposition 4.2.2, we built a symbol q such that $A\,\mathrm{Op}(q) - \mathbb{1}$ and $\mathrm{Op}(q)A - \mathbb{1}$ are smoothing operators. From Lemma 4.3.2, we deduce that it would suffice to improve $\mathrm{Op}(q)$ only with respect to the action on the zeroth Fourier coefficient in the cusps. Since the symbol q was built using symbolic calculus, we deduce directly that $\mathrm{Op}(q)$ is $\mathbb{R}$-L^2-admissible. Consider an open interval I which is a connected component of

$$\mathbb{R} \setminus \{\Re\lambda \mid \lambda \text{ is an indicial root}\},$$

and the corresponding inverse S_I of $I_Z(A)$. Then set

$$Q_I := \mathrm{Op}(q) + \sum_Z \mathcal{E}_Z \chi_C \left[S_I - I_Z(\mathrm{Op}(q))\right] \chi_C \mathcal{P}_Z,$$

which is now a $I-L^2$-admissible pseudodifferential operator (indeed, we have corrected the indicial part of $\mathrm{Op}(q)$, by an I-L^2 admissible smoothing operator). We now write $Q_I A = \mathbb{1} + R_I'$ and we aim to prove that R_I' is compact on $H^{s,\rho_0-d/2,\rho_\perp}$ for all $s \in \mathbb{R}$, $\rho_0 \in I$, $\rho_\perp \in \mathbb{R}$. By stability by composition of admissible pseudodifferential operators (see Proposition 4.3.2), we know that R_I' is a smoothing admissible operators. Moreover, the operator Q_I was chosen so that $I_Z(R_I') = 0$ (this can be checked using the calculation rules of Lemma 4.3.6). As a consequence, thanks to Lemma 4.3.2, the proof of Theorem 4.1.1 (except the Fredholm properties) now boils down to the following Lemma :

Lemma 4.3.8. *Let $A \in \Psi^{-m}(N, \mathcal{L}(L))$ be a (ρ_-, ρ_+)-L^2 admissible pseudodifferential operator such that $I_Z(A) = 0$. Then A is bounded from $H^{s,\rho_0-d/2,\rho_\perp}$ to $H^{s+m,\rho_0'-d/2,\rho_\perp}$ for $\rho_0, \rho_0' \in (\rho_-, \rho_+)$, $\rho_\perp \in \mathbb{R}$.*

Proof. Let χ be a smooth cutoff function in the cusp. Then :

$$\begin{aligned} Af &= (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) A (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) f \\ &\quad + \mathcal{E}_Z \chi \mathcal{P}_Z A (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) f + (\mathbb{1} - \mathcal{E}_Z \chi \mathcal{P}_Z) A \mathcal{E}_Z \chi \mathcal{P}_Z f + \mathcal{E}_Z \chi \mathcal{P}_Z A \mathcal{E}_Z \chi \mathcal{P}_Z f \end{aligned}$$

By definition of being admissible, the first three terms directly satisfy the announced bounds. The last one also does since we have assumed that $I_Z(A) = 0$. □

Corollary 4.3.1. *The set of $\rho_0 \in (\rho_-, \rho_+)$ for which one cannot build such a parametrix is given by the real part of the set*

$$\{\lambda \mid \Re\lambda \in (\rho_-, \rho_+),\ I_Z(A, \lambda) \text{ is not invertible}\}.$$

4.3.6 Fredholm index of operators

We start with the following

Lemma 4.3.9. *For all $s, \rho_0, \rho_\perp \in \mathbb{R}$, one can identify via the L^2 scalar product the spaces $(H^{s,\rho_0,\rho_\perp})' \simeq H^{-s,-\rho_0,-\rho_\perp}$.*

Proof. We have to prove that the bilinear map

$$C_c^\infty(N, L) \times C_c^\infty(N, L) \ni (u, v) \mapsto \langle u, v \rangle = \int_N g^L(u, v) d\,\mathrm{vol}_N(z) \tag{4.3.3}$$

extends boundedly as a map $H^{s,\rho_0,\rho_\perp} \times H^{-s,-\rho_0,-\rho_\perp} \to \mathbb{C}$. Up to a smoothing order modification of Λ_s which we denote by Λ_s', we can assume that $\Lambda_{-s}\Lambda_s' = \mathbb{1}$. Then, for $u, v \in C_c^\infty(N, L)$, one has $\langle u, v \rangle = \langle \Lambda_{-s}\Lambda_s' u, v \rangle = \langle \Lambda_s u, \Lambda_{-s}' v \rangle$. By Lemma 4.3.5, since $\Lambda_{\pm s}$ is admissible, $\Lambda_{\pm s} : H^{\pm s,\rho_0,\rho_\perp} \to H^{0,\rho_0,\rho_\perp}$ is bounded. The boundedness of (4.3.3) on $H^{0,\rho_0,\rho_\perp} \times H^{0,-\rho_0,-\rho_\perp} \to \mathbb{C}$ is immediate (these are L^2 spaces with weight y^{ρ_0} on the zeroth Fourier mode and $y^{\rho_\perp}$ on the non-zero modes) and thus :

$$|\langle \Lambda_s u, \Lambda_{-s}' v \rangle| \lesssim \|\Lambda_s u\|_{H^{0,\rho_0,\rho_\perp}} \|\Lambda_{-s}' v\|_{H^{0,-\rho_0,-\rho_\perp}} \lesssim \|u\|_{H^{s,\rho_0,\rho_\perp}} \|v\|_{H^{-s,-\rho_0,-\rho_\perp}}.$$

We then conclude by density of $C_c^\infty(N, L)$. □

In the following, we will denote by P^* the formal adjoint of a pseudodifferential operator P. An immediate computation shows that

$$I_Z(P^*, \lambda) = I_Z(P, d - \overline{\lambda})^*. \tag{4.3.4}$$

As a consequence, λ is an indicial root of P if and only if $d - \overline{\lambda}$ is an indicial root of P^*.

Proposition 4.3.3. *Let P be a (ρ_-, ρ_+)-L^2 admissible elliptic pseudodifferential operator of order $m \in \mathbb{R}$. Let I be a connected component in (ρ_-, ρ_+) not containing the real part of any indicial root. Then P is Fredholm as a bounded operator $H^{s+m,\rho_0-d/2,\rho_\perp} \to H^{s,\rho_0-d/2,\rho_\perp}$ with $s \in \mathbb{R}, \rho_0 \in I$, $\rho_\perp \in \mathbb{R}$. The index does not depend on $s, \rho_0, \rho_\perp$ in that range.*

Proof. We write $I = (\rho_-^I, \rho_+^I)$. First, from the parametrix construction, and the compactness of the relevant spaces, we deduce that the kernel of P is finite dimensional on each of those spaces (and is actually always the same). Indeed, we have

$$QP = \mathbb{1} + K,$$

with K mapping $H^{-N,\rho_+^I-\epsilon-d/2,\rho_\perp}$ to $H^{N,\rho_-^I+\epsilon-d/2,\rho_\perp}$ for any $N > 0$, any $\epsilon > 0$ small enough and any $\rho_\perp \in \mathbb{R}$. In particular, by the compact embeddings of Lemma 4.3.2, we know that K is compact on $H^{s,\rho_0-d/2,\rho_\perp}$, for any $s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R}$. We deduce that the kernel of $\mathbb{1} + K$ is finite dimensional. Moreover, by Lemma 4.3.3 given $N > 0$ and $\rho_\perp \in \mathbb{R}$, we have for $N' > N$ large enough $\rho'_\perp > \rho_\perp$ large enough that

$$H^{N',\rho_0-d/2,\rho_\perp} \hookrightarrow H^{N,\rho_0-d/2,\rho'_\perp}.$$

and this implies that the kernel of P is contained in the intersection of all the spaces $H^{s,\rho_0-d/2,\rho_\perp}$, $s \in \mathbb{R}, \rho_0 \in I, \rho_\perp \in \mathbb{R}$. In particular, the kernel of P, which is contained in the kernel of $\mathbb{1} + K$ satisfies the same result, and its dimension does not depend on the space. Eventually, using Lemma 4.3.9, we can consider the same argument for the adjoint P^* (to obtain the codimension of the image of P), and this closes the proof. □

4.3.7 Crossing indicial roots

Let A be an elliptic $\mathbb{R}$-admissible (both on L^2 and L^∞) pseudodifferential operator of order $m > 0$. We want to investigate what happens when one crosses an indicial root : the operator may fail to be injective and/or surjective. For the sake of simplicity, we assume that the operator A has no indicial root on $\Re(\lambda) = d/2$ and that it is an isomorphism as a map $H^{s,\rho,\rho_\perp} \to H^{s-m,\rho,\rho_\perp}$ for all $s \in \mathbb{R}, \rho_\perp \in \mathbb{R}$ and ρ in a neighbourhood of 0. Let us investigate its kernel : we consider $u \in H^{0,\rho_0,\rho_\perp}$ such that $Au = 0$, where $\rho_0 > 0$ and we assume that $\rho_0 + d/2$ is not an indicial root. By ellipticity, it implies in particular that $u \in H^{+\infty,\rho_0,\rho_\perp}$ and we recall that this notation means that $u \in H^{N,\rho_0,\rho_\perp}$ for all $N \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} Au = 0 = &(\mathbb{1} - \mathcal{E}_Z\chi\mathcal{P}_Z\chi)A(\mathbb{1} - \mathcal{E}_Z\chi\mathcal{P}_Z\chi)u \\ &+ \mathcal{E}_Z\chi\mathcal{P}_Z\chi A(\mathbb{1} - \mathcal{E}_Z\chi\mathcal{P}_Z\chi)u + (\mathbb{1} - \mathcal{E}_Z\chi\mathcal{P}_Z\chi)A\mathcal{E}_Z\chi\mathcal{P}_Z\chi u + \mathcal{E}_Z\chi\mathcal{P}_Z\chi A\mathcal{E}_Z\chi\mathcal{P}_Z\chi u \end{aligned}$$

Since A is $\mathbb{R}$-admissible, the first three terms are respectively in

$$H^{+\infty,-\infty,\rho_\perp}, H^{+\infty,-\infty,-\infty}, H^{+\infty,-\infty,-\infty}.$$

In particular, this implies that

$$\chi\mathcal{P}_Z A\mathcal{E}_Z\chi\mathcal{P}_Z\chi u = I_Z(A)\mathcal{P}_Z\chi u + \mathcal{O}_{y^{-\infty}H^\infty}(1) = \mathcal{O}_{y^{-\infty}H^\infty}(1),$$

that is $I_Z(A)\mathcal{P}_Z\chi u = \mathcal{O}_{y^{-\infty}H^\infty}(1)$. Since $\rho_0+d/2$ was assumed not to be an indicial root, $I_Z(A)$ is invertible on $e^{\rho_0+d/2}H^{s+m} \to e^{\rho_0+d/2}H^s$, for all $s \in \mathbb{R}$ with inverse $S_{\rho_0+d/2}(A)$ and the Schwartz kernel of this inverse does not depend on a small perturbation on ρ_0. By Lemma 4.3.4, we have $f := \mathcal{P}_Z\chi u \in e^{(\rho_0+d/2)r}H^\infty$ and thus $S_{\rho_0+d/2}(A)I_Z(A)f = f$. On the other hand, we know by a classical contour integration argument (note that we have to consider $m > 0$ in order to perform this argument, that is we have to use $\|I_Z(A,\lambda)\|_{L^2\to L^2} = \mathcal{O}(\langle\Re(\lambda)^{-m}\rangle))$ that

$$S_{\rho_0+d/2}(A) = S_{d/2}(A) + \sum_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} \Pi_\lambda.$$

This implies, using the boundedness properties of $S_{d/2}$ that

$$\begin{aligned} f &= S_{d/2}(A)I_Z(A)f + \sum_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} \Pi_\lambda I_Z(A)f \\ &= \mathcal{O}_{e^{d/2r}H^\infty}(1) + \sum_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} \Pi_\lambda I_Z(A)f. \end{aligned}$$

Going back to u and writing $u = (\mathbb{1} - \chi\mathcal{E}_Z\mathcal{P}_Z\chi)u + \chi\mathcal{E}_Z f$, we eventually obtain that $u = u_0 + u_1$, where $u_0 \in H^{+\infty,0,\rho_\perp}$ and

$$u_1 = \chi\mathcal{E}_Z \sum_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} \Pi_\lambda I_Z(A)\mathcal{P}_Z\chi u,$$

which lives in a finite-dimensional space. Also observe by the same contour integral argument that $I_Z(A)\Pi_\lambda = 0$ on all the space $e^{r\rho}H^s$ for $\rho < \Re(\lambda), s \in \mathbb{R}$, where λ is an indicial root such that $\Re(\lambda) \in]d/2,\rho_0[$. This implies by a rather straightforward computation that :

$$Au = 0 = Au_0 + \underbrace{Au_1}_{=\mathcal{O}_{H^{+\infty,-\infty,-\infty}}(1)}$$

By invertibility of A for functions in $H^{s,0,\rho_\perp}, s \in \mathbb{R}, \rho_\perp \in \mathbb{R}$, we obtain that $u_0 = -A^{-1}(Au_1)$. To sum up the discussion, we have proved the

Lemma 4.3.10. *Assume that $Au = 0, u \in H^{0,\rho_0,\rho_\perp}$ with $\rho_\perp \in \mathbb{R}$ and $\rho_0 + d/2$ not being the real part of an indicial root. Then $u = u_0 + u_1$ with*

$$\begin{aligned} u_1 &= \chi\mathcal{E}_Z \sum_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} \Pi_\lambda I_Z(A)\mathcal{P}_Z\chi u \\ &\in \bigoplus_{\substack{\lambda \text{ indicial root of } A\\ \Re\lambda\in]d/2,\rho_0[}} H^{+\infty,\Re(\lambda)-d/2,-\infty}, \end{aligned}$$

and $Au_1 \in H^{+\infty,-\infty,-\infty}$, and $u_0 = -A^{-1}(Au_1) \in H^{+\infty,0,\rho_\perp}$. In particular, u lives in a finite-dimensional space contained in the range of an explicit finite-rank operator.

We also have a similar statement for the resolution of equation $Au = v$ on smaller spaces than $H^{s,0,\rho_\perp}$.

Lemma 4.3.11. *Let $\rho_0 < 0$ and assume that $\rho_0 + d/2$ is not the real part of an indicial root (in particular, there is no indicial root on $(\rho_0 - \varepsilon, \rho_0 + \varepsilon)$ for some $\varepsilon > 0$). Then, there exists $S \in \Psi^{-m}$, an $(\rho_0 - \varepsilon, \rho_0 + \varepsilon)$-admissible operator both on L^2 and L^∞, a linear mapping $G : H^{s,\rho_0,\rho_\perp} \to e^{\rho r} H^{+\infty}$, bounded on these spaces for all $s, \rho, \rho_\perp \in \mathbb{R}$, such that for all $v \in H^{s,\rho_0,\rho_\perp}$, $s \in \mathbb{R}, \rho_\perp \in \mathbb{R}$, one has :*

$$A^{-1}v = Sv + \chi \mathcal{E}_Z \sum_{\substack{\lambda \text{ indicial root of } A \\ \Re\lambda \in]\rho_0, d/2[}} \Pi_\lambda (\mathcal{P}_Z \chi + G) v.$$

Moreover, one has $A \chi \mathcal{E}_Z \Pi_\lambda : e^{r\rho} H^s \to H^{+\infty,-\infty,-\infty}$ for all $\rho < \Re(\lambda)$.

Proof. Since A is assumed to be invertible on the spaces $H^{s,0,\rho_\perp}$, $s \in \mathbb{R}, \rho_\perp \in \mathbb{R}$, given $v \in H^{s,\rho_0,\rho_\perp}$ for $\rho_0 < 0$, the equation $Au = v$ admits a solution $u \in H^{s+m,0,\rho_\perp}$ and one needs to prove that u is actually more decreasing than this. The proof follows the same arguments as the ones given in the proof of Lemma 4.3.10, namely one has to solve in the full cusp the equation $I_Z(A)\tilde{u} = \tilde{f}$, where $\tilde{f} \in e^{r(\rho_0 + d/2)} H^s$ and $\tilde{u}$ is a priori in $e^{rd/2} H^{s+m}$. □

Finally, putting together Lemmas 4.3.10 and 4.3.11, we deduce that the Fredholm index of A acting on $H^{s,\rho,\rho_\perp}$ is given by the

Lemma 4.3.12. *When $\rho > 0$ is not the real part of an indicial root, $s, \rho_\perp \in \mathbb{R}$, one has :*

$$\operatorname{ind}(A_{|H^{s,\rho,\rho_\perp}}) = \sum_{\Re\lambda \in]d/2, d/2+\rho[} \operatorname{rank}(\Pi_\lambda).$$

If $\rho < 0$, this is minus the sum for $\Re\lambda \in]d/2 + \rho, d/2[$.

4.4 Pseudo-differential operators on cusps for Hölder-Zygmund spaces

In this section, we are going to prove that the class of pseudodifferential operators defined in the previous section is bounded on the Hölder-Zygmund spaces C_*^s (see below for a definition). On a compact manifold, this is a well-known fact and we refer to the arguments before [Tay97, Equation (8.22)] for more details. In our case, there are subtleties coming from the non-compactness of the manifold. First, just as for the scale of Sobolev spaces H^s (built from the Laplacian induced by the metric), we need to correctly define the Hölder-Zygmund spaces so that they take into account the geometry at infinity of the manifold, namely the hyperbolic cusps. This is done via a Littlewood-Paley decomposition that encapsulates the hyperbolic behaviour. At this stage, we insist on the fact that the euclidean Littlewood-Paley decomposition is rather remarkable insofar as it only involves Fourier multipliers (and not "real" pseudo-differential operators), which truly simplify all the computations. This is not the case in the hyperbolic world and some rather tedious integrals have to be estimated.

Then, we will be able to prove that the previously defined pseudodifferential operators of order $m \in \mathbb{R}$ map continuously C_*^{s+m} to C_*^s, just as in the compact setting. Since we can always split the operator in different parts that are properly supported in cusps or in a fixed compact subset of the manifold (modulo a smoothing operator), we can directly restrict ourselves to operators supported in a cusp as long as we know that smoothing operators enjoy the boundedness property. Finally, we will prove Theorem 4.1.1 in the L^∞ case.

4.4.1 Definitions and properties

In the paper [Bon16], only Sobolev spaces were considered. So we will have to prove several basic results of boundedness of the calculus, acting now on Hölder-Zygmund spaces. We will give the proofs in the case of cusps, and leave the details of extending to products of cusps with compact manifolds to the reader.

We consider a smooth cutoff function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(s) = 1$ for $|s| \leq 1$ and $\psi(s) = 0$ for $|s| \geq 2$. We define for $j \in \mathbb{N}^*$,

$$\varphi_j(x, \xi) = \psi(2^{-j}\langle\xi\rangle) - \psi(2^{-j+1}\langle\xi\rangle), \tag{4.4.1}$$

where $\langle\xi\rangle := \sqrt{1 + y^2|\xi|^2}$ and here $|\xi|$ is the euclidean norm of the vector $\xi \in \mathbb{R}^{d+1}$. Observe that

$$\operatorname{supp} \varphi_j \subset \left\{(x, \xi) \in \mathbb{H}^{d+1} \times \mathbb{R}^{d+1} \mid 2^{j-1} \leq \langle\xi\rangle \leq 2^{j+1}\right\}.$$

Then, with $\varphi_0 = \psi(\langle\xi\rangle)$, $\sum_{j=0}^{+\infty} \varphi_j(x, \xi) = 1$. We introduce the

Definition 4.4.1. We define the *Hölder-Zygmund space of order* $s > 0$ as :

$$C_*^s(Z) := \left\{u \in L^\infty(Z) \mid \|u\|_{C_*^s} < \infty\right\},$$

where :

$$\|u\|_{C_*^s} := \sup_{j \in \mathbb{N}} 2^{js} \| \operatorname{Op}(\varphi_j) u\|_{L^\infty(Z)}.$$

For $s \leq 0$, we define

$$C_*^s(Z) := \left\{u \in \Delta^N L^\infty(Z) + L^\infty(Z) \mid \|u\|_{C_*^s} < \infty, N > (|s| + d + 1)/2\right\}.$$

The distinction between $s \leq 0$ and $s \geq 0$ is due to the fact that we need to assume *a priori* that u is a distribution in some rough functional space. This will appear in the computations. One can check that the definition of these spaces do not depend on the choice of the initial function ψ (as long as it satisfies the aforementioned properties). This mainly follows from Lemma 4.4.3. Note that, although a cutoff function χ around the "diagonal" $y = y'$ has been introduced in (4.2.2) in the quantization Op, we still have $\mathbb{1} = \sum_{j\in\mathbb{N}} \operatorname{Op}(\varphi_j)$. Thus, given $u \in C_*^s$ with $s > 0$, one has $u = \sum_{j\in\mathbb{N}} \operatorname{Op}(\varphi_j) u$, with normal convergence in L^∞ and

$$\|u\|_{L^\infty} \leq \sum_{j\in\mathbb{N}} \| \operatorname{Op}(\varphi_j) u\|_{L^\infty} \leq \sum_{j\in\mathbb{N}} 2^{-js} \underbrace{2^{js} \| \operatorname{Op}(\varphi_j) u\|_{L^\infty}}_{\leq \|C_*^s\|} \lesssim \|u\|_{C_*^s}$$

It can be checked that this definition locally coincides with the usual definition of Hölder-Zygmund spaces on a compact manifold, that is for [2] $s \notin \mathbb{N}$, C_*^s contains the functions that have $[s]$ derivatives which are locally L^∞ and such that the $[s]$-th derivatives are $s - [s]$ Hölder continuous. Indeed, if we choose a function f that is localized in a strip $y \in [a, b]$, then the size of the annulus in the Paley-Littlewood decomposition is uniform in y and can be estimated in terms of a and b, so the definition of the Hölder-Zygmund spaces boils down to that of $\mathbb{R}^{d+1}$. This will be made precise in Proposition 4.4.2.

2. For $s \in \mathbb{N}$, this does not exactly coincide with the set of functions that have exactly $[s]$ derivatives in L^∞.

Definition 4.4.2. We will say that an operator R is small Zygmund-smoothing, and write $R \in \Psi^{-\infty,L^\infty}_{\text{small}}(N,L)$ if

$$R : y^\rho C^s_*(N,L) \to y^\rho C^{s'}_*(N,L)$$

is bounded for any $\rho \in \mathbb{R}, s, s' \in \mathbb{R}$. We will denote by $\Psi^{m,L^\infty}_{small}(N,L)$ the operators that decompose as $\mathrm{Op}(\sigma) + R$, with $\sigma \in S^m$ and $R \in \Psi^{-\infty,L^\infty}_{small}(N,L)$.

We have the equivalent of Proposition 4.2.1 :

Proposition 4.4.1. *Let $P = \mathrm{Op}(\sigma)$ be a pseudodifferential operator in the class $\Psi^m(N, L_1 \to L_2)$. Then :*

$$P : y^\rho C^{s+m}_*(N,L_1) \to y^\rho C^s_*(N,L_2),$$

is bounded for $s \in \mathbb{R}$. If $\sigma' \in S^{m'}$ is another symbol,

$$\mathrm{Op}(\sigma)\,\mathrm{Op}(\sigma') = \mathrm{Op}(\sigma\sigma') + \mathcal{O}_{\Psi^{m+m'-1,L^\infty}_{small}}(1).$$

As usual, since we added a cutoff function on the kernel of the operator around the diagonal $y = y'$, the statement boils down to $\rho = 0$, which we are going to prove in the next paragraph.

4.4.2 Basic boundedness

The first step here is to derive a bound on L^∞ spaces. We follow the notations in [Bon16], $\tilde{}$ denoting the lifting of functions on Z to periodic functions in $\mathbb{H}^{d+1}$. If f is a function on the full cusp Z, then for $P = \mathrm{Op}(\sigma)$, one has :

$$Pf(x) = \int_{\mathbb{H}^{d+1}} \chi(y'/y - 1)\left(\frac{y}{y'}\right)^{\frac{d+1}{2}} K^w_\sigma(y,\theta,y',\theta')\tilde{f}(y',\theta')dy'd\theta',$$

where the kernel K^w_σ can be written :

$$K^w_\sigma(x,x') = \int_{\mathbb{R}^{d+1}} e^{i\langle x-x',\xi\rangle}\sigma\left(\frac{x+x'}{2},\xi\right)d\xi$$

If $P : L^\infty(Z) \to L^\infty(Z)$ is bounded, then :

$$\begin{aligned}\|P\|_{\mathcal{L}(L^\infty,L^\infty)} &\le \sup_{(y,\theta)\in\mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)\left(\frac{y}{y'}\right)^{\frac{d+1}{2}} |K^w_\sigma(y,\theta,y',\theta')|dy'd\theta' \\ &\lesssim \sup_{(y,\theta)\in\mathbb{H}^{d+1}} \int_{y'=y/C}^{y'=Cy}\int_{\theta'\in\mathbb{R}^d} |K^w_\sigma(y,\theta,y',\theta')|dy'd\theta'.\end{aligned} \tag{4.4.2}$$

Thus, we will look for bounds on $|K^w_\sigma(y,\theta,y',\theta')|$. A rather immediate computation shows that :

$$\frac{x_i - x'_i}{i\frac{y+y'}{2}} K^w_\sigma = K^w_{X_i\sigma}, \tag{4.4.3}$$

where $x = (x_0, x_1, ..., x_d) = (y, \theta)$ and $X_0 = y^{-1}\partial_Y, X_i = y^{-1}\partial_{J_i}$ for $i = 1, ..., d$ and we will iterate many times this equality, denoting $X^\alpha = X_0^{\alpha_0}\dots X_d^{\alpha_d}$ for each multiindices α. Since

$$|K^w_\sigma(y,\theta,y',\theta')| \lesssim \int_{\mathbb{R}^{d+1}} |\sigma((x+x')/2,\xi)|d\xi,$$

we also get

$$|K^w_\sigma(y,\theta,y',\theta')| \lesssim \left|\frac{x-x'}{y+y'}\right|^{-\alpha}\int_{\mathbb{R}^{d+1}} |X^\alpha\sigma|d\xi.$$

Lemma 4.4.1. *Let $\sigma \in S^{-m}$ with $m > d+1$. Then $\mathrm{Op}(\sigma)$ is bounded on L^∞.*

Proof. Under the assumptions, σ is integrable in ξ, and so are its derivatives. In particular, we get for all multiindices α,

$$|K_\sigma^w(y,\theta,y',\theta')| \lesssim \frac{C_\alpha}{(y+y')^{d+1}} \left|\frac{y+y'}{x-x'}\right|^\alpha .$$

From this we deduce

$$|K_\sigma^w(y,\theta,y',\theta')| \lesssim \frac{1}{(y+y')^{d+1}} \frac{1}{1+\left|\frac{\theta-\theta'}{y+y'}\right|^{d+1}}$$

and

$$\begin{aligned}\|\mathrm{Op}(\sigma)\|_{L^\infty\to L^\infty} &\lesssim \sup_y \int_{y/C}^{yC} dy' \int_{\mathbb{R}^d} d\theta \frac{1}{(y+y')^{d+1}} \frac{1}{1+\left|\frac{\theta}{y+y'}\right|^{d+1}} \\ &\lesssim \sup_y \int_{y/C}^{yC} dy' \frac{1}{y+y'} < \infty.\end{aligned}$$

□

We now use the previous dyadic partition of unity. Given a symbol $\sigma \in S^m$, we define $\sigma_j := \sigma\varphi_j \in S^{-\infty}$. Observe that

$$P = \mathrm{Op}(\sigma) = \sum_{j=0}^{+\infty} \mathrm{Op}(\underbrace{\sigma\varphi_j}_{=\sigma_j}) = \sum_{j=0}^{+\infty} P_j,$$

where $P_j := \mathrm{Op}(\sigma_j)$. We will need the following refined version of the previous lemma :

Lemma 4.4.2. *Assume that $\sigma \in S^m$. Then, $\|P_j\|_{\mathcal{L}(L^\infty,L^\infty)} \lesssim 2^{jm}$*

In particular, if $u \in L^\infty$, we find that $u \in C^0_*$ (but the converse is not true!).

Proof. The proof is similar to the proof of Lemma 4.4.1, but we have to be careful to obtain the right bound in terms of power of 2^j. Since φ_j has support in $\{2^{j-1} \leq \langle\xi\rangle \leq 2^{j+1}\}$, the kernel $K_{\sigma_j}^w$ of P_j satisfies :

$$|K_{\sigma_j}^w(x,x')| \lesssim \int_{\{2^{j-1}\leq\langle\xi\rangle\leq 2^{j+1}\}} \langle\xi\rangle^m d\xi \lesssim \frac{2^{j(m+d+1)}}{(y+y')^{d+1}} \tag{4.4.4}$$

Differentiating in ξ, we get for all multiindices α,

$$|K_{\sigma_j}^w| \lesssim \left|\frac{y+y'}{x-x'}\right|^\alpha \frac{2^{j(m-|\alpha|+d+1)}}{(y+y')^{d+1}}, \tag{4.4.5}$$

Combining with (4.4.3) (we iterate the equality k' times in y and k times in θ that is in each θ_i coordinate), we obtain :

$$|K_{\sigma_j}^w(x,x')| \lesssim \frac{2^{j(m+d+1)}}{(y+y')^{d+1}\left(1+2^{jk'}\left|\frac{y-y'}{y+y'}\right|^{k'} + 2^{jk}\left|\frac{\theta-\theta'}{y+y'}\right|^k\right)} \tag{4.4.6}$$

Then, integrating in (4.4.2), we obtain :

$$\begin{aligned}
&\|P_j\|_{\mathcal{L}(L^\infty,L^\infty)} \\
&\lesssim \sup_{(y,\theta)\in\mathbb{H}^{d+1}} \int_{y'=y/C}^{y'=Cy}\int_{\theta'\in\mathbb{R}^d} |K_\sigma^w(y,\theta,y',\theta')|dy'd\theta' \\
&\lesssim 2^{j(m+d+1)} \sup_{(y,\theta)\in\mathbb{H}^{d+1}} \int_{y'=y/C}^{y'=Cy}\int_{\theta'\in\mathbb{R}^d} \frac{dy'd\theta'}{(y+y')^{d+1}\left(1+2^{jk'}\left|\frac{y-y'}{y+y'}\right|^{k'}+2^{jk}\left|\frac{\theta-\theta'}{y+y'}\right|^k\right)} \\
&\lesssim 2^{j(m+d+1)} \sup_{(y,\theta)\in\mathbb{H}^{d+1}} 2^{-jd}\int_{y'=y/C}^{y'=Cy} \frac{dy'}{(y+y')\left(1+2^{jk'}\left|\frac{y-y'}{y+y'}\right|^{k'}\right)^{1-d/k}} \\
&\lesssim 2^{j(m+1)} \int_{1/C}^{C} \frac{1}{(1+u)\left(1+2^{jk'}\left|\frac{u-1}{u+1}\right|^{k'}\right)^{1-d/k}}du,
\end{aligned}$$

where we have done the change of variable $u = y'/y$. We let $v = 2^j\frac{1-u}{1+u}$, so that $u = (1-2^{-j}v)/(1+2^{-j}v)$,

$$1/(1+u) = (1+2^{-j}v)/2, \quad du = -\frac{2^{1-j}}{(1+2^{-j}v)^2}dv.$$

and we get the bound

$$\begin{aligned}
&\int_{1/C}^{C} \frac{1}{(1+u)\left(1+2^{jk'}\left|\frac{u-1}{u+1}\right|^{k'}\right)^{1-d/k}}du \\
&\lesssim 2^{-j}\int_{-2^j(C-1)/(C+1)}^{2^j(C-1)/(C+1)} \frac{1}{(1+|v|^{k'})^{1-d/k}}\frac{dv}{1+2^{-j}v}.
\end{aligned}$$

Let now $k = d+1$ and $k' = d+2$. We can bound the term $1/(1+2^{-j}v)$ by $(C+1)/2$, and we get

$$\|P_j\|_{\mathcal{L}(L^\infty,L^\infty)} \lesssim 2^{jm}\int_{\mathbb{R}} \frac{dv}{(1+|v|^{d+2})^{1/(d+1)}} \lesssim 2^{jm}.$$

Here, it was crucial that the kernel is uniformly properly supported. □

Lemma 4.4.3. *Let $\sigma \in S^m$. For all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that for all integers $j, k \in \mathbb{N}$ such that $|j-k| \geq 3$,*

$$\|P_j \operatorname{Op}(\varphi_k)\|_{\mathcal{L}(L^\infty,L^\infty)}, \|\operatorname{Op}(\varphi_k)P_j\|_{\mathcal{L}(L^\infty,L^\infty)} \leq C_N 2^{-N\max(j,k)},$$

where $P_j = \operatorname{Op}(\sigma\varphi_j)$.

Proof. This is a rather tedious computation and we only give the key ingredients. It is actually harmless to assume that $\sigma = 1$, which we will assume to hold for the sake of simplicity. We use [Bon16, Proposition 1.19]. We know that

$$\operatorname{Op}(\varphi_j)\operatorname{Op}(\varphi_k)f(x) = \int_{x'\in\mathbb{H}^{d+1}} \left(\frac{y}{y'}\right)^{\frac{d+1}{2}} K^w_{\varphi_j\sharp\varphi_k}(x,x')f(x')dx'$$

where, by definition,

$$K^w_{\varphi_j\sharp\varphi_k}(x,x') = \int e^{i\langle x-x',\xi\rangle}\varphi_j\sharp\varphi_k\left(\frac{x+x'}{2},\xi\right)d\xi \tag{4.4.7}$$

and

$$\begin{aligned}\varphi_j\sharp\varphi_k(x,\xi) = 2^{-2d-2}\int & e^{2i(-\langle x-x_1,\xi-\xi_1\rangle+\langle x-x_2,\xi-\xi_2\rangle)} \\ & \varphi_j(x_2,\xi_1)\varphi_k(x_1,\xi_2)\chi(y,y_1,y_2)dx_1dx_2d\xi_1d\xi_2,\end{aligned} \tag{4.4.8}$$

where, for fixed y, $\chi(y,\cdot,\cdot)$ is supported in the rectangle $\{y/C \leq y_{1,2} \leq yC\}$ (C not depending on y). To prove the claimed boundedness estimate, it is thus sufficient to prove that

$$\sup_{x\in\mathbb{H}^{d+1}}\int_{x'\in\mathbb{H}^{d+1}}\left(\frac{y}{y'}\right)^{\frac{d+1}{2}}|K^w_{\varphi_j\sharp\varphi_k}(x,x')|dx' \lesssim C_N 2^{-N\max(j,k)},$$

and we certainly need bounds on the kernel $K^w_{\varphi_j\sharp\varphi_k}$. First observe that it is supported in some region $\{y/C' \leq y' \leq yC'\}$ so, as before, the term $(y/y')^{\frac{d+1}{2}}$ is harmless in the integral. Then, we follow the same strategy as in the proof of Lemma 4.4.2. We deduce that it suffices to obtain bounds of the form

$$|K^w_{X^\alpha(\varphi_j\sharp\varphi_k)}|, |K^w_{\varphi_j\sharp\varphi_k}| \lesssim C_N\frac{2^{-N\max(j,k)}}{(y+y')^{d+1}}.$$

for $|\alpha| \leq d+2$.

For the sake of simplicity, we only deal with the bound on $|K^w_{\varphi_j\sharp\varphi_k}|$, the others being similar. To obtain a bound on this kernel, it is sufficient to prove that $|\varphi_j\sharp\varphi_k(x,\xi)| \lesssim C_N 2^{-N\max(j,k)}\langle\xi\rangle^{-N}$ (where N has to be chosen large enough). Indeed, one then obtains :

$$|K^w_{\varphi_j\sharp\varphi_k}(x,x')| \lesssim C_N 2^{-N\max(j,k)}\int_{\mathbb{R}^{d+1}}\frac{d\xi}{\left(1+\left(\frac{y+y'}{2}\right)^2|\xi|^2\right)^{N/2}} \lesssim C_N\frac{2^{-N\max(j,k)}}{(y+y')^{d+1}}.$$

We denote by $y_1 D_{x_{1,i}} := \frac{y_1}{2i}\partial_{x_{1,i}}$ the operator of derivation and we use in (4.4.8) the identity

$$(1+y_1^2|\xi-\xi_1|^2)^{-N}(1+y_1^2D^2_{x_1})^N(e^{2i\langle x-x_1,\xi-\xi_1\rangle}) = e^{2i\langle x-x_1,\xi-\xi_1\rangle} \tag{4.4.9}$$

where $D^2_{x_1} = \sum_i D^2_{x_{1,i}}$. In terms of Japanese bracket, this can be rewritten shortly $\langle\xi-\xi_1\rangle^{-2N}\langle D_{x_1}\rangle^{2N}(e^{2i\langle x-x_1,\xi-\xi_1\rangle}) = e^{2i\langle x-x_1,\xi-\xi_1\rangle}$. We thus obtain :

$$\begin{aligned}\varphi_j\sharp\varphi_k(x,\xi) =& 2^{-2d-2}\int e^{2i(\langle x-x_1,\xi-\xi_1\rangle+\langle x-x_2,\xi-\xi_2\rangle)}\langle\xi-\xi_1\rangle^{-2N}\langle\xi-\xi_2\rangle^{-2N} \\ & \langle D_{x_1}\rangle^{2N}\langle D_{x_2}\rangle^{2N}\left(\varphi_j(x_2,\xi_1)\varphi_k(x_1,\xi_2)\chi(y,y_1,y_2)\right)dx_1dx_2d\xi_1d\xi_2,\end{aligned}$$

We also need to use this trick in the x variable (more precisely on the θ variable) to ensure absolute convergence of this integral. This yields the formula :

$$\begin{aligned}\varphi_j\sharp\varphi_k(x,\xi) = 2^{-2d-2}\int & e^{2i(\langle x-x_1,\xi-\xi_1\rangle+\langle x-x_2,\xi-\xi_2\rangle)} \\ & \langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\langle D_{J_1}\rangle^{2M}\langle D_{J_2}\rangle^{2M} \\ & \left[\langle\xi-\xi_1\rangle^{-2N}\langle\xi-\xi_2\rangle^{-2N}\langle D_{x_1}\rangle^{2N}\langle D_{x_2}\rangle^{2N}(\varphi_j(x_2,\xi_1)\varphi_k(x_1,\xi_2)\chi(y,y_1,y_2))\right] \\ & dx_1dx_2d\xi_1d\xi_2,\end{aligned}$$

where M is chosen large enough. We here need to clarify a few things. First of all, the notation is a bit hazardous insofar as $\langle \theta - \theta_1 \rangle^2 := 1 + \frac{|\theta-\theta_1|^2}{y_1^2}$ this time. This comes from the fact that the natural operation of differentiation (which preserves the symbol class) is $\langle D_{J_1} \rangle^2 := 1 + \sum_{i=1}^d (y_1^{-1}\partial_{J_{1,i}})^2$. If ones formally develops the previous formula, one obtains a large number of terms involving derivatives — coming from the brackets

$$\langle D_{J_1}\rangle^{2M}\langle D_{J_2}\rangle^{2M}\langle D_{x_1}\rangle^{2N}\langle D_{x_2}\rangle^{2N}$$

— of φ_j and φ_k. These derivatives obviously do not change the supports of these functions and can only better the estimate (there is a 2^{-j} that pops up out of the formula each time one differentiates, stemming from the very definition of φ_j). As a consequence, it is actually sufficient to bound the integral if one forget about these brackets of differentiation. We are thus left to bound

$$\int e^{2i(\langle x-x_1,\xi-\xi_1\rangle+\langle x-x_2,\xi-\xi_2\rangle)}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}$$
$$\langle\xi-\xi_1\rangle^{-2N}\langle\xi-\xi_2\rangle^{-2N}\varphi_j(x_2,\xi_1)\varphi_k(x_1,\xi_2)\chi(y,y_1,y_2)dx_1dx_2d\xi_1d\xi_2.$$

We can now assume without loss of generality that $k \geq j+3$. Then, φ_j and φ_k are supported in two distinct annulus whose interdistance is bounded below by $2^{k-1}-2^{j+1} \geq 2^{k-2}$. Using this fact, one can bound the integrand by

$$\begin{aligned}\langle\xi-\xi_1\rangle^{-2N}\langle\xi-\xi_2\rangle^{-2N}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\chi(y,y_1,y_2)\\ \lesssim C_N 2^{-Nk}\langle\xi\rangle^{-4N}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\chi(y,y_1,y_2),\end{aligned}$$

where the last bracket is $\langle\xi\rangle := \sqrt{1+y^2|\xi|^2}$. (The estimates actually come out with a Japanese bracket in terms of $y_{1,2}$ but these are uniformly comparable to the Japanese bracket in terms of y because χ is supported in the region $\{y/C \leq y_{1,2} \leq yC\}$.) We thus obtain :

$$\Bigg|\int e^{2i(\langle x-x_1,\xi-\xi_1\rangle+\langle x-x_2,\xi-\xi_2\rangle)}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}$$
$$\langle\xi-\xi_1\rangle^{-2N}\langle\xi-\xi_2\rangle^{-2N}\varphi_j(x_2,\xi_1)\varphi_k(x_1,\xi_2)\chi(y,y_1,y_2))dx_1dx_2d\xi_1d\xi_2\Bigg|$$
$$\lesssim C_N 2^{-Nk}\langle\xi\rangle^{-4N}\int_{\substack{x_1\in\mathbb{R}^{d+1}\\ x_2\in\mathbb{R}^{d+1}\\ 2^{j-1}\leq\langle\xi_1\rangle\leq 2^{j+1}\\ 2^{k-1}\leq\langle\xi_2\rangle\leq 2^{k+1}}}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\chi(y,y_1,y_2)d\xi_1d\xi_2dx_1dx_2$$

We simply use a volume bound of the annulus (the ball in which it is contained actually) for the ξ_1, ξ_2 integrals which provides :

$$\int_{2^{j-1}\leq\langle\xi_1\rangle\leq 2^{j+1}} d\xi_1 \lesssim 2^{j(d+1)}/y_1^{d+1}$$

As a consequence, the bound in the previous integral becomes :

$$\begin{aligned}&C_N\frac{2^{-Nk}}{\langle\xi\rangle^{4N}}\int_{\substack{x_1\in\mathbb{R}^{d+1}\\ x_2\in\mathbb{R}^{d+1}\\ 2^{j-1}\leq\langle\xi_1\rangle\leq 2^{j+1}\\ 2^{k-1}\leq\langle\xi_2\rangle\leq 2^{k+1}}}\frac{\chi(y,y_1,y_2)d\xi_1d\xi_2dx_1dx_2}{\langle\theta-\theta_1\rangle^{2M}\langle\theta-\theta_2\rangle^{2M}}\\ &\lesssim C_N\frac{2^{-Nk+(j+k)(d+1)}}{\langle\xi\rangle^{4N}}\int_{\substack{x_1\in\mathbb{R}^{d+1}\\ x_2\in\mathbb{R}^{d+1}}}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\chi(y,y_1,y_2)\frac{dx_1dx_2}{y_1^{d+1}y_2^{d+1}}\end{aligned}$$

Now, the last integral can be bounded by

$$\int_{y_1=y/C}^{Cy}\int_{\theta_1\in\mathbb{R}^d}\int_{y_2=y/C}^{Cy}\int_{\theta_2\in\mathbb{R}^d}\langle\theta-\theta_1\rangle^{-2M}\langle\theta-\theta_2\rangle^{-2M}\frac{dx_1dx_2}{y_1^{d+1}y_2^{d+1}}\lesssim 1,$$

where M is large enough, which eventually yields the estimate

$$|\varphi_j\sharp\varphi_k(x,\xi)|\lesssim C_N 2^{-Nk}2^{(j+k)(d+1)}\langle\xi\rangle^{-4N}.$$

Since N was chosen arbitrary, we can always take it large enough so that it swallows the term $2^{(j+k)(d+1)}$. In the end, concluding by symmetry of j and k, we obtain the sought estimate

$$|\varphi_j\sharp\varphi_k(x,\xi)|\lesssim C_N 2^{-N\max(j,k)}\langle\xi\rangle^{-N}. \tag{4.4.10}$$

This implies the estimate on the kernel $K^w_{\varphi_j\sharp\varphi_k}$ and concludes the proof. □

Remark 4.4.1. Following the same scheme of proof, one can also obtain the independence of the definition of the Hölder-Zygmund spaces with respect to the cutoff function ψ chosen at the beginning. If $\widetilde{\psi}\in C_0^\infty(\mathbb{R})$ is another cutoff function such that $\widetilde{\psi}\equiv 1$ on $[-a,a]$ and $\widetilde{\psi}\equiv 0$ on $\mathbb{R}\setminus[-b,b]$ (and $0<a<b$), we denote by $\operatorname{Op}(\widetilde{\varphi}_j)$ the operators built from $\widetilde{\psi}$ like in (4.4.1). Then, in order to show the equivalence of the C^s_*- and $\widetilde{C}^s_*$-norms respectively built from ψ or $\widetilde{\psi}$, one has to compute quantities like $\|\operatorname{Op}(\varphi_j)\operatorname{Op}(\widetilde{\varphi}_k)\|_{\mathcal{L}(L^\infty,L^\infty)}$. If $k\in\mathbb{N}$ is fixed, then the terms $\operatorname{Op}(\varphi_j)\operatorname{Op}(\widetilde{\varphi}_k)$ ‘interact” (in the sense that one will not be able to obtain a fast decay estimate like (4.4.10)) for $j\in[k-1+\lfloor\log_2(a)\rfloor, k+1+\lceil\log_2(b)\rceil]$. We can improperly call these terms “diagonal terms”. Note that the number of such terms is independent of both j and k. The content of Lemma 4.4.3 can be interpreted by saying that when taking the same cutoff function (that is $\psi=\widetilde{\psi}$), the diagonal terms are $\{j,k\in\mathbb{N}\mid |j-k|\le 2\}$. In the following, we will use the definition of Hölder-Zygmund spaces with the rescaled cutoff functions $\widetilde{\psi}_h:=\psi(h\cdot)$. The diagonal terms are then shifted by $\log_2(h^{-1})$.

A consequence of the previous Lemma is the following estimate. Note that it is not needed for the proof of Proposition 4.4.1 but will appear shortly after when comparing the Hölder-Zygmund spaces C^s_* with the usual spaces C^s.

Lemma 4.4.4. *Let $P=\operatorname{Op}(\sigma)$ for some $\sigma\in S^m, m\in\mathbb{R}$ and let $0<s<m$. Then, there exists a constant $C>0$ such that for all $j\in\mathbb{N}$:*

$$\|P\operatorname{Op}(\varphi_j)\|_{\mathcal{L}(C^s_*,L^\infty)}\le C2^{-j(s-m)}$$

Proof. This is a rather straightforward computation, using Lemma 4.4.3 :

$$\begin{aligned}\|P\operatorname{Op}(\varphi_j)f\|_{L^\infty} &\lesssim \sum_{k\in\mathbb{N}}\|P_k\operatorname{Op}(\varphi_j)f\|_{L^\infty}\\ &\lesssim \sum_{|k-j|\ge 3}\|P_k\operatorname{Op}(\varphi_j)f\|_{L^\infty}+\sum_{|k-j|\le 2}\|P_k\operatorname{Op}(\varphi_j)f\|_{L^\infty}\\ &\lesssim \sum_{|k-j|\ge 3}C_N2^{-N\max(j,k)}\|f\|_{L^\infty}+2^{jm}\|\operatorname{Op}(\varphi_j)f\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty}+2^{-j(s-m)}\underbrace{2^{js}\|\operatorname{Op}(\varphi_j)f\|_{L^\infty}}_{\lesssim\|f\|_{C^s_*}}\\ &\lesssim 2^{-j(s-m)}\|f\|_{C^s_*},\end{aligned}$$

where $N\ge 1$ is arbitrary. □

We can now start the proof of Proposition 4.4.1.

Proof of Proposition 4.4.1, case $s+m>0$, $s>0$. We look at :

$$\|\operatorname{Op}(\varphi_j)Pu\|_{L^\infty} \lesssim \sum_{|j-k|\geq 3} \|\operatorname{Op}(\varphi_j)P_k u\|_{L^\infty} + \|\operatorname{Op}(\varphi_j)\sum_{|j-k|\leq 2} P_k u\|_{L^\infty}$$

The first term can be bounded using Lemma 4.4.3 and for $N \geq [s]+1$:

$$\begin{aligned}\sup_{j\in\mathbb{N}} 2^{js} \sum_{|j-k|\geq 3} \|\operatorname{Op}(\varphi_j)P_k u\|_{L^\infty} &\leq \sup_{j\in\mathbb{N}} 2^{js} C_N \sum_{|j-k|\geq 3} 2^{-N\max(j,k)}\|u\|_{L^\infty} \\ &\lesssim \|u\|_{L^\infty} \lesssim \|u\|_{C_*^{s+m}}\end{aligned}$$

Concerning the second term, we use the same trick, writing $u_k := \operatorname{Op}(\varphi_k)u$.

$$\begin{aligned}\|\operatorname{Op}(\varphi_j)\sum_{|j-k|\leq 2} P_k u\|_{L^\infty} &\lesssim \|\sum_{|j-k|\leq 2} P_k u\|_{L^\infty} \\ &\lesssim \sum_{|j-k|\leq 2}\sum_{|j-l|\geq 5} \|P_k u_l\|_{L^\infty} + \sum_{|j-k|\leq 2}\sum_{|j-l|\leq 4} \|P_k u_l\|_{L^\infty}\end{aligned}$$

The first term can be bounded just like before, using Lemma 4.4.3. As to the second term, we use Lemma 4.4.2, which gives that

$$\sup_{j\in\mathbb{N}} 2^{js} \sum_{|j-k|\leq 2}\sum_{|j-l|\leq 4} \|P_k u_l\|_{L^\infty} \lesssim \sup_{j\in\mathbb{N}} 2^{js}2^{jm} \sum_{|j-l|\leq 4} \|\operatorname{Op}(\varphi_l)u\|_{L^\infty} \lesssim \|u\|_{C_*^{s+m}}$$

Combining the previous inequalities, we obtain the desired result. Observe that the proof above also gives that for $P\in\Psi^m$, $m\in\mathbb{R}$,

$$\|Pu\|_{C_*^{-m}} \lesssim \|u\|_{L^\infty}.$$

□

Next, we want to deal with the case of negative s. To this end, we need to have some rough spaces on which our operators are bounded. Consider the space of distributions (for some constant $h>0$ small enough).

$$C^{-2n} := (-h^2\Delta + \mathbb{1})^n L^\infty.$$

equipped with the norm

$$\|u\| := \inf\{\|v\|_{L^\infty} \mid (-h^2\Delta+\mathbb{1})^n v = u\}.$$

Lemma 4.4.5. *For $n\geq 1$ and h small enough, $s>0$, and $\sigma\in S^{-2n+1-s}$, $\operatorname{Op}(\sigma)$ is bounded on C^{-2n}. Also, for $n>n'$, $C^{-2n'}\subset C^{-2n}$.*

Proof. First of all, we prove that $L^\infty \subset C^{-2n}$. To this effect, we consider parametrices

$$(-h^2\Delta+\mathbb{1})^n \operatorname{Op}(q_n) = \mathbb{1} + h^N \operatorname{Op}'(r_n),$$

with q_n of order $-2n$, and r_n of order $-N$. Taking N larger than $d+1$, by Lemma 4.4.1, $\operatorname{Op}(r_n)$ is bounded on L^∞ and $\operatorname{Op}(q_n)$ is bounded from L^∞ to $C_*^{2n}\subset L^\infty$ by the previous Lemma. We get that for $v\in L^\infty$,

$$(-h^2\Delta+\mathbb{1})^n \underbrace{\operatorname{Op}(q_n)(\mathbb{1}+h^N\operatorname{Op}(r_n))^{-1}}_{:=P_n} v = v,$$

the inverse being defined by Neumann series for h small enough and P_n is of order $-2n$ so $P_n v \in C_*^{2n} \subset L^\infty$. The inclusion $C^{-2n'} \subset C^{-2n}$ follows decomposing $(-h^2\Delta + \mathbb{1})^n = (-h^2\Delta + \mathbb{1})^{n'}(-h^2\Delta + \mathbb{1})^{n-n'}$.

For $f = (-h^2\Delta + \mathbb{1})^n \widetilde{f} \in C^{-2n}$ (with $\widetilde{f} \in L^\infty$), observe that

$$\mathrm{Op}(\sigma) f = \mathrm{Op}(\sigma)(-h^2\Delta + \mathbb{1})^n \widetilde{f} = \mathrm{Op}'(\sigma_h')\widetilde{f} + (-h^2\Delta + \mathbb{1})^n \,\mathrm{Op}(\sigma)\widetilde{f},$$

with $\sigma \in S^{-2n+1-s}$ — here Op' is a quantization with cutoffs around the diagonal with a larger support and $\sigma_h' \in S^{-s}$. By the last remark in the proof of the previous lemma, this is in $C_*^s + (-h^2\Delta + \mathbb{1})^n C_*^{2n+s-1} \subset L^\infty + (-h^2\Delta + \mathbb{1})^n L^\infty \subset C^{-2n}$.

□

Proof of Proposition 4.4.1, general case. Given $p \in S^m$ and n, we can build parametrices

$$(-h^2\Delta + \mathbb{1})^k \,\mathrm{Op}(q_k)\,\mathrm{Op}(p) = \mathrm{Op}(p) + \mathrm{Op}(r_k),$$

with $q_n \in S^{-2k}$, $r_n \in S^{-2n-d-1}$. With $k \geq n + (m+d+1)/2$, we get that for $u \in C^{-2n}$,

$$\mathrm{Op}(p)u = (-h^2\Delta + \mathbb{1})^k \,\mathrm{Op}(q_k)\,\mathrm{Op}(p)u - \mathrm{Op}(r_k)u \in C^{-2(n+k)} + C^{-2n} = C^{-2(n+k)}.$$

In particular, $\mathrm{Op}(p)$ is continuous from C^{-2n} to $C^{-4n-2\lceil (m-d-1)/2\rceil}$. Next, inspecting the proof of Lemma 4.4.3, we find that it also applies to the spaces C^{-2n}. In particular, we obtain that for all $n \geq 0$, and every $s \in \mathbb{R}$,

$$\|\mathrm{Op}(p)u\|_{C_*^s} \leq C\|u\|_{C_*^{s+m}} + C\|u\|_{C^{-2n}}. \tag{4.4.11}$$

So far, we have proved that for $n \geq 0$, $s \in \mathbb{R}$, $m \in \mathbb{R}$, $\mathrm{Op}(p)$ is continuous as a map

$$\{u \in C^{-2n} \mid \|u\|_{C_*^{s+m}} < \infty\} \to \{u \in C^{-4n-2\lceil (m-d-1)/2\rceil} \mid \|u\|_{C_*^s} < \infty\}.$$

We would like to replace $-4n - 2\lceil (m-d-1)/2\rceil$ by a number that only depends on s. To this end, we pick $u \in C^{-4n-2\lceil (m-d-1)/2\rceil}$ such that $\|u\|_{C_*^s} < \infty$. First off, if $s > 0$, then $u \in L^\infty$. So we assume that $s \leq 0$. Then for all $\epsilon > 0$, using the estimate (4.4.11),

$$\|\mathrm{Op}(\langle\xi\rangle^{s-\epsilon})u\|_{C_*^\epsilon} < \infty.$$

Using parametrices again, we can find $r_N \in S^{-s-\epsilon}$ and $q \in S^{s+\epsilon}$ so that

$$u = \mathrm{Op}(q_{s+\epsilon})\,\mathrm{Op}(\langle\xi\rangle^{-s-\epsilon})u + \mathrm{Op}(r_N)u.$$

Since $\mathrm{Op}(r_N)u, \mathrm{Op}(\langle\xi\rangle^{-s-\epsilon})u \in L^\infty$, we can apply the first part of the proof and obtain $u \in C^{-2\lceil (s+\epsilon+d+1)/2\rceil}$. □

4.4.3 Correspondance between Hölder-Zygmund spaces and usual Hölder spaces

We prove that the Hölder-Zygmund spaces $C_*^s(Z)$ coincide with the usual spaces $C^s(Z)$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$.

Proposition 4.4.2. *For all $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $C_*^s(Z) = C^s(Z)$ and more precisely*

$$\|f\|_{C_*^s(Z)} \asymp \|f\|_{C^s(Z)}.$$

For the sake of simplicity, we prove the previous proposition in the case $s \in (0,1)$, the general case being handled in a similar fashion. This will require a preliminary

Lemma 4.4.6. *There exists a constant $C > 0$ such that for all $j \in \mathbb{N}$:*

$$\| \operatorname{Op}(\varphi_j)\mathbf{1}\|_{L^\infty} \leq C2^{-j}.$$

Proof. Let us start by giving an explicit expression :

$$\operatorname{Op}(\varphi_j)\mathbf{1} = \int_{\mathbb{H}^{d+1}} \int_{\mathbb{R}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} e^{i\langle x-x',\xi\rangle}\sigma_j\left(\frac{y+y'}{2},\xi\right) d\xi dy' d\theta'.$$

Since there is no dependence in θ, we can remove θ and J and get

$$\int_{y'=0}^{+\infty}\int_{Y=-\infty}^{+\infty} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} e^{i\langle y-y',Y\rangle}\sigma_j\left(\frac{y+y'}{2},Y,J=0\right) dY dy'$$

That is,

$$\int_{y'=0}^{+\infty}\int_{Y=-\infty}^{+\infty} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} e^{i\langle y-y',Y\rangle} \left[\psi\left(2^{-j}\sqrt{1+\left(\frac{y+y'}{2}Y\right)^2}\right) - \psi\left(2^{-j+1}\sqrt{1+\left(\frac{y+y'}{2}Y\right)^2}\right)\right] dY dy'$$

Making the change of variables $u = \frac{y+y'}{2}$, we get the following expression :

$$2\times\int_{u=y/2}^{+\infty}\int_{Y=-\infty}^{+\infty} \chi(2u/y-2)\left(\frac{y}{2u-y}\right)^{\frac{d+1}{2}} e^{2i\langle y-u,Y\rangle} \left[\psi\left(2^{-j}\sqrt{1+(uY)^2}\right) - \psi\left(2^{-j+1}\sqrt{1+(uY)^2}\right)\right] dY du. \tag{4.4.12}$$

It is sufficient to prove that each term in this difference is bounded by $C2^{-j}$. Let us deal with the first one for instance. For the sake of simplicity, we also forget about the factors

$$\chi(2u/y-2)\left(\frac{y}{2u-y}\right)^{\frac{d+1}{2}}$$

since, in the end, this will amount to integrating in the y' variable for $y' \in [y/C', yC']$, for some uniform constant $C' > 0$. Using the identity

$$\left(\frac{i\partial_Y}{2u}\right)\left(e^{2i\langle y-u,Y\rangle}\right) = e^{2i\langle y-u,Y\rangle},$$

we can thus estimate the first term in (4.4.12)

$$\begin{aligned}
&\int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty} e^{2i\langle y-u,Y\rangle}\psi\left(2^{-j}\sqrt{1+(uY)^2}\right)dY\,du \\
&= \int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty}\left(\frac{i\partial_Y}{2u}\right)^2\left(e^{2i\langle y-u,Y\rangle}\right)\psi\left(2^{-j}\sqrt{1+(uY)^2}\right)dY\,du \\
&= \int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty} e^{2i\langle y-u,Y\rangle}\left(\frac{i\partial_Y}{2u}\right)^2\left[\psi\left(2^{-j}\sqrt{1+(uY)^2}\right)\right]dY\,du \\
&= \int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty} e^{2i\langle y-u,Y/u\rangle}\left(\frac{i\partial_Y}{2}\right)^2\left[\psi\left(2^{-j}\sqrt{1+Y^2}\right)\right]dY\,du/u \\
&= -\frac{2^{-j}}{4}\int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty} e^{2i\langle y-u,Y/u\rangle} \\
&\left[\frac{1}{(1+Y^2)^{3/2}}\psi'\left(2^{-j}\sqrt{1+Y^2}\right)+2^{-j}\frac{Y}{\sqrt{1+Y^2}}\psi''\left(2^{-j}\sqrt{1+Y^2}\right)\right]dY\,du/u
\end{aligned}$$

Once again, we only estimate the first term in the previous sum, the second one being handled in the same fashion. By definition, ψ is supported in the ball of radius 2, thus :

$$\begin{aligned}
&\left|2^{-j}\int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty} e^{2i\langle y-u,Y/u\rangle}\frac{1}{(1+Y^2)^{3/2}}\psi'\left(2^{-j}\sqrt{1+Y^2}\right)dY\,du/u\right| \\
&\lesssim 2^{-j}\int_{u=y/C'}^{yC'}\int_{Y=-\infty}^{+\infty}\frac{1}{(1+Y^2)^{3/2}}\psi'\left(2^{-j}\sqrt{1+Y^2}\right)dY\,du/u \\
&\lesssim 2^{-j}\int_{u=y/C'}^{yC'}\int_{|Y|\le 2\cdot 2^j}\frac{dY}{(1+Y^2)^{3/2}}du/u \lesssim 2^{-j}\int_{u=y/C'}^{yC'}du/u \lesssim 2^{-j}
\end{aligned}$$

This concludes the proof of the Lemma. □

We can now prove Proposition 4.4.2 :

Proof. We first prove that there exists $C>0$ such that for all functions $f\in C^s_*$, $\|f\|_{C^s}\le C\|f\|_{C^s_*}$. For $x,x'\in Z$ such that $d(x,x')\le 1$, we write :

$$\begin{aligned}
|f(x)-f(x')| &= \left|\sum_{j\in\mathbb{N}}(\mathrm{Op}(\varphi_j)f)(x)-(\mathrm{Op}(\varphi_j)f)(x')\right| \\
&\le \sum_{j\in\mathbb{N}}|(\mathrm{Op}(\varphi_j)f)(x)-(\mathrm{Op}(\varphi_j)f)(x')|
\end{aligned}$$

Let $N\in\mathbb{N}\setminus\{0\}$ be the unique integer such that $2^{-N}\le d(x,x')\le 2^{-N+1}$. We split the previous sum between $j\ge N$ and $j<N$. First :

$$\begin{aligned}
\sum_{j\ge N}|(\mathrm{Op}(\varphi_j)f)(x)-(\mathrm{Op}(\varphi_j)f)(x')| &\lesssim \sum_{j\in\mathbb{N}}\|\mathrm{Op}(\varphi_j)f\|_{L^\infty} \\
&\lesssim \sum_{j\ge N}2^{-js}\|f\|_{C^s_*} \\
&\lesssim 2^{-sN}\|f\|_{C^s_*}\lesssim\|f\|_{C^s_*}d(x,x')^s
\end{aligned}$$

Now, using Lemma 4.4.4 with $P = \nabla$ (note that $0 < s < m = 1$), one has :

$$\sum_{j<N} |(\mathrm{Op}(\varphi_j)f)(x) - (\mathrm{Op}(\varphi_j)f)(x')| \lesssim \sum_{j<N} \|\nabla \mathrm{Op}(\varphi_j)f\|_{L^\infty} d(x,x') \\ \lesssim 2^{-j(s-1)} \|f\|_{C^s_*} d(x,x') \lesssim \|f\|_{C^s_*} d(x,x')^s$$

Eventually, using the obvious estimate $\|f\|_{L^\infty} \lesssim \|f\|_{C^s_*}$, one obtains $\|f\|_{C^s} \lesssim \|f\|_{C^s_*}$.

Let us now prove the other estimate. We start with :

$$\begin{aligned}\mathrm{Op}(\varphi_j)f(x) &= \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} K^w_{\varphi_j}(x,x')f(x')dx' \\ &= \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} K^w_{\varphi_j}(x,x')(f(x')-f(x))dx' \\ &\quad + f(x)\,\mathrm{Op}(\varphi_j)\mathbf{1}\end{aligned}$$

According to Lemma 4.4.6, the last term is bounded by $\lesssim \|f\|_{L^\infty} 2^{-j} \lesssim \|f\|_{C^s} 2^{-j}$. As to the first term, using the Hölder property of f :

$$\left| \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} K^w_{\varphi_j}(x,x')(f(x')-f(x))dx' \right| \\ \lesssim \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} \left|K^w_{\varphi_j}(x,x')\right| d(x,x')^s dx' \|f\|_{C^s}$$

Now, following the exact same arguments as the ones developed in Lemma 4.4.2 and using the crucial fact that on the support of the kernel of the pseudodifferential operator (namely for $y' \in [y/C, yC]$) one can bound the distance $d(x,x') \lesssim |\log(y/y')| + \frac{|\theta-\theta'|}{y}$, one can prove the estimate

$$\sup_{x\in\mathbb{H}^{d+1}} \int_{\mathbb{H}^{d+1}} \chi(y'/y-1)(y/y')^{\frac{d+1}{2}} \left|K^w_{\varphi_j}(x,x')\right| d(x,x')^s dx' \lesssim 2^{-js}$$

The sought estimate $\|f\|_{C^s_*} \lesssim \|f\|_{C^s}$ then follows immediately.

□

4.4.4 Embedding estimates

Using the Paley-Littlewood decompositions in the cusps, we are going to prove the embedding estimates. We can actually strengthen them to the following two Lemmas :

Lemma 4.4.7. *For all $s, s' \in \mathbb{R}$ such that $s' > s$, $\rho, \rho' \in \mathbb{R}$ such that $\rho' > \rho - d/2$,*

$$y^\rho C^{s'}_*(N,L) \hookrightarrow y^{\rho'} H^s(N,L)$$

is a continuous embedding.

In our notations, $y^{\rho'} H^s = H^{s,\rho',\rho'}$.

Lemma 4.4.8. *For all $s, \rho \in \mathbb{R}$,*

$$y^\rho H^s(N,L) \hookrightarrow y^{\rho+d/2} C^{s-(d+1)/2}_*(N,L)$$

is a continuous embeddings.

Observe that the two lemmas are locally true so that it it is sufficient to prove them when the function is supported on a single fibered cusp. The key lemma here is the following

Lemma 4.4.9. *For all $s \in \mathbb{R}$,*

$$\|u\|^2_{H^s(N)} \asymp \sum_{j\in\mathbb{N}} \|\operatorname{Op}(\varphi_j)u\|^2_{L^2(Z)} 4^{js}$$

Proof. The proof is done using semiclassical estimates and then concluding by equivalence of norms when h is bounded away from 0. For $h > 0$, we start from

$$\begin{aligned}
\|u\|^2_{H^s_h(N)} &\asymp \|\operatorname{Op}_h(\langle\xi\rangle^s)u\|^2_{L^2} \\
&= \sum_{j,k} \langle \operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u, \operatorname{Op}_h(\langle\xi\rangle^s\varphi_k)u\rangle \\
&= \sum_{|j-k|\leq 2} \langle \operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u, \operatorname{Op}_h(\langle\xi\rangle^s\varphi_k)u\rangle + \sum_{|j-k|\geq 3} \langle \operatorname{Op}_h(\langle\xi\rangle^s\varphi_k)\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u, u\rangle.
\end{aligned}$$

The first term is obviously bounded by $\lesssim \sum_j \|\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u\|^2_{L^2(Z)}$. To bound the last term we can first use the estimate (4.4.10) in the proof of Lemma 4.4.3 which yields

$$\langle \operatorname{Op}_h(\langle\xi\rangle^s\varphi_k)\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u, u\rangle \leq C_N 2^{-N\max(j,k)}\|u\|^2_{H_h^{-N}},$$

where $N > |s|$ is taken arbitrary large and thus $\sum_{|j-k|\geq 3}\langle \operatorname{Op}_h(\langle\xi\rangle^s\varphi_k)\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u, u\rangle \lesssim \|u\|^2_{H_h^{-N}(Z)}$. Now, we also have that

$$\begin{aligned}
\|u\|^2_{H_h^{-N}} &= \|\operatorname{Op}_h(\langle\xi\rangle^{-N})u\|^2_{L^2} \\
&= \|\sum_j \operatorname{Op}_h(\langle\xi\rangle^{-N}\varphi_j)u\|^2_{L^2} \\
&\lesssim \sum_j 2^{-j}\|\operatorname{Op}_h(2^j\langle\xi\rangle^{-N-s}\langle\xi\rangle^s\varphi_j)u\|^2_{L^2} \\
&\lesssim \sum_j 2^{-j}(\|\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u\|^2_{L^2} + h\|u\|^2_{H_h^{-N}(Z)}) \\
&\lesssim \sum_j \|\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u\|^2_{L^2} + h\|u\|^2_{H_h^{-N}(Z)},
\end{aligned}$$

where the peniultimate inequality follows from Gärding's inequality [GW17, Lemma A.15] for symbols of order $-(2N-1)$ since $2^j\langle\xi\rangle^{-N-s}\langle\xi\rangle^s\varphi_j \in S^{-(2N-1)}$ is controlled by $\lesssim \langle\xi\rangle^s\varphi_j$. For h small enough, we can swallow the term $h\|u\|^2_{H_h^{-N}(Z)}$ in the left-hand side and we eventually obtain that $\|u\|^2_{H^s_h} \lesssim \sum_j \|\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u\|^2_{L^2}$, where the constant hidden in the $\lesssim$ notation is independent of h. Actually, since $\langle\xi\rangle^s\varphi_j \lesssim 2^{js}\varphi_j$, the same arguments involving Gärding's inequality also yield

$$\|u\|^2_{H^s_h} \lesssim \sum_j \|2^{js}\operatorname{Op}_h(\varphi_j)u\|^2_{L^2}.$$

On the other hand,

$$\sum_j \|\operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)u\|^2_{L^2(Z)} = \langle \sum_j \operatorname{Op}_h(\langle\xi\rangle^s\varphi_j)^2u, u\rangle.$$

Using expansions for products, we find that this is $\lesssim \langle \mathrm{Op}_h(\langle\xi\rangle^{2s}\sum\varphi_j^2)u,u\rangle$. This itself is controlled by the H^s_h norm. Eventually, we conclude by equivalence of norms when h is bounded away from 0 (see Remark 4.4.1). □

We will also need the following observation : $\mathrm{Op}(\varphi_j)(y^\rho u) = y^\rho\,\mathrm{Op}'(\varphi_j)(u)$ for some other quantization Op' (the cutoff function $\chi(y'/y-1)$ in the quantization Op is changed to $(y'/y)^\rho\chi(y'/y-1)$). In the following proof, we will denote by Op' and Op'' other quantizations than Op which are produced by multiplying the cutoff function χ by some power of y'/y. Eventually, one last remark is that Proposition 4.4.2 imply in particular that the spaces $C^s_*(N)$ defined for $s\in\mathbb{R}_+\setminus\mathbb{N}$ do not depend on the choice of the cutoff function χ in the quantization (insofar as they can be identified to the usual Hölder spaces $C^s(N)$).

Proof of Lemma 4.4.7. We fix $\rho<\rho'+d/2$ and $\varepsilon>0$ small enough so that $\rho<\rho'+d/2-\varepsilon$. Then :

$$\begin{aligned}
\|u\|^2_{y^{\rho'}H^s} &= \sum_{j\in\mathbb{N}}\|\mathrm{Op}(\varphi_j)(y^{-\rho'}u)\|^2_{L^2}4^{js}\\
&\lesssim \sum_{j\in\mathbb{N}}\|y^{-\rho'}\mathrm{Op}'(\varphi_j)u\|^2_{L^2}4^{js}\\
&\lesssim \sum_{j\in\mathbb{N}}\|y^{-\rho'-d/2+\varepsilon}\mathrm{Op}'(\varphi_j)u\|^2_{L^\infty}4^{js}\\
&\lesssim \sum_{j\in\mathbb{N}}\|\mathrm{Op}''(\varphi_j)(y^{-\rho'-d/2+\varepsilon}u)\|^2_{L^\infty}4^{js}\\
&\lesssim \sum_{j\in\mathbb{N}}4^{j(s-s')}\underbrace{\|\mathrm{Op}''(\varphi_j)(y^{-\rho'-d/2+\varepsilon}u)\|^2_{L^\infty}4^{js'}}_{\leq\|u\|^2_{y^{\rho'+d/2-\varepsilon}C^{s'}_*}} \lesssim \|u\|^2_{y^{\rho'+d/2-\varepsilon}C^{s'}_*}\lesssim\|u\|^2_{y^\rho C^{s'}_*},
\end{aligned}$$

since $s<s'$. □

Proof of Lemma 4.4.8. Let us sketch the proof for the embedding $y^{-d/2}H^{(d+1+\varepsilon)/2}\hookrightarrow C^0$, the general case being handled in the same fashion with a little bit more work. We start by computing a $L^1\to L^\infty$ norm for $\mathrm{Op}(\sigma)$ when $\sigma\in S^{-(d+1+\epsilon)}$. We find

$$\|\mathrm{Op}(\sigma)\|^2_{y^\rho L^1\to L^\infty}\leq\sup_{x,x'}y^{d+1}y'^{2\rho}\left|\sum_{\gamma\in\Lambda}K^w_\sigma(x,y',\theta'+\gamma)\right|.$$

Going through the arguments of proof for equation (4.4.6), we deduce that

$$|K^w_\sigma(x,y',\theta')|\leq\left[(y+y')^{d+1}\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}+\left|\frac{\theta-\theta'}{y+y'}\right|^k\right)\right]^{-1}.$$

As a consequence, we have to estimate :

$$\begin{aligned}
\sum_{\gamma\in\Lambda}|K^w_\sigma(x,y',\theta'+\gamma)| &\leq\sum_{\gamma\in\Lambda}\left[(y+y')^{d+1}\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}+\left|\frac{\theta-\theta'+\gamma}{y+y'}\right|^k\right)\right]^{-1}\\
&\leq\left[(y+y')^{d+1}\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}\right)\right]^{-1}\sum_{\gamma\in\Lambda}\left[1+\frac{\left|\frac{\theta-\theta'+\gamma}{y+y'}\right|^k}{1+\left|\frac{y-y'}{y+y'}\right|^{k'}}\right]^{-1}
\end{aligned}$$

Since $y + y' > a$ the function in the sum has bounded variation, so we can apply a series-integral comparison, and replace it by the integral.

$$\leq \frac{C(y+y')^d}{(y+y')^{d+1}\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}\right)} \int_{\gamma\in\mathbb{R}^d} \left[1+\frac{\left|\frac{\theta-\theta'}{y+y'}+|\gamma|\right|^k}{1+\left|\frac{y-y'}{y+y'}\right|^{k'}}\right]^{-1}$$

$$\leq \frac{1}{y+y'}\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}\right)^{-1+d/k}.$$

We deduce that

$$\|\operatorname{Op}(\varphi_j)\|^2_{y^\rho L^1\to L^\infty} \leq \sup_{x,x'} y^{d+1}y'^{\rho}\left[(y+y')\left(1+\left|\frac{y-y'}{y+y'}\right|^{k'}\right)^{1-d/k}\right]^{-1}.$$

This is bounded for $\rho = -d$. We conclude that $\operatorname{Op}(\sigma)$ is bounded from $y^{-d}L^1$ to L^∞. Now, we recall that for $h > 0$ small enough, $(-\Delta + h^{-2})^{-(d+1+\epsilon)/2} = \operatorname{Op}(\sigma_{d+1+\epsilon}) + R$, with R smoothing, and $\sigma_{d+1+\epsilon} \in S^{-d-1-\epsilon}$. For $f \in y^{-d}W^{d+1+\epsilon,1}$, writing

$$f = (-\Delta + h^{-2})^{-(d+1+\epsilon)/2}\underbrace{(-\Delta + h^{-2})^{+(d+1+\epsilon)/2}f}_{\in y^{-d}L^1},$$

we deduce that $y^{-d}W^{d+1+\epsilon,1} \hookrightarrow C^0$ for $\epsilon > 0$. By interpolation, we then deduce that $y^{-d/2}W^{(d+1+\epsilon)/2,2} = y^{-d/2}H^{(d+1+\varepsilon)/2} \hookrightarrow C^0$. □

4.4.5 Improving parametrices II

In this section, we will explain how one can prove Theorem 4.1.1 in the case of operators acting on Hölder-Zygmund spaces on cusps. Let us gather the conditions for an operator to be $\mathbb{R}$-L^∞-admissible.

Definition 4.4.3. Let $A \in \Psi^{m,L^\infty}_{small}(N, L)$, and for each cusp Z, $I_Z(A) \in \Psi^m(\mathbb{R}\times F_Z, L_Z)$ a pseudo-differential convolution operator. We will say that A is $\mathbb{R}$-L^∞-admissible with indicial operator $I_Z(A)$ if the following holds. There exist some cutoff function $\chi \in C^\infty([a, +\infty[)$, such that $\chi \equiv 1$ on $y > C$ for some $C > 2a$,

$$\chi[\partial_\theta, A]\chi \text{ and } \mathcal{E}_Z\chi\left[\mathcal{P}_Z A\mathcal{E}_Z - I_Z(A)\right]\chi\mathcal{P}_Z, \tag{4.4.13}$$

are operators bounded from $y^N C^{-N}_*$ to $y^{-N}C^N_*$, for all $N \in \mathbb{N}$. The operator $I_Z(A)$ is independent of χ.

In the proof of Theorem 4.1.1 in the case of L^2-admissible operators, the main ingredients were the existence of the inverse of the indicial operator and the compactness of some injections. Translating the proof to the case of Hölder-Zygmund spaces, the compactness of the corresponding injections is still assured.

Lemma 4.4.10. *For any $\rho \in \mathbb{R}, s > s'$, the restriction of the injection $y^\rho C^s(N, L) \hookrightarrow y^\rho C^{s'}(N, L)$ to non-constant Fourier modes is compact. In other words, if χ is a smooth cutoff function such that $\chi \equiv 1$ for $y > 2a$ and vanishing around $y = a$, then*

$$\mathbb{1} - \mathcal{E}_Z\chi\mathcal{P}_Z : y^\rho C^s(N, L) \to y^\rho C^{s'}(N, L)$$

is compact.

Proof. We follow the proof of Lemma 4.3.1. Like in that proof, it is sufficient to prove that $\|(1-\psi_n)f\|_{C^0_*} \leq C/n\|f\|_{C^{s_0}_*}$ for some $s_0 > 0, C > 0$ and then to conclude by interpolation. Since $L^\infty \hookrightarrow C^0_*$ and $C^{1+\epsilon}_* \hookrightarrow C^1$ (for any $\epsilon > 0$), it is therefore sufficient to prove that $\|(1-\psi_n)f\|_{L^\infty} \leq C/n\|f\|_{C^1}$. By Poincaré-Wirtinger's inequality, there exists a constant $C > 0$ (only depending on the lattice Λ) such that for any f such that $\int f d\theta = 0$, $\|f(y)\|_{L^\infty(\mathbb{T}^d)} \leq C\|\partial_\theta f(y)\|_{L^\infty(\mathbb{T}^d)}$, for all $y > a$. Thus, $\|(1-\psi_n)f(y)\|_{L^\infty(\mathbb{T}^d)} \leq C/n\|y\partial_\theta f(y)\|_{L^\infty(\mathbb{T}^d)}$ and passing to the supremum in y, we obtain the sought result. □

The fact that the indicial operator has a bounded inverse is however a bit more subtle. For simplicity, assume there are no indicial roots in $\{\Re\lambda \in I\} \supset i\mathbb{R}$, and consider the action of

$$S_I = \int_{i\mathbb{R}} e^{\lambda(r-r')}(I_Z(A,\lambda))^{-1}d\lambda, \tag{4.4.14}$$

on $C^s_*(\mathbb{R}\times F_Z)$. While the action of convolution operators on L^2 spaces is very convenient to analyze, it is not so easy for Hölder-Zygmund spaces. First, from the computations in the proof of Lemma 4.3.4, we deduce that the C^s_* spaces of $L \to N$, correspond with the usual C^s_* spaces of $L_Z \to \mathbb{R}\times F_Z$.

Next, we prove the following lemma

Lemma 4.4.11. *Assume that* $\mathrm{Op}(\sigma)$ *is admissible. Then* $I_Z(\mathrm{Op}(\sigma),\lambda)$ *is* $\langle\Im\lambda\rangle^{-1}$*-semi-classically elliptic, i.e it can be written as* $\mathrm{Op}_h(\tilde\sigma_\lambda) + \mathcal{O}(h^\infty)$*, where* $h = \langle\Im\lambda\rangle^{-1}$*, the remainder is a smoothing operator, and both* $\tilde\sigma_\lambda$ *and* $1/\tilde\sigma_\lambda$ *are symbols.*

Proof. Let us express the kernel of $I_Z(\mathrm{Op}(\sigma))$ (in local charts in F_Z) as

$$\int e^{i\Phi(r,r',\lambda,z,\eta)}\chi(r-r')\tilde\sigma\left(\frac{z+z'}{2},\lambda,\eta\right)\frac{2e^{(r+r')/2}}{e^r+e^{r'}}\frac{d\eta d\lambda}{(2\pi)^{1+k}}.$$

with

$$\Phi = \langle z-z',\eta\rangle + 2\lambda\tanh\frac{r-r'}{2}.$$

As a consequence, $I_Z(\mathrm{Op}(\sigma),\lambda) = \mathrm{Op}(\sigma_\lambda)$ with

$$\sigma_\lambda = \frac{1}{2\pi}\int e^{-\lambda u + 2i\mu\tanh\frac{u}{2}}\frac{\chi(u)}{\cosh\frac{u}{2}}\tilde\sigma(z,\mu,\eta)dud\mu.$$

This integral is stationary at $\mu = i\Im\lambda$, $u = 0$, with compact support in u, and symbolic estimates in μ. So we get $\sigma_\lambda \in S^m$, with the refined estimates

$$|\partial_z^\alpha\partial_\eta^\beta\sigma_\lambda| \leq C_{\alpha,\beta}(1+|\Im\lambda|^2+|\eta|^2)^{(m-|\beta|)/2}, \tag{4.4.15}$$

with constants $C_{\alpha,\beta}$ locally uniform in $\Re\lambda$. We deduce from this that $I_Z(\mathrm{Op}(\sigma),\lambda)$ is semi-classical with parameter $h = \langle\Im\lambda\rangle^{-1}$. Since σ was elliptic, we also get for $|\lambda|^2 + |\eta|^2 > 1/c^2$:

$$\sigma_\lambda = \sigma(z,\lambda,\eta)\left(1+\mathcal{O}\left(\frac{|\Re\lambda|}{(|\Im\lambda|^2+|\eta|^2)^{1/2}}\right)\right).$$

As a consequence, $I_Z(\mathrm{Op}(\sigma),\lambda)$ is elliptic for all λ, and is semi-classical elliptic as $h \to 0$, so it is invertible for h small enough. □

From this, we deduce that $I_Z(A,\lambda)^{-1}$ is also pseudo-differential, and $\langle \Im\lambda\rangle^{-1}$-semi-classically elliptic and that S_I is pseudo-differential. More precisely, we recall from the proof of Lemma 4.3.7 that if $QA = \mathbb{1} + R$ is a first parametrix for A, then we can write for $|\Im\lambda| \gg 0$ large enough

$$\begin{aligned} I_Z(A,\lambda)^{-1} &= I_Z(Q,\lambda)(\mathbb{1} + I_Z(R,\lambda))^{-1} \\ &= I_Z(Q,\lambda) + I_Z(Q,\lambda)I_Z(R,\lambda)(\mathbb{1} + I_Z(R,\lambda))^{-1} \\ &= \mathrm{Op}(\sigma_\lambda) + R_\lambda, \end{aligned}$$

where $\sigma_\lambda \in S^{-m}$ satisfies the symbolic estimates (4.4.15) with m replaced by $-m$ and R_λ is a $\mathcal{O}(\langle \Im\lambda\rangle^{-\infty})$ smoothing operator. Note that, in (4.4.15), σ_λ also satisfies the symbolic estimate when differentiating with respect to λ. Writing $\widetilde{\sigma}(\lambda, z, \eta) := \sigma_\lambda(z, \eta)$, we have that $\widetilde{\sigma} \in S^{-m}(\mathbb{R} \times F_Z)$ (and is independent of r).

We write $S_I = S_I^{(1)} + S_I^{(2)}$, the operators respectively obtained from the contributions of $\mathrm{Op}(\sigma_\lambda)$ and R_λ in the formula (4.4.14). Choosing local patches in F_Z, we can write

$$S_I^{(1)} f(r,\zeta) = \int_{\mathbb{R}\times\mathbb{R}^n} e^{i\lambda(r-r')} e^{i\langle \zeta-\zeta',\eta\rangle} \widetilde{\sigma}(\lambda, z, \xi) f(r',\zeta') dr' d\zeta' d\lambda d\eta,$$

and this is a classical pseudodifferential operator of order $-m$ on $R \times F_Z$ which is bounded as a map $C_*^s(\mathbb{R} \times F_Z) \to C_*^{s+m}(\mathbb{R} \times F_Z)$.

It remains to study $S_I^{(2)}$. For the sake of simplicity, we will confuse in our notations the operator and its kernel. We pick $z, z' \in F_Z$ and $r > 1$. When $|\rho| < \epsilon$,

$$S_I^{(2)}(r, z, z') = \int_{\mathbb{R}} e^{itr} R_{it}(z, z') dt = e^{\rho r} \int_{\mathbb{R}} e^{itr} R_{it+\rho}(z, z') dt,$$

where $R_{it+\rho}$ is $\mathcal{O}(\langle t\rangle^{-\infty})$ in $C^\infty(F_Z \times F_Z)$, for $|\rho| < \epsilon$. We deduce that $S_I^{(2)}(r, z, z')$ is $\mathcal{O}(e^{-\epsilon|r|})$ in $C^\infty(\mathbb{R} \times F_Z \times F_Z)$. In particular, $S_I^{(2)}$ acts boundedly as a map $C_*^s(\mathbb{R} \times F_Z) \to C_*^{s+m}(\mathbb{R} \times F_Z)$. Now that we have checked that S_I is bounded on the appropriate spaces, the proof of Section §4.3.5 applies. This finishes the proof of Theorem 4.1.1.

4.4.6 Fredholm index of elliptic operators II

We now state a result concerning the Fredholm index of elliptic operators acting on Hölder-Zygmund spaces. It is similar to Proposition 4.3.3.

Proposition 4.4.3. *Let P be a (ρ_-, ρ_+)-L^∞ and $-L^2$ admissible elliptic pseudo-differential operator of order $m \in \mathbb{R}$. Let I be a connected component in (ρ_-, ρ_+) not containing any indicial root. Then, the Fredholm index of the bounded operator $P : y^\rho C_*^{s+m} \to y^\rho C_*^s$ is independent of $s \in \mathbb{R}, \rho \in I$. Moreover, the Fredholm index coincides with that of Proposition 4.3.3, that is of P acting on Sobolev spaces $H^{s+m,\rho-d/2,\rho_\perp} \to H^{s,\rho-d/2,\rho_\perp}$, for $s, \rho_\perp \in \mathbb{R}$.*

Proof. This is a rather straightforward consequence of Proposition 4.3.3 combined with the embedding estimates of Lemma 4.4.7 and Lemma 4.4.8. □

This concludes the proof of Theorem 4.1.1.

Chapitre 5

Linear perturbation theory

« *Des ciels gris de cristal. Un bizarre dessin de ponts (...)* »

Les Ponts, *Illuminations*, Arthur Rimbaud

This chapter contains the last part of the article *Local rigidity of manifolds with hyperbolic cusps I. Linear theory and pseudodifferential calculus*, written in collaboration with Yannick Guedes Bonthonneau.

Sommaire

In this chapter, we use the formalism developed in the previous chapter in order to study three operators on manifolds with hyperbolic cusps : ∇_S (the gradient of the Sasaki metric on the unit tangent bundle SM), D (the operator of differentiation of symmetric tensors) and D^*D (the Laplacian on 1-forms). This will allow us to prove that such manifolds are *spectrally rigid* that is rigid for infinitesimal perturbations of the marked length spectrum. This result will be a first step towards the proof of the local rigidity in the next chapter.

5.1 Spectral rigidity of cusp manifolds

In this chapter, we are interested in the linear version of the marked length spectrum rigidity problem, namely the question of *infinitesimal spectral rigidity* as it was originally studied in [GK80a]. We recall that a manifold (M, g) is said to be *spectrally rigid* if any smooth isospectral deformation $(g_\lambda)_{\lambda\in(-1,1)}$ of the metric g is trivial, namely there exists an isotopy $(\phi_\lambda)_{\lambda\in(-1,1)}$ such that $\phi_\lambda^* g_\lambda = g$. In the case of a closed manifold, this usually boils down to proving that the X-ray transform I_2 — that is, the integration of symmetric 2-tensors along closed geodesics in (M, g) — on symmetric *solenoidal* or *divergence-free* 2-tensors is injective. This will be called *solenoidal injectivity* in the rest of the paper.

As mentioned in Chapter 2, the solenoidal injectivity of this operator I_2 was first obtained for negatively-curved closed *surfaces* by Guillemin-Kazhdan in their celebrated paper [GK80a] and then extended by [CS98, PSU14a, Gui17a]. In this chapter, we are interested in the solenoidal injectivity of I_2^g on noncompact complete manifolds of negative curvature whose ends are real hyperbolic cusps. This does not seem to have been considered before in the literature and will be a stepping stone in the proof of the local marked length spectrum rigidity of such manifolds in the next chapter. More precisely, the case we will consider will be that of a complete negatively-curved Riemannian manifold (M, g) with a finite numbers of ends of the form

$$Z_{a,\Lambda} = [a, +\infty[_y \times (\mathbb{R}^d/\Lambda)_\theta,$$

where $a > 0$, and Λ is a crystallographic group with covolume 1. On this end, we have the hyperbolic metric

$$g = \frac{dy^2 + d\theta^2}{y^2}.$$

We recall that such a manifold (M, g) is called a cusp manifold. The sectional curvature of g is constant equal to -1, and the volume of $Z_{a,\Lambda}$ is finite. All ends with finite volume and curvature -1 take this form. In dimension two, all cusps are the same (we must have $\Lambda = \mathbb{Z}$). However, in higher dimensions, if Λ and Λ' are not in the same orbit of $SO(d, \mathbb{Z})$, $Z_{a,\Lambda}$ and $Z_{a',\Lambda'}$ are never isometric. In the following, we will sometimes call *cusp manifolds* such manifolds. Up to taking a finite cover, we can always assume that each Λ is a lattice in $\mathbb{R}^d$.

In our case, we denote by $\mathcal{C}$ the set of hyperbolic free homotopy classes on M, which is in one-to-one correspondance with the set of hyperbolic conjugacy classes of $\pi_1(M, \cdot)$. In each such class $c \in \mathcal{C}$ of C^1 curves on M, there is a unique $\gamma_g(c)$ which is a *geodesic* for g. If h is a symmetric 2-tensor, we define its X-ray transform by

$$I_2 h(c) = \frac{1}{\ell(\gamma_g(c))} \int_0^{\ell(\gamma_g(c))} h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))dt,$$

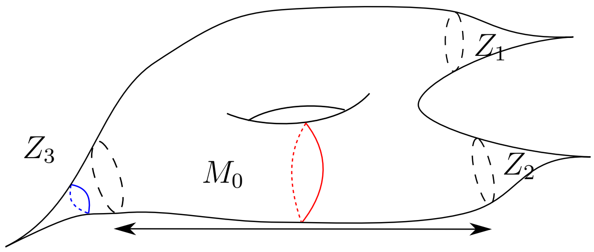

FIGURE 5.1 – A surface with three cusps.

where γ is a parametrization by arc-length. We will prove the

Theorem 5.1.1. *Let (M^{d+1}, g) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Let $-\kappa_0 < 0$ be the maximum of the sectional curvature. Then, for all $\alpha \in (0,1)$ and $\beta \in (0, \sqrt{\kappa_0}\alpha)$, the X-ray transform I_2 is injective on*

$$y^\beta C^\alpha(M, S^2T^*M) \cap H^1(M, S^2T^*M) \cap \ker D^*$$

Here, D^* denotes the divergence on 2-tensors : as usual, a tensor f is declared to be *solenoidal* if and only if $D^*f = 0$. It is defined as the formal adjoint (for the L^2-scalar product) of the operator $D := \sigma \circ \nabla$ acting on 1-forms, where ∇ is the Levi-Civita connection and σ is the operator of symmetrization of 2-tensors. In turn, the previous Theorem 5.1.1 implies the spectral rigidity for smooth compactly supported isospectral deformations.

Corollary 5.1.1. *Let (M^{d+1}, g) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Let $(g_\lambda)_{\lambda \in (-1,1)}$ be a smooth isospectral deformation of $g = g_0$ with compact support in M. Then, there exists an isotopy $(\phi_\lambda)_{\lambda \in (-1,1)}$ such that $\phi_\lambda^* g_\lambda = g$.*

Theorem 5.1.1 is the first step towards proving the local rigidity of the marked length spectrum on such manifolds, as the X-ray transform on symmetric 2-tensors turns out to be the differential of the marked length spectrum. This program will be carried out in the following chapter.

In order to prove Theorem 5.1.1, we will need — together with an approximate Livsic-type theorem which does not really differ from the compact case — to study the decomposition of symmetric 2-tensors into a potential part and a solenoidal part. Namely, we will need to prove that any symmetric 2-tensor f can be written as $f = Dp + h$, where p is a 1-form and h is solenoidal. The existence of such a decomposition relies on the analytic properties of the elliptic differential operator D^*D and in particular on the existence of a parametrix with compact remainder. Since the manifold M is not compact, this theory is made harder (smoothing operators are no longer compact) and one has to resort to a careful analysis of the behaviour of the operator on the infinite ends of the manifold. This will heavily rely on the previous chapter.

5.2 X-ray transform and symmetric tensors

5.2.1 Gradient of the Sasaki metric

A first step towards the Livsic Theorem 5.2.1 is the analytic study of the gradient ∇_S induced by the Sasaki metric g_S (itself induced by g) on the unit tangent bundle

SM of (M, g). We recall (see Appendix B) that the tangent bundle to SM can be decomposed according to :

$$T(SM) = \mathbb{V} \oplus^{\perp} \mathbb{H} \oplus^{\perp} \mathbb{R}X,$$

where $\mathbb{H}$ is the *horizontal bundle*, $\mathbb{V}$ is the *vertical bundle* and SM is endowed with the Sasaki metric g_S. If $\pi : TM \to M$ denotes the projection on the base, then $d\pi : \mathbb{H} \oplus^{\perp} \mathbb{R}X \to TM$ is an isomorphism, and there also exists an isomorphism $\mathcal{K} : \mathbb{V} \to TM$ called the *connection map*. We denote by ∇_S the Levi-Civita connection induced by the Sasaki metric g_S on SM. Given $u \in C^{\infty}(SM)$, one can decompose its gradient according to :

$$\nabla_S u = \nabla^v u + \nabla^h u + Xu \cdot X, \tag{5.2.1}$$

where $\nabla^{v,h}$ are the respective vertical and horizontal gradients (the orthogonal projection of the gradient on the vertical and horizontal bundles), i.e. $\nabla^v u \in \mathbb{V}, \nabla^h u \in \mathbb{H}$.

Lemma 5.2.1. *The gradient $\nabla_S : C^{\infty}(SM) \to C^{\infty}(SM, T(SM))$ is an elliptic $\mathbb{R}$-L^2 and $\mathbb{R}$-L^{∞} admissible differential operator of order 1. Its only indicial root is 0. Moreover, there exists two $]0, +\infty[$-L^2 and L^{∞} admissible pseudodifferential operators Q, R of order $-1, -\infty$ such that :*

$$Q\nabla_S = \mathbb{1} + R$$

with R bounded from $H^{-N,\rho-d/2,\rho_{\perp}}$ to $H^{N,-d/2+\epsilon,\rho_{\perp}}$ and from $y^{\rho}C^s_*$ to $y^{\epsilon}C^s_*$ for all $d/2 > \epsilon > 0, N \in \mathbb{N}, \rho > 0, \rho_{\perp} \in \mathbb{R}$.

Proof. The fact that ∇_S is an elliptic admissible differential operator of order 1 is immediate. We compute its indicial operator. Let $TZ \simeq [a, +\infty) \times \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$ be a global trivialization of the tangent space to the cusp with coordinates $(y, \theta, v_y, v_{\theta})$. Let $f \in C^{\infty}(\mathbb{R}^{d+1})$ be a smooth 0-homogeneous function. Then :

$$y^{-\lambda}\nabla_S(fy^{\lambda}) = \nabla^v f + \lambda f d\pi^{-1}(y\partial_y) + \sum_{i=1}^{d}(R_i f)d\pi^{-1}(y\partial_{\theta_i})$$

where $R_i := -v_{\theta_i}\partial_{v_y} + v_y\partial_{v_{\theta_i}}$ and ∇_S actually denotes the gradient on the whole tangent bundle TM. We set $I(Q, \lambda)(Z) := \lambda^{-1}g_S(Z, d\pi^{-1}(y\partial_y))$. Then :

$$I(Q, \lambda)I(\nabla_S, \lambda)f = f$$

The only indicial root of ∇_S is thus $\lambda = 0$. ☐

5.2.2 Exact Livsic Theorem

We recall that $\mathcal{C}$ is the set of hyperbolic free homotopy classes on M and that for each such class $c \in \mathcal{C}$ of C^1 curves on M, there is a unique representant $\gamma_g(c)$ which is a *geodesic* for g. In this section, we prove an exact *Livsic theorem* asserting that a function whose integrals over closed geodesic vanish is a *coboundary*, namely a derivative in the flow direction. For $f \in C^0(SM)$, we can define

$$I^g f(c) = \frac{1}{\ell(\gamma_g(c))}\int_0^{\ell(\gamma_g(c))} f(\gamma(t), \dot{\gamma}(t))dt,$$

for $c \in \mathcal{C}$.

Theorem 5.2.1 (Livsic Theorem). *Let (M^{d+1}, g) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. Denote by $-\kappa_0$ the maximum of the sectional curvature. Let $0 < \alpha < 1$ and $0 < \beta < \sqrt{\kappa_0}\alpha$. Let $f \in y^\beta C^\alpha(SM) \cap H^1(SM)$ such that $I^g f = 0$. Then there exists $u \in y^\beta C^\alpha(SM) \cap H^1(SM)$ such that $f = Xu$. Moreover, $\nabla^v Xu, \nabla_X \nabla^v u \in L^2(SM)$ and u thus satisfies the Pestov identity (Lemma 5.2.2).*

We will denote by $N_\perp$ the subbundle of $TM \to SM$ whose fiber at $(x, v) \in SM$ is given by $N_\perp(x, v) := \{v\}^\perp$. Using the maps $d\pi$ and $\mathcal{K}$, the vectors $\nabla^{v,h} u$ can be identified with elements of $N_\perp$, i.e. $\mathcal{K}(\nabla^v u), d\pi(\nabla^h u) \in N_\perp$. For the sake of simplicity, we will drop the notation of these projection maps in the following and consider $\nabla^{v,h} u$ as elements of $N_\perp$. Before starting with the proof of the Livsic Theorem 5.2.1, we recall the celebrated *Pestov identity* :

Lemma 5.2.2 (Pestov identity). *Let (M^{d+1}, g) be a cusp manifold. Let $u \in H^2(SM)$. Then*

$$\|\nabla^v Xu\|^2 = \|\nabla_X \nabla^v u\|^2 - \int_{SM} \kappa(v, \nabla^v u)\|\nabla^v u\|^2 d\mu(x, v) + d\|Xu\|^2,$$

where κ is the sectional curvature and μ is the Liouville measure.

In the compact case, the proof is based on the integration of local commutator formulas and clever integration by parts (see [PSU15, Proposition 2.2] or Appendix B). Since the manifold has finite volume and no boundary, the proof is identical and we do not reproduce it here. By a density argument and using the fact that the sectional curvature is pinched negative, assuming only $\nabla^v Xu \in L^2(SM)$, we deduce that $\nabla_X \nabla^v u, \nabla^v u \in L^2(SM)$ and

$$\|\nabla_X \nabla^v u\|, \ \|\nabla^v u\| \lesssim \|\nabla^v Xu\|.$$

Proof of Theorem 5.2.1. In this proof, we will first build u, and then determine its exact regularity. For the construction, we follow the usual tactics, but we give the details since we want to let the Hölder constant grow at infinity. For the sake of simplicity, we will denote by $y : M \to \mathbb{R}_+$ a smooth extension of the height function (initially defined in the cusps) to the whole unit tangent of the manifold, such that $0 < c < y$ is uniformly bounded from below and $y \leq a$ on $M \setminus \cup_\ell Z_\ell$. The case of uniformly Hölder functions was dealt with in [PPS15, Remark 3.1]. Since the flow is transitive, we pick a point with dense orbit x_0, and define

$$u(\varphi_t(x_0)) = \int_0^t f(\varphi_s(x_0)) ds.$$

Obviously, we have $Xu = f$, so it remains to prove that it is locally uniformly Hölder to consider the extension of u to SM. Pick $x_1 = \varphi_t(x_0)$ and $x_2 = \varphi_{t'}(x_0)$, with $t' > t$. Pick $\epsilon > 0$, and assume that $d(x_1, x_2) = \epsilon$. By the Shadowing Lemma, there is a periodic point x' with $d(x_1, x') < \epsilon$ and period $T < |t' - t| + C\epsilon$, for some uniform constant $C > 0$ depending on the dynamics, which shadows the segment $(\varphi_s(x_0))_{s \in [t, t']}$. Moreover, there exists a time $\tau \leq C\epsilon$ such that we have the following estimate :

$$d(\varphi_s(\varphi_\tau(x_1)), \varphi_s(x')) \leq C\epsilon e^{-\sqrt{\kappa_0}\min(s, |t'-t|-s)} \tag{5.2.2}$$

This is a classical bound in hyperbolic dynamics (see [HF, Proposition 6.2.4] for instance). The constant $\sqrt{\kappa_0}$ follows from the fact the maximum of the curvature is related to the lowest expansion rate of the flow (see [Kli95, Theorem 3.9.1] for instance).

Then, using the assumption that $\int_0^T f(\varphi_s(x'))ds = 0$, we write :

$$\begin{aligned} u(x_2) - u(x_1) &= \int_0^{t'-t} f(\varphi_s(x_1))ds \\ &= \int_0^{t'-t-\tau} f(\varphi_s(\varphi_\tau(x_1))) - f(\varphi_s(x'))ds - \int_{t'-t-\tau}^{T} f(\varphi_s(x'))ds + \int_0^\tau f(\varphi_s(x_1))ds \end{aligned}$$

The last two terms are immediately bounded by $\lesssim \epsilon y(x_1)^\beta$. As to the first one, it is controlled by $\lesssim \int_0^{t'-t} y(\varphi_s(x'))^\beta d(\varphi_s(x_1), \varphi_s(x'))^\alpha$ using the assumption on f. Let us find an upper bound on $y(\varphi_s(x'))$. Of course, when a segment of the trajectory $(\varphi_s(x'))_{s\in[0,T]}$ is included in a compact part of the manifold (say of height $y \leq a$), $y(\varphi_s(x'))$ is uniformly bounded by a, so the only interesting part is when the trajectory is contained in the cusps. In time $|t'-t|$, the segment $(\varphi_s(x'))_{s\in[0,T]}$ has started and returned at height $y(x_1)$. Thus, it can only go up to a height

$$y(\varphi_s(x')) \leq e^{\min(s,|t'-t|-s)} y(x_1). \tag{5.2.3}$$

Combining (5.2.2) and (5.2.3), this leads to :

$$\begin{aligned} &\int_0^{t'-t} y(\varphi_s(x'))^\beta d(\varphi_s(x_1), \varphi_s(x'))^\alpha \\ &\qquad \lesssim \int_0^{t'-t} y(x_1)^\beta e^{\beta \min(s,|t'-t|-s)} d(x_1,x_2)^\alpha e^{-\alpha\sqrt{\kappa_0}\min(s,(t'-t)-s)} ds \\ &\qquad \lesssim y(x_1)^\beta d(x_1,x_2)^\alpha \int_0^{t'-t} e^{(\beta-\alpha\sqrt{\kappa_0})\min(s,|t'-t|-s)} ds \end{aligned}$$

As long as $\sqrt{\kappa_0}\alpha > \beta$, this is uniformly bounded as $|t'-t| \to +\infty$. In particular, we conclude that u is $y^\beta C^\alpha$, and we can thus extend it to a global $y^\beta C^\alpha$ function on SM.

We now have to prove that $u \in H^1(SM)$ and to this end, we will use a kind of bootstrap argument. Since $f \in H^1(SM)$ and $f = Xu$, we obtain that $\nabla^v X u \in L^2(SM)$. Moreover, as discussed after the Pestov identity, we obtain directly that $\nabla_X \nabla^v u, \nabla^v u \in L^2(SM)$.

By using the commutator identity $[X, \nabla^v] = -\nabla^h$ (see [PSU15, Lemma 2.1]), we deduce $\nabla^h u \in L^2(SM)$. Thus, $\nabla_S u \in L^2$. By Lemma 5.2.1, we deduce that $u \in H^1(SM)$ □

5.2.3 X-ray transform and symmetric tensors

Like in the compact setting, we introduce the

Definition 5.2.1. The X-ray transform on symmetric m-tensors is defined in the same way as for C^0 functions on SM : if h is a symmetric m-tensor,

$$I_m h(c) = \frac{1}{\ell(\gamma_g(c))} \int_0^{\ell(\gamma_g(c))} \pi_m^* h(\gamma(t), \dot\gamma(t)) dt,$$

where $t \mapsto \gamma(t)$ is a parametrization by arc-length, $c \in \mathcal{C}$.

In the following, we will restrict our study to 1- and 2-tensors but it is very likely that most of the results still hold for tensors of general order $m \in \mathbb{N}$. As in the compact case, we obtain :

Lemma 5.2.3. *The symmetric derivative D is $\mathbb{R}$-L^2 and $\mathbb{R}$-L^∞ admissible. Its only indicial root is -1. Additionally, it is injective on $y^\rho H^s$ and $y^\rho C^s_*$ for all $\rho, s \in \mathbb{R}$. In particular, there is a $]-1, +\infty[$-L^2 (resp. $]-1, +\infty[$-L^∞) admissible pseudo-differential operators Q, R of order $-1, -\infty$ such that*

$$QD = \mathbb{1} + R,$$

with $R : H^{-N,\rho-d/2-1,\rho_\perp} \to H^{N,-d/2-1-\epsilon,\rho_\perp}$ and $R : y^N C^{-N}_* \to y^{-1+\epsilon} C^N_*$ bounded for all $N \in \mathbb{N}, \rho > 0, \rho_\perp \in \mathbb{R}, \epsilon > 0$. In particular, the image of D is closed.

Proof. Since D is a differential operator, it makes no difference to work with Sobolev or Hölder-Zygmund spaces. The first step is to prove that D is uniformly elliptic. We deal with the general case $m \geq 0$. By taking local coordinates around a point $(x, \xi) \in T^*M \setminus \{0\}$ for instance, one can compute the principal symbol of the operator D which is $\sigma(D)(x, \xi) : u \mapsto \sigma(\xi \otimes u)$, where $u \in \otimes^m_S T^*_x M$ (see [Sha94, Theorem 3.3.2]). Then, using the fact that the antisymmetric part of $\xi \otimes u$ vanishes in the integral :

$$\|\sigma(D)u\|^2 \geq C_m^{-1} \int_{\mathbb{S}^d} \langle \xi, v\rangle^2 \pi_m^* u^2(v) dv = C_m^{-1} |\xi|^2 \int_{\mathbb{S}^d} \langle \xi/|\xi|, v\rangle^2 \pi_m^* u^2(v) dv > 0,$$

unless $u \equiv 0$. Since $\otimes^m_S T^*_x M$ is finite dimensional, the map

$$(u, \xi/|\xi|) \mapsto \|\sigma(D)(x, \xi/|\xi|)u\|,$$

defined on the compact set $\{u \in \otimes^m_S T^*_x M, |u|^2 = 1\} \times \mathbb{S}^d$ is bounded and attains its lower bound $C^2 > 0$ (which is independent of x). Thus $\|\sigma(x, \xi)u\| \geq C|\xi|\|u\|$, so the operator is uniformly elliptic.

Next, let us give a word on the injectivity of D. Consider a 1-form f such that $Df = 0$, and f is either in some $y^\rho H^s$ or some $y^\rho C^s_*$. Then f is smooth by the elliptic regularity Theorem. As a consequence $\pi_1^* f$ is a smooth function on SM. Recall that $X\pi_1^* f = \pi_2^* Df = 0$. Additionally, the geodesic flow admits a dense orbit ; we deduce that $\pi_1^* f$ is a constant. However, since f is a 1-form, $\pi_1^* f(x, -v) = -\pi_1^* f(x, v)$ for all $(x, v) \in SM$, thus $f = 0$.

Now, we recall the results from Section §4.3. Since D is a differential operator that is invariant under local isometries, it is a $\mathbb{R}$ admissible elliptic operator. In particular, it suffices to determine whether its associated indicial operator $I_Z(D, \lambda)$ has a left inverse. In the present case, since D is an operator on sections of a bundle over M, the indicial operator is just a matrix. We consider a 1-form α in the cusp in the form

$$y^\lambda \left[a\frac{dy}{y} + \sum b_i \frac{d\theta_i}{y} \right]$$

Then we find that

$$D\alpha = y^\lambda \left[a \left(\lambda \frac{dy^2}{y^2} - \sum \frac{d\theta_i^2}{y^2} \right) + \sum b_i(\lambda + 1) \frac{d\theta_i dy + dy d\theta_i}{y^2} \right].$$

The matrix $I_Z(D, \lambda)$ is thus the transpose of

$$\begin{pmatrix} \lambda & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(\lambda+1) & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2(\lambda+1) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 2(\lambda+1) \end{pmatrix}$$

In particular, with

$$J(\lambda) = \begin{pmatrix} (\lambda+1)^{-1} & -(\lambda+1)^{-1} & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & (2(\lambda+1))^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & (2(\lambda+1))^{-1} \end{pmatrix}$$

we get

$$J(\lambda)I_Z(D,\lambda) = \mathbb{1}; \quad \|J^{-1}\| = \mathcal{O}(|\lambda|^{-1}) \text{ as } \Im\lambda \to \pm\infty.$$

We deduce that D has -1 for sole indicial root. As a consequence, we can apply Theorem 4.1.1 :

$$QD = 1 + R, \tag{5.2.4}$$

with R bounded from $H^{-N,\rho,\rho_\perp}$ to $H^{N,-d/2-1+\epsilon,\rho_\perp}$ and from $y^\rho C_*^{-N}$ to $y^\rho C_*^N$, for all $d/2 > \epsilon > 0$, $N > 0$, $\rho > -d/2+1$, $\rho_\perp \in \mathbb{R}$. □

5.2.4 Projection on solenoidal tensors

In this section, we will study the symmetric Laplacian on 1-forms, that is the operator $\Delta := D^*D$ acting on sections of $T^*M \to M$. This will allow us to define the projection on symmetric solenoidal tensors. We will denote by $\lambda_d^\pm = d/2 \pm \sqrt{d + d^2/4}$.

Lemma 5.2.4. *For all $s \in \mathbb{R}, \rho \in]\lambda_d^-, \lambda_d^+[, \rho_\perp \in \mathbb{R}$, the operator Δ is invertible on the spaces $H^{s,\rho-d/2,\rho_\perp}(M, T^*M)$ and on $y^\rho C_*^s(M, T^*M)$. Its inverse Δ^{-1} is a pseudodifferential operator of order -2.*

Proof. The operator $\Delta = D^*D$ is elliptic since D is elliptic, and it is also invariant under local isometries, and differential. In particular, it is $\mathbb{R}$-L^2 and $\mathbb{R}$-L^∞ admissible, so we can apply Theorem 4.1.1. Let us compute its indicial operator : we find

$$\begin{aligned} I(\Delta,\lambda)\left(a\frac{dy}{y}\right) &= (\lambda^2 - \lambda d - d)a\frac{dy}{y} \\ I(\Delta,\lambda)\left(b_i\frac{d\theta_i}{y}\right) &= \frac{1}{2}(\lambda+1)(\lambda-(d+1))b_i\frac{d\theta_i}{dy} \end{aligned}$$

$I(\Delta,\lambda)$ is a diagonal matrix which is invertible for

$$\lambda \notin \left\{ -1, d+1, \underbrace{d/2 \pm \sqrt{d+d^2/4}}_{=\lambda_d^\pm} \right\}$$

The interval $]\lambda_d^-, \lambda_d^+[$ does not contain other any roots, so we can apply directly Theorem 4.1.1, and get a pseudo-differential operator of order -2, Q, bounded on the relevant Sobolev and Hölder-Zygmund spaces such that

$$Q\Delta = \mathbb{1} + K, \tag{5.2.5}$$

with K bounded from $y^\rho H^{-N}$ to $y^{-\rho}H^N$, $y^{\rho+d/2}C_*^{-N}$ to $y^{d/2-\rho}C_*^N$ for all $\rho \in [0, \lambda_d^+ - d/2[$. We can also do this on the other side :

$$\Delta Q = \mathbb{1} + K', \tag{5.2.6}$$

K' satisfying the same bounds. We deduce that Δ is Fredholm and the index is constant on the window with weight $\rho \in]\lambda_d^-, \lambda_d^+[$ (see Propositions 4.3.3 and 4.4.3). Additionally, from the parametrix equation, we find that any element of its kernel (on any Sobolev or Hölder-Zygmund space we are considering) has to lie in $L^2(SM)$. However, on L^2, $\Delta u = 0$ implies $Du = 0$, and $u = 0$. Additionally, on L^2 (which corresponds to the weight $\rho = d/2$), Δ is self-adjoint so it is invertible and its Fredholm index is 0.

□

As a consequence, we obtain the

Lemma 5.2.5. *$\pi_{\ker D^*} = \mathbb{1} - D\Delta^{-1}D^*$ is the orthogonal projection on solenoidal tensors. It is a $]\lambda_d^-, \lambda_d^+[$-$L^2$ admissible operator operator of order* 0.

In other words, this proves that any tensor in the spaces $y^\rho C_*^s$ (resp. $H^{s,\rho-d/2,\rho_\perp}$) for $s \in \mathbb{R}, \rho \in]\lambda_d^-, \lambda_d^+[, \rho_\perp \in \mathbb{R}$ admits a unique decomposition in solenoidal and potential tensors $f = Dp + h$ (see Appendix B), just like in the compact setting, with $h \in \ker D^*$ and $p \in y^\rho C_*^{s+1}$ (resp. $H^{s+1,\rho-d/2,\rho_\perp}$) and $h \in y^\rho C_*^s$ (resp. $H^{s,\rho-d/2,\rho_\perp}$).

5.2.5 Solenoidal injectivity of the X-ray transform

We now prove Theorem 5.1.1. As usual, the proof relies on the *Pestov identity* combined with the Livsic theorem. It follows exactly that of [CS98] ; nevertheless, we thought it was wiser to include it insofar as we only work in H^1 regularity on a noncompact manifold (where as [CS98] is written in smooth regularity on a compact manifold).

We recall that there exists a canonical splitting

$$T_{(x,v)}(TM) = \mathbb{V}_{(x,v)} \oplus^\perp \mathbb{H}_{(x,v)},$$

where $(x, v) \in TM$ which is orthogonal for the Sasaki metric. We insist on the fact that *we now work on the whole tangent bundle TM and no longer on the unit tangent bundle SM*. As a consequence, the horizontal space $\mathbb{H}$ is the same but the vertical space $\mathbb{V}$ sees its dimension increased by 1. These two spaces are identified to the tangent vector space T_xM via the maps $d\pi$ and $\mathcal{K}$.

Given $u \in C^\infty(TM)$, we can write $\nabla_S u = \nabla^v u + \nabla^h u$, where $\nabla^v u \in \mathbb{V}, \nabla^h u \in \mathbb{H}$. We denote by $\mathrm{div}^{v,h}$ the formal adjoints of the operators $\nabla_S^{v,h}$.

Proof. We first start with an elementary inequality. Let $u \in C^\infty(SM)$. We extend u to $TM \setminus \{0\}$ by 1-homogeneity. The local Pestov identity [CS98, Equation (2.14)] at $(x, v) \in TM$ reads :

$$2\langle \nabla^h u, \nabla^v (Xu)\rangle = |\nabla^h u|^2 + \mathrm{div}^h Y + \mathrm{div}^v Z - \langle R(v, \nabla^v u)v, \nabla^v u\rangle$$

where

$$Y := \langle \nabla^h u, \nabla^v u\rangle v - \langle v, \nabla^h u\rangle \nabla^v u \qquad Z := \langle v, \nabla^h u\rangle \nabla^h u$$

Moreover, $\langle v, Z\rangle = |Xu|^2$. Integrating over SM and using the Green-Ostrogradskii formula [Sha94, Theorem 3.6.3] together with the assumption that the curvature is nonpositive, we obtain :

$$\int_{SM} \|\nabla^h u\|^2 d\mu \leq 2 \int_{SM} \langle \nabla^h u, \nabla^v (Xu)\rangle d\mu - (3 + d) \int_{SM} \underbrace{\langle v, Z\rangle}_{=|Xu|^2} d\mu \tag{5.2.7}$$

Note that by a density argument, the previous formula extends to functions $u \in H^1(SM)$ such that $\nabla^v(Xu) \in L^2(SM)$.

We now consider the case where $\pi_2^* f = Xu$ with $f \in H^1$ (and thus $u \in H^1$ and $\nabla^v(Xu) \in L^2$ by the arguments given in the proof of Livsic theorem). Following [CS98, Equation (2.18)], one obtains the following equality almost-everywhere in TM :

$$2\langle \nabla^h, \nabla^v(Xu)\rangle = \operatorname{div}^h W - 4 \times u\pi_m^*(D^* f),$$

with $W(x,v) = 4u(x,v)(f_x(\cdot, v, ..., v))^\sharp$ (where $\sharp : T^*M \to TM$ is the musical isomorphism). In (5.2.7), this yields

$$\int_{SM} \left(|\nabla^h u|^2 + (3+d)|Xu|^2\right) d\mu \leq -4 \int_{SM} u\pi_m^*(D^* f) d\mu \tag{5.2.8}$$

We now assume that f is a symmetric 2-tensor in

$$y^\beta C^\alpha(M, \otimes_S^m T^*M) \cap H^1(M, \otimes_S^m T^*M),$$

such that $D^* f = 0$ and $I_2 f = 0$. By the Livsic Theorem 5.2.1, there exists $u \in y^\beta C^\alpha(SM) \cap H^1(SM)$ such that $\pi_2^* f = Xu$. By (5.2.8), we obtain $Xu = 0$, thus $f = 0$.
□

Chapitre 6

The marked length spectrum of manifolds with hyperbolic cusps

« *Le binôme de Newton est aussi beau que la Vénus de Milo. - Le fait est qu'il y a bien peu de gens pour s'en aviser.* »

Le Gardeur de troupeaux,
Fernando Pessoa

This chapter contains the article *Local rigidity of manifolds with hyperbolic cusps II. Nonlinear theory*, written in collaboration with Yannick Guedes Bonthonneau.

Sommaire

In this chapter, we extend the local rigidity of the marked length spectrum proved in the compact case in Chapter 3 to the case of manifolds with hyperbolic cusps. We deal with the nonlinear version of the problem and prove that such manifolds are locally rigid for nonlinear perturbations of the metric that decrease sufficiently at infinity. Our proof relies on the linear theory addressed in the previous chapter and on two new ingredients : an *approximate version of the Livsic Theorem* and a careful analytic study of the operator Π_2, the *generalized X-ray transform*. In particular, we prove that the latter fits into the microlocal theory introduced in [Bon16] and developed in [GW17] and Chapter 4.

6.1 Introduction

We will be interested in the marked length spectrum rigidity question on noncompact manifolds whose ends are real hyperbolic cusps, like the ones introduced in the previous chapter. Like before, we denote by $\mathcal{C}$ the set of hyperbolic free homotopy classes on M, which is in one-to-one correspondance with the set of hyperbolic conjugacy classes of $\pi_1(M, \cdot)$ and we know that for each such class $c \in \mathcal{C}$ of C^1 curves on M, there is a unique representant $\gamma_g(c)$ which is a *geodesic* for g (see Lemma 6.4.1 for a more extensive discussion about this). This is still true for small perturbations g' of a cusp metric of reference g. The marked length spectrum of such a manifold (M, g') is then defined as the map

$$L_g : \mathcal{C} \to \mathbb{R}_+, \qquad L_g(c) = \ell_g(\gamma_g(c)).$$

Of course, like in the previous setting, this map is invariant under the action of the group of diffeomorphisms that are homotopic to the identity, namely if ϕ is a smooth diffeomorphism on M (satisfying some mild assumptions at infinity), one has $L_{\phi^* g'} = L_{g'}$. In the case of a smooth compact manifold, given a fixed metric g, the space of isometry classes of metrics (that is the orbits under the action of the group of diffeomorphisms homotopic to the identity) in a neighbourhood of g can be easily described (see Lemma B.1.7 and [Ebi68] for the historical result) : there exists a small $C^{k,\alpha}$-neighbourhood $\mathcal{U}$ (here $k \geq 2, \alpha \in (0,1)$) around g such that for any metric $g' \in \mathcal{U}$, there exists a unique $C^{k+1,\alpha}$-diffeomorphism close to the identity in this topology such that $\phi^* g' - g$ is solenoidal (with respect to g). For the sake of simplicity, we will now write $C_*^{k+\alpha}$ instead of $C^{k,\alpha}$ for the regularity spaces. Thus, isometry classes in a neighbourhood around g are in 1-to-1 correspondance with (small) divergence-free symmetric 2-tensors. In the case of a cusp manifold, this is no longer the case and we will prove (see Proposition 6.4.1) that for $N \geq 1$ large enough, isometry classes of metrics g' such that $\|g' - g\|_{y^{-N}C_*^N}$ is small (these are metrics g' which differ from g by a fast-decaying term, y being a height function in the cusp) are in 1-to-1 correspondance with *almost solenoidal* (also called *almost divergence-free*) symmetric 2-tensors in $y^{-N}C_*^N$, which are tensors f such that $(\mathbb{1} - P)D_g^* f = 0$, P being a finite rank operator of rank 1. For the sake of simplicity, given a metric g' close to g, we will denote by $[g']$ its isometry class, identified with its almost solenoidal symmetric 2-tensor given by this correspondance. We will prove the following local rigidity result.

Theorem 6.1.1. *Let (M^{d+1}, g) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. There exists $N \geq 1$ large enough, $\varepsilon > 0$ small enough and a*

1-codimensional submanifold $\mathcal{N}_{\mathrm{iso}}$ of the space of isometry classes

$$\left\{[g'] \mid \|g'-g\|_{y^{-N}C_*^N} < \epsilon\right\},$$

such that the following holds. Let g' be a metric such that $\|g'-g\|_{y^{-N}C_*^N} < \epsilon$, $g' \in \mathcal{N}_{\mathrm{iso}}$ and assume that the marked length spectrum of g and g' coincide i.e. $L_g = L_{g'}$. Then g' is isometric to g. More precisely, there exists a unique diffeomorphism ϕ close enough to the identity in the $y^{-N}C_*^{N+1}$-topology such that $\phi^* g' = g$.

Remark 6.1.1. If the Theorem is proved in the case of cusps defined with lattices, it follows for the general case. Indeed, we can take a finite cover for which the Theorem applies. Then we observe that the differomorphism ψ commutes with the corresponding group of isometries, so it factors to the quotient.

While we have not tracked down precisely the number N it should be possible to express it in terms of the Lyapunov exponents of the metric g, so it should be controlled by uniform bounds on the sectional curvature of g. We strongly believe that the introduction of the codimension 1 submanifold emerges as an artifact from the proof (which is of very analytical nature, whereas the problem is essentially geometric) but we were unable to relax this assumption. For surfaces of finite area, following the works of [Cro90, Ota90], the conjecture of Burns-Katok was globally addressed by [Cao95] and our result is not new. However, in dimension ≥ 3, this is the first non-linear result concerning the conjecture obtained allowing variable curvature on non-compact manifolds.

Like in Chapter 3, the previous Theorem is actually a corollary of a stronger result which quantifies the distance between isometry classes in terms of the marked length spectrum in a neighborhood of a metric of reference g. This statement is new even in dimension 2.

Theorem 6.1.2. *Let (M^{d+1}, g) be a negatively-curved complete manifold whose ends are real hyperbolic cusps. There exists $N \geq 1$ large enough, $\varepsilon, s > 0$ small enough, $\gamma > 0$ and a 1-codimensional submanifold $\mathcal{N}_{\mathrm{iso}}$ of the space of isometry classes*

$$\left\{[g'] \mid \|g'-g\|_{y^{-N}C_*^N} < \epsilon\right\},$$

such that the following holds. Let g' be a metric such that $\|g'-g\|_{y^{-N}C_*^N} < \epsilon$, $g' \in \mathcal{N}_{\mathrm{iso}}$. Then, there exists a diffeomorphism $\phi : M \to M$ such that :

$$\|\phi^* g' - g\|_{H^{-1-s}} \lesssim \|L_{g'}/L_g - \mathbf{1}\|^{\gamma}_{\ell^\infty(\mathcal{C})} \|g'-g\|^{1-\gamma}_{y^{-N}C_*^N}.$$

The diffeomorphism ϕ is of the form $\phi = e_V \circ T_u$, where $e_V(x) := \exp_x(V(x))$, for some vector field $V \in y^{-N}C_*^{N+1}(M, TM)$ and $T_u(y,\theta) := (y, \theta + \chi u \cdot \partial_\theta)$, for some $u \in \mathbb{R}^d$.

Of course, assuming that $L_{g'} = L_g$, one recovers the statement of Theorem 6.1.1. The strategy of the proof is rather similar to that developed in Chapter 3. We prove an approximate Livsic theorem and introduce the generalized X-ray transform Π_2 which turns out to fit in the calculus developed in the two previous chapters : the combination of these two tools and the injectivity of the X-ray transform on solenoidal 2-tensors proved in Theorem 5.1.1 will allow us to deduce a stability estimate on the X-ray transform I_2 as in Theorem 2.1.4 (see Theorem 6.3.2). After a first gauge transform and using the second-order Taylor expansion of the marked length spectrum, we will obtain a quadratic control of the X-ray transform of the difference of the two metrics which will allow us to conclude in the end by an interpolation argument.

6.2 Approximate Livsic Theorem

We prove an approximate version of the Livsic theorem, like Theorem 2.1.3 in the closed case, which will be crucial in the proof of the main theorem.

Theorem 6.2.1 (Approximate Livsic theorem)**.** *There exists* $s_0 \in]0,1[$*, and* $\nu > 0$ *such that for all* $\delta > 0$*, there exists a constant* $C > 0$ *such that : if* $f \in C^1(SM)$*, one can find* $u, h \in H^{s_0}(SM)$ *so that* $f = Xu + h$ *and*

$$\|h\|_{H^{s_0,-d/2+\delta,0}} \leq C\|f\|_{C^1}^{1-\nu\delta}\|I^g f\|_{\ell^\infty}^{\nu\delta}.$$

If we can prove this result with the additional condition that $\|I^g f\|_{\ell^\infty} \leq \varepsilon_0\|f\|_{C^1}$, then the full result is proved, because in the case $\|I^g f\|_{\ell^\infty} > \varepsilon_0\|f\|_{C^1}$, taking $h = f$, this lemma is a consequence of $C^1 \hookrightarrow H^{s_0,-d/2+\delta,0}$. From now on, we can and will thus assume that $\|f\|_{C^1} \leq 1$, and $\|I^g\|_{\ell^\infty}$ is small.

Let us briefly explain the mechanism behind the proof. The idea is to divide the manifold $\mathcal{M} := SM$ into a compact part $\mathcal{M}_\varepsilon$ and a non-compact part $\mathcal{M} \setminus \mathcal{M}_\varepsilon$ whose volume is controlled by some power of $\varepsilon > 0$. In the compact part, the arguments roughly follow that given in the proof of Theorem 2.1.3, to prove the approximate Livsic theorem on a closed manifold. The idea is to construct a coboundary Xu by defining u (as a primitive of f) on an orbit which is both sufficiently dense and sufficiently "separated" (see the definition in §6.2.2) so that one can control the Hölder norm of the difference $h := f - Xu$. In the non-compact part, however, the control of the H^s-norm of h is obtained thanks to the estimate on the volume of $\mathcal{M} \setminus \mathcal{M}_\varepsilon$. One could be much more precise on the exponents appearing, however, there does not seem to be anything to be gained by such precision.

6.2.1 General remarks on cusps

Since we will be considering the geodesic flow on cusp manifolds, it is convenient to introduce some coordinates on SZ. Given a vector in TZ,

$$v = v_y y\partial_y + v_\theta \cdot y\partial_\theta,$$

one has that $|v|^2 = v_y^2 + v_\theta^2$. In particular, we can take spherical (ϕ, u) coordinates in SZ. Here, $\phi \in [0, \pi]$ and $u \in \mathbb{S}^{d-1}$, and (y, θ, ϕ, u) denotes the point

$$\cos(\phi)y\partial_y + \sin(\phi)u \cdot y\partial_\theta.$$

The geodesic vector field over Z is then given by

$$X = \cos(\phi)y\partial_y + \sin(\phi)\partial_\phi + y\sin(\phi)u \cdot \partial_\theta. \tag{6.2.1}$$

Observe that u is invariant under the geodesic flow of the cusp.

Hyperbolic dynamics. Since the curvature is globally assumed to be negative, the geodesic flow φ_t on $\mathcal{M} := SM$ is Anosov, in the sense that there exists a continuous flow-invariant splitting

$$T_z(\mathcal{M}) = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{6.2.2}$$

where $E_s(z)$ (resp. $E_u(z)$) is the *stable* (resp. *unstable*) vector space at $z \in \mathcal{M}$, such that

$$\begin{aligned} |d\varphi_t(z) \cdot \xi|_{\varphi_t(z)} &\leq Ce^{-\lambda|t|}|\xi|_z, \ \forall t > 0, \xi \in E_s(z), \\ |d\varphi_t(z) \cdot \xi|_{\varphi_t(z)} &\leq Ce^{-\lambda|t|}|\xi|_z, \ \forall t < 0, v \in E_u(z), \end{aligned} \tag{6.2.3}$$

for some uniform constants $C, \lambda > 0$. The norm, here, is given in terms of the Sasaki metric on $\mathcal{M} = SM$. Observe that the Sasaki metric is uniformly equivalent on SZ to the product metric given by $SZ \simeq Z \times \mathbb{S}^d$. We define the *global stable* and *unstable manifolds* $W_s(z), W_u(z)$ by :

$$\begin{aligned} W_s(z) &= \{z' \in \mathcal{M}, d(\varphi_t(z), \varphi_t(z')) \to_{t\to+\infty} 0\} \\ W_u(z) &= \{z' \in \mathcal{M}, d(\varphi_t(z), \varphi_t(z')) \to_{t\to+\infty} 0\} \end{aligned}$$

For $\varepsilon > 0$ small enough, we define the *local stable* and *unstable manifolds* $W_s^\varepsilon(z) \subset W_s(z), W_u^\varepsilon(z) \subset W_u(x)$ by :

$$\begin{aligned} W_s^\varepsilon(z) &= \{z' \in W_s(z), \forall t \geq 0, d(\varphi_t(z), \varphi_t(z')) \leq \varepsilon\} \\ W_u^\varepsilon(z) &= \{z' \in W_u(z), \forall t \geq 0, d(\varphi_{-t}(z), \varphi_{-t}(z')) \leq \varepsilon\} \end{aligned}$$

We fix once for all such an ε_0 small enough.

Example 6.2.1. In the cusp Z_ℓ, in the usual coordinates $(y, \theta, \phi, u) \in [a, +\infty) \times \mathbb{T}^d \times \mathbb{S}^1 \times \mathbb{S}^d$, we consider a point $z = (y_0, \theta_0, 0, 0)$. Then, $W^s(z) = \left\{(y_0, \theta, 0, 0), \theta \in \mathbb{T}^d\right\}$.

Exit time in the cusp. It is convenient to think of cusps as (non-compact) manifolds with (geodesically) *strictly convex* boundary. We will denote by

$$\partial_- SZ = \left\{(a, \theta, \phi, u), \theta \in \mathbb{T}^d, \phi \in [0, \pi/2[, u \in \mathbb{S}^d\right\},$$

the *incoming boundary* and correspondingly $\partial_+ SZ$ the *outgoing boundary*. Given $z \in SZ$, $\ell_+(z) \leq +\infty$ will denote its exit time from the cusp in the future, and $-\infty \leq \ell_-(z)$ its exit time in the past.

From the expression of X in SZ, we see that the angle ϕ evolves according to the ODE $\dot{\phi} = \sin(\phi)$. Given $z := (x, \phi, u) \in \partial_- SZ$, its exit angle satisfies $\phi(\varphi_{\ell_+}(z)) = \pi - \phi$. Thus, a direct integration of the ODE, gives that :

$$z \in \partial_- SZ, \ \ell_+(z) = -2 \ln |\tan(\phi/2)| \tag{6.2.4}$$

6.2.2 Covering a cusp manifold

Transverse sections in the cusps. We now fix $\eta > 0$ small enough so that the closing lemma is satisfied at this scale. For the sake of simplicity, we will write the proof as if there were a single cusp : this is just a matter of notation and does not affect the content of the proof. By this means, we hope to simplify the reading.

We consider on the cusp the following transverse sections to the geodesic flow

$$\begin{aligned} \Sigma_{out} &= \left\{(a, \theta, \phi, u), \theta \in \mathbb{T}^d, \phi \in [0, \pi/4], u \in \mathbb{S}^{d-1}\right\}, \\ \Sigma_{in} &= \left\{(a, \theta, \phi, u), \theta \in \mathbb{T}^d, \phi \in [3\pi/4, \pi], u \in \mathbb{S}^{d-1}\right\}. \end{aligned}$$

Note that, up to taking a larger $a' > a$ and readjusting the constants, we can always assume that $\Sigma_{out,in}$ have diameter less than η. We consider the flowboxes

$$\mathcal{U}_{out} = \{\varphi_t z,\ z \in \Sigma_{out},\ t \geq -\eta,\ \phi(\varphi_t z) \leq \pi/2 + \eta\},$$
$$\mathcal{U}_{in} = \{\varphi_t z,\ z \in \Sigma_{in},\ t \leq \eta,\ \phi(\varphi_t z) \geq \pi/2 - \eta\}.$$

Their union covers the whole cusp. It will also be convenient to give a name to the incoming unstable manifold

$$D_\infty := \left\{(y, \theta, \pi, u),\ y \geq a,\ \theta \in \mathbb{T}^d,\ u \in \mathbb{S}^{d-1}\right\}.$$

In $\mathcal{U}_{out}$ (resp. $\mathcal{U}_{in}$), we denote by π the map $\pi(z) = \varphi_{\ell_-(z)}(z)$ (resp. $\pi(z) := \varphi_{\ell_+(z)}(z)$).

Lemma 6.2.1. *There exists a constant $C > 0$ such that for any point $z \in \mathcal{U}_{out}$, $\|\nabla \ell_-(z)\| \leq C$ and $\|d_z\pi\| \leq Ce^{|\ell_-(z)|}$.*

Proof. Let $z \in \mathcal{U}_e$. By construction, one has : $y(\varphi_{\ell_-(z)}(z)) = a$. Thus, by differentiating with respect to z, one gets for any $Z \in T_z\mathcal{M}$:

$$dy\left(d_z\varphi_{\ell_-(z)}(Z) + (\nabla \ell_-(z) \cdot Z)X(\varphi_{\ell_-(z)}(z))\right) = 0$$

In other words, if we write $\varphi_{\ell_-(z)}(z) = (a, \theta, \phi, u)$ and use the expression (6.2.1) :

$$|(\nabla \ell_-(z) \cdot Z)| = |\cos(\phi)|^{-1} \left|\frac{dy}{y}\left(d_z\varphi_{\ell_-(z)}(Z)\right)\right|$$

Now, by definition of the section Σ_{out}, there exists a uniform lower bound $|\cos(\phi)| \geq \cos(\pi/4) = 1/\sqrt{2} > 0$. Since the equation for $y_t := y(\varphi_t(z))$ is

$$\dot{y} = y\cos\phi,$$

we deduce that

$$\frac{dy_t}{y_t} = \frac{dy_0}{y_0} + \int_0^t \frac{\partial \cos\phi_s}{\partial \phi_0} ds d\phi_0.$$

For $\phi_0 < \pi/2 + \eta$, and in negative time, $|\partial\phi_s/\partial\phi_0| \leq Ce^{-Cs}$, so that (since dy/y is unitary with respect to the dual metric) we get :

$$\forall Z \in T_z\mathcal{M},\ |(\nabla \ell_-(z) \cdot Z)| \leq C|Z|$$

This provides the sought result.

As to the differential of the projection π, one has to write $\pi(z) = \varphi_{\ell_-(z)}(z)$ and differentiate with respect to z. The result then follows from the previous arguments. □

Covering the unit tangent bundle. We now choose a finite number of smooth transverse sections $(\Sigma_i)_{1\leq i\leq N}$ to the flow of diameter less than η so that the flowboxes $\mathcal{U}_{out} \cup \mathcal{U}_{in} \cup_{i=1}^N \mathcal{U}_i$ form a cover of $\mathcal{M}$, where $\mathcal{U}_i = \varphi_{(-\eta,\eta)}(\Sigma_i)$. We then fix a partition of unity $\mathbf{1} = \sum_i \theta_i$ associated to this cover. Note that this can be done so that the function θ_{out} is such that $X\theta_{out}$ is C^∞-bounded. Indeed, one first picks a cutoff χ_{out} on Σ_{out} (equal to 1 in a neighborhood of $\mathcal{N} := \left\{(a, \theta, 0, u), \theta \in \mathbb{T}^d, u \in \mathbb{S}^{d-1}\right\}$) and then pushes this function by the flow in order to obtain a function χ_{out} on $\mathcal{U}_{out}$. It remains to multiply χ_{out} by a smooth functions $\chi_{out}^{height}(y)$ and $\chi_{out}^{angle}(\phi)$, equal to 1 respectively for $y \geq a$ and $\varphi \leq \pi/2$. A similar construction is available for $\mathcal{U}_{in}$ and θ_{in}.

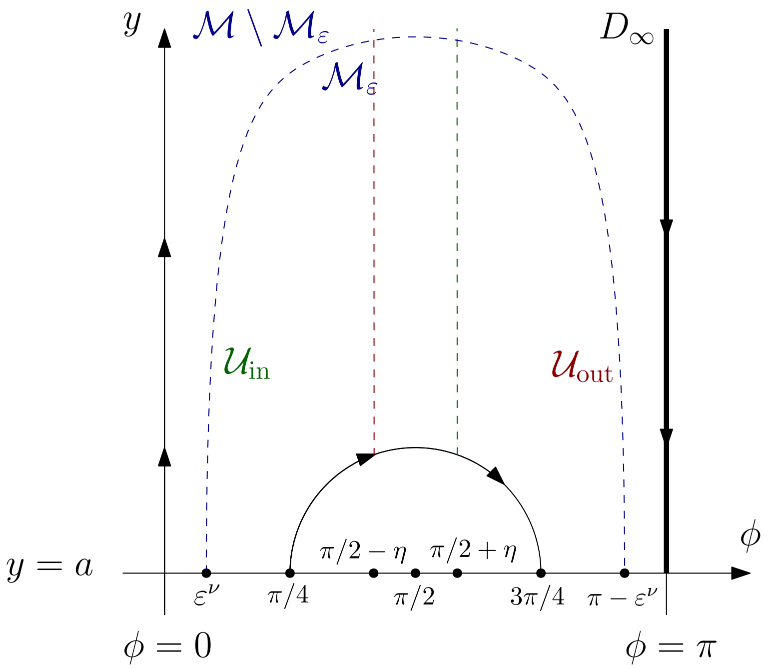

FIGURE 6.1 – The partition of a cusp.

We set $\mathcal{M}_0 := SM_0$ and

$$\mathcal{M}_\varepsilon = \mathcal{M} \setminus \left(\left\{ \varphi_t z; 0 \leq t \leq \ell_+(z), z = (a, \theta, \phi, u) \in \Sigma_{out}, \phi \in [0, \varepsilon^\nu] \right\} \cup D_\infty \right)$$

where $\nu, \varepsilon > 0$ will be chosen small enough at the end. We will pay attention to the fact that the different constants appearing in the following paragraphs do not depend on ν, unless explicitly stated. Note that by construction, any point in $\mathcal{M}_\varepsilon$ will exit the cusp (either in the future or in the past) by a time which is bounded above by $C + \nu |\log \varepsilon|$, which we state as a

Lemma 6.2.2. *There exists a constant $C > 0$ such that for all $z \in \mathcal{M}_\varepsilon$, there exists a time t such that $|t| \leq C + \nu |\log \varepsilon|$ and $\varphi_t z \in \mathcal{M}_0$.*

This is a rather elementary computation following from §6.2.1 which we do not detail.

A well-designed periodic orbit. As mentioned in the introduction to this section, the proof heavily relies on the fact that one can find a suitable orbit, which will be used in order to define an approximate coboundary. In the following, we will denote by $W_\theta(z) = \cup_{w \in W_u^\theta(z)} W_s^\theta(w)$ for $\theta > 0$. This is a Hölder section which is transverse to the flow. We will say that a segment of orbit S is θ-*transversally separated* if for all $z \in \mathcal{M}$, S intersects $W_\theta(z)$ at most in one point. We also say that a segment of orbit is $\eta > 0$ dense in Ω if its η-neighbourhood contains Ω.

Lemma 6.2.3. *There are constants $\beta_t > 1 > \beta_d > 0$ such that for all $\varepsilon > 0$ small enough, there exists a periodic point z_0 with period $T \leq \varepsilon^{-1/2}$, such that in $\mathcal{M}_\varepsilon$ its orbit is ε^{β_t}-tranversally separated and $(\varphi_t z_0)_{0 \leq t \leq T-1}$ is ε^{β_d}-dense. Moreover, there exists a segment of length $\leq C$ which is η-dense in $\mathcal{M}_0$.*

Proof. The proof is rather identical to that of [GL19a] so we skip it. The main difference is that, for any $z, w \in \mathcal{M}_\varepsilon$, the non-compactness does not allow to find a segment of orbit $\gamma_{z,w}$ joining a ball of radius ρ around z to a ball of radius ρ around w in a time $T(\rho)$ which is independent of ε. However, thanks to Lemma 6.2.2, one can prove that this time $T(\rho, \varepsilon)$ is bounded by $C + \nu|\log \varepsilon|$, which is harmless for the rest of the proof. We refer to the proof in [GL19a] for further details. □

6.2.3 Proof of the approximate Livsic Theorem

We first construct the coboundary Xu and then show that it satisfies the required estimate. Recall that $\|f\|_{C^1} \leq 1$, and $\varepsilon := \|I^g f\|_{L^\infty}$ is assumed to be small. It will only be required to be small enough so that we can apply Lemma 6.2.3, and get a corresponding good orbit $\varphi_t z_0$.

Construction of the coboundary. On the periodic orbit of z_0, we define the function $\tilde{u}$ by $\tilde{u}(\varphi_t z_0) = \int_0^t f(\varphi_s z_0) ds$. Note that it may not be continuous at z_0. To circumvent this problem, we will rather define $\tilde{u}$ only on the set $\mathcal{O}(z_0) := (\varphi_t x_0)_{0 \leq t \leq T-1}$ (which satisfies the desired properties of density and transversality).

Lemma 6.2.4. *For $\beta := (2\beta_t)^{-1} < 1/2$, there exists $C > 0$, independent of ε, such that $\|\tilde{u}\|_\beta \leq C$.*

Here $\|f\|_\beta := \sup_{z,z'} \frac{|f(z)-f(z')|}{d(z,z')^\beta}$ denotes the Hölder part of the C^β-norm.

Proof. If z, z' are close enough and on the same piece of local orbit, the result is obvious. We can thus assume $z, z' \in \mathcal{O}(z_0)$ and $z' \in W_\eta(z)$. Then, by separation of the orbit, we know that $d(z, z') \geq \varepsilon^{\beta_t}$. Without loss of generality, we can assume that $z' = \varphi_{t'} z_0$ and $z = \varphi_t z_0$ with $t > t'$ and thus

$$\tilde{u}(z) - \tilde{u}(z') = \int_0^t f(\varphi_s z') ds.$$

By the Anosov closing lemma, we can close the segment of orbit $(\varphi_s z')_{0 \leq s \leq t}$, that is there exists a periodic point z_p such that $d(z', z_p) \leq C d(z, z')$ and of period $t_p = t + \tau$, where $|\tau| \leq C d(z, z')$ which shadows the segment. Then :

$$|\tilde{u}(z') - \tilde{u}(z)| \leq \underbrace{\left| \int_0^t f(\varphi_s z') ds - \int_0^{t_p} f(\varphi_s z_p) ds \right|}_{=(\mathrm{I})} + \underbrace{\left| \int_0^{t_p} f(\varphi_s z_p) ds \right|}_{=(\mathrm{II})}$$

The first term (I) is bounded by $C d(z, z')^\beta$ by hyperbolicity, with C depending only on the global hyperbolicity of the flow. The second term (II) is bounded — by assumption — by εt_p. But

$$\varepsilon t_p \leq 2\varepsilon t \leq 2\varepsilon T \leq 2\varepsilon^{1/2} \leq 2 d(z, z')^{1/(2\beta_t)}.$$

This finishes the proof. □

We consider $i \in \{out, in, 1, ..., N\}$. Given $z \in \Sigma_i \cap \mathcal{O}(z_0)$, we define $\tilde{u}_i(z) := \tilde{u}(z)$. We have the

Lemma 6.2.5. *There exists a constant $C > 0$ such that, $\|\tilde{u}_i\|_{C^\beta} \leq C$.*

Proof. Since all the sections Σ_i for $i \in \{out, in, 1, ..., N\}$ are included in $\mathcal{M}_0$, this amounts to studying the C^β norm of $\tilde{u}$ in $\mathcal{M}_0$. The β-Hölder part of the C^β-norm follows from the previous Lemma. All we have to prove is that $\tilde{u}$ is bounded for the C^0-norm in $\mathcal{M}_0$. But we know that there exists a segment of the orbit $\mathcal{O}(z_0)$ — call it S — of length $\leq C$ which is η-dense in $\mathcal{M}_0$. Any point $z \in \mathcal{M}_0$ can be joined by a curve in $W_\eta(z_0)$, a piece of the segment S which we denote by $[w; w']$ and a curve in $W_\eta(z)$. Then :

$$|u(z)| = |u(z) - u(z_0)| \leq \underbrace{|u(z_0) - u(w)|}_{=(\text{I})} + \underbrace{|u(w) - u(w')|}_{=(\text{II})} + \underbrace{|u(w') - u(z)|}_{=(\text{III})}$$

The terms (I) and (III) are controlled by a constant $\leq C$, using the Hölder regularity provided by the previous Lemma. The control of the term (II) follows from the fact that S has length $\leq C$ and that $\|f\|_{C^0} \leq \|f\|_{C^1} \leq 1$. □

For $i \in \{out, in, 1, ..., N\}$, we then extend $\tilde{u}_i$ to Σ_i by the formula

$$u_i(x) := \sup\{\tilde{u}(y) - \|u\|_{C^\beta \mathcal{O}(z_0)} d(x,y)^\beta \; | y \in \mathcal{O}(z_0) \cap \Sigma_i\}.$$

One finds that $\|u_i\|_{C^\beta} \leq C\|\tilde{u}_i\|_{C^\beta} \leq C$. We then push the function u_i by the flow in order to define it on $\mathcal{U}_i$ by setting for $z \in \Sigma_i, \varphi_t z \in \mathcal{U}_i$:

$$u_i(\varphi_t z) := u_i(z) + \int_0^t f(\varphi_s z) ds.$$

We now set $u := \sum_i u_i \theta_i$ and $h := f - Xu = -\sum_i u_i X\theta_i$.

Regularity of the coboundary. By construction, the functions $X\theta_i$ are uniformly bounded in C^∞, independently of ε. Thus, for $i \in \{1, ..., N\}$, the functions $u_i X\theta_i$ are in C^β with a Hölder norm independent of $\varepsilon > 0$. However, this is not the case of the function $u_{out} X\theta_{out}$, $u_{in} X\theta_{in}$. We have local results. First, let us introduce $\overline{u}$ and $\overline{h}$ the averages with respect to the θ variable in the cusps.

Lemma 6.2.6. *We have the following estimates in the cusps :*

$$|u(z)|, \; |h(z)| \leq C\|f\|_{C^\beta} + (\log y + \eta)\|f\|_{C^0}$$

$$\sup_{d(z,z') \leq 1} \frac{|u(z) - u(z')|}{d(z,z')^\beta}, \; \frac{|h(z) - h(z')|}{d(z,z')^\beta} \leq Cy^\beta.$$

$$\sup_{d(z,z') \leq 1} \frac{|\overline{u}(z) - \overline{u}(z')|}{d(z,z')^\beta}, \; \frac{|\overline{h}(z) - \overline{h}(z')|}{d(z,z')^\beta} \leq C.$$

Proof. Of course, since $\theta_{in,out}$ do not depend on θ, the estimates on u imply those on h. It is thus sufficient to control u_{out}. We first control the C^0-norm. For $z \in \Sigma_{out}$, $t \geq -\eta$, $\phi(\varphi_t z) < \pi + \eta$, we have :

$$\begin{aligned} |u_{out}(\varphi_t z)| = \left| u_{out}(z) + \int_0^t f(\varphi_s z) ds \right| &\leq \underbrace{\|u_e|_{\Sigma_e}\|_{C^0}}_{\leq C, \text{Lemma } 6.2.5} + |t| \|f\|_{C^0} \\ &\leq C + (|\ell_-(z)| + \eta)\|f\|_{C^0}. \end{aligned}$$

As to the β-Hölder norm, we have for $z, z' \in \mathcal{U}_{out}$ (such that $d(z,z') \leq 1$) assuming without loss of generality that $|\ell_-(z')| \geq |\ell_-(z)|$:

$$|u_{out}(z) - u_{out}(z')| \leq \underbrace{|u_{out}(\pi(z)) - u_{out}(\pi(z'))|}_{:=(\mathrm{I})} + \underbrace{\int_0^{|\ell_-(z)|} |f(\varphi_{-s}(z)) - f(\varphi_{-s+\ell_-(z')-\ell_-(z)}(z'))| ds}_{:=(\mathrm{II})} + \underbrace{2|\ell_-(z) - \ell_-(z')| \|f\|_{C^0}}_{:=(\mathrm{III})} \tag{6.2.5}$$

By Lemma 6.2.1, successively, using that $|\ell_-(z)| \leq C + |\log y|$:

$$\begin{aligned}
(\mathrm{I}) &\leq \|u_{out}|_{\Sigma_{out}}\|_{C^{\beta_2}} d(\pi(z), \pi(z'))^\beta \leq Ce^{|\ell_-(z)|\beta_2} d(z,z')^\beta, \\
(\mathrm{II}) &\leq \int_0^{|\ell_-(z)|} \|f\|_{C^\beta} d(\varphi_{-s} z, \varphi_{-s+\ell_-(z')-\ell_-(z)} z')^\beta ds \\
&\leq C \int_0^{|\ell_-(z)|} e^{\lambda_{max}\beta s} d(z, \varphi_{\ell_-(z')-\ell_-(z)} z')^\beta ds \leq \frac{C}{\lambda_{max}\beta} e^{\lambda_{max}\beta|\ell_-(z)|} d(z,z')^\beta,
\end{aligned}$$

and :

$$(\mathrm{III}) \leq Cd(z,z').$$

Here, λ_{max} is the maximal Lyapunov exponent of the flow in the cusp, which is just 1.

Let us now deal with $\overline{u}$ and $\overline{h}$. Then, in (6.2.5), terms (I) and (II) become much better. Indeed, the estimates are formally the same, except that we have assumed that $\theta(z) = \theta(z')$. In that case, we are considering trajectories in an unstable manifold, in negative time, so

$$\begin{aligned}
d(\pi(z), \pi(z')) &\leq Ce^{-|\ell_-(z)|} d(z,z'), \\
d(\varphi_{-s}(z), \varphi_{-s}(z')) &\leq Cd(z,z').
\end{aligned}$$

□

Now, we claim that h vanishes on $\mathcal{O}(z_0)$: indeed, for $i \in \{out, in, 1, ..., N\}$, on $\mathcal{U}_i \cap \mathcal{O}(z_0)$ one has $u_i \equiv \tilde{u}$ and thus $h = -\tilde{u} \sum_i X\theta_i = -\tilde{u} X \sum_i \theta_i = -\tilde{u} X \mathbf{1} = 0$. Next, recall that $y \leq C\varepsilon^{-\nu}$ in $\mathcal{M}_\varepsilon$, so that by Lemma 6.2.6, $\|h|_{\mathcal{M}_\varepsilon}\|_{C^\beta} \leq C\varepsilon^{-\beta\nu}$. Combining this with the fact that $\mathcal{O}(z_0)$ is ε^{β_d}-dense in $\mathcal{M}_\varepsilon$, we deduce :

Lemma 6.2.7. *The coboundary satisfies*

$$\|h|_{\mathcal{M}_\varepsilon}\|_{C^0} \leq C\varepsilon^{\beta(\beta_d - \nu)}.$$

We can now end the proof of Theorem 6.2.1.

Proof of Theorem 6.2.1. By Lemma 6.2.6, we have that $u, h \in y^\beta C^\beta(SM) \subset H^s(SM)$ for $0 < s < \beta$ (since $\beta < 1/2 \leq d/2$). On the other hand, the zeroth Fourier mode is much better, with C^β estimates. Using Lemma 4.4.7, we deduce that $u, h, Xu \in H^{s,-d/2+\delta,0}(SM)$, for any $\delta > 0$ small enough, $0 < s < \beta$. Moreover, we can decompose

$$\|y^\rho h\|^2_{L^2(SM)} = \underbrace{\int_{\mathcal{M}_\varepsilon} y^{2\rho} h^2 d\mu}_{=(\mathrm{I})} + \underbrace{\int_{\mathcal{M}\setminus\mathcal{M}_\varepsilon} y^{2\rho} h^2 d\mu}_{=(\mathrm{II})},$$

where $|(\mathrm{I})| \leq C\varepsilon^{2\beta(\beta_d-\nu)}$ by Lemma 6.2.7 as long as $\rho < d/2$. For $|(\mathrm{II})|$, using the logarithmic bound on h given by Lemma 6.2.6, we get

$$\begin{aligned} |(\mathrm{II})| &\leq C \int_0^{\pi/2} \sin^{d-1}\phi d\phi \int_{(\frac{a}{y}\sin\phi)<\varepsilon^\nu, y>a} \frac{(1+\log y)}{y^{d+1-2\rho}} dy \\ &\leq C \int_0^{\pi/2} \sin^{d-1}\phi d\phi \left(\mathbb{1}_{\phi<\varepsilon^\nu} a^{2\rho-d}\log a + \mathbb{1}_{\phi>\varepsilon^\nu} (a\frac{\sin\phi}{\varepsilon^\nu})^{2\rho-d} \log\frac{a\sin\phi}{\varepsilon^\nu}\right) \\ &\leq C\varepsilon^{\nu(d-2\rho)}. \end{aligned}$$

As a consequence, setting $\beta_3 := \min(\beta(\beta_d-\nu), \nu(d/2-\rho))$, we obtain that

$$\|y^\rho h\|_{L^2} \leq C\varepsilon^{\beta_3},$$

that is $\|h\|_{H^{0,-\rho,-\rho}} \leq C\varepsilon^{\beta_3}$ and thus in particular, $\|h\|_{H^{0,-\rho,0}} \leq C\varepsilon^{\beta_3}$. To conclude the proof, it suffices to interpolate between $H^{0,-d/2+\delta,0}$ and $H^{s,-d/2+\delta,0}$. □

6.3 The normal operator

6.3.1 Definition and results

Like in Chapter 2, we introduce the normal operator which will be crucial to our analysis of the problem. In the article [GW17], a scale of *anisotropic Hilbert spaces* $H^{r\mathbf{m},\rho}(SM)$ *was introduced to analyze the meromorphic continuation of the resolvent* $R_\pm(\tau) = (X \pm \tau)^{-1}$ *of* X. This scale took the form

$$H^{r\mathbf{m},\rho}(SM) = \mathrm{Op}(e^{rG})^{-1} H^{0,\rho,0}(SM).$$

Here, G is a log order symbol of the form $G \sim \mathbf{m}\log|\xi|$, where $\mathbf{m}$ is an order 0 symbol. To obtain the meromorphic continuation of $(X-\tau)^{-1}$, as usual, the criterion is a sign condition on the subprincipal symbol of X acting on thoses spaces (there was also a special ingredient relating to inversion of an indicial operator). In particular, the arguments from [GW17] apply to the spaces $H^{r\mathbf{m},\rho,\rho_\perp}(SM)$, and we find that $(X-\tau)^{-1}$ continues from $\Re s > 0$ to $\Re s > -\delta$ as a bounded operator on $H^{r\mathbf{m},\rho,\rho_\perp}(SM)$ if $Cr > \max(|\rho|, |\rho_\perp|) + \delta$, for some constant $C > 0$ depending only on $\mathbf{m}$.

Since one has for some $C > 0$,

$$H^{s,\rho,\rho_\perp}(SM) \subset H^{Cs\mathbf{m},\rho,\rho_\perp}(SM) \subset H^{-s,\rho,\rho_\perp}(SM),$$

we obtain the following :

Lemma 6.3.1. *Let (M,g) be a cusp manifold. Given $s > 0$, $\rho \in]-d/2, d/2[$ and $|\rho_\perp| \leq |\rho|$, there is a $\delta > 0$ such that seen as an operator from $H^{s+C|\rho|,\rho,\rho_\perp}(SM)$ to $H^{-s-C|\rho|,\rho,\rho_\perp}(SM)$, $R_\pm(\tau)$ has a meromorphic continuation from $\{\tau \in \mathbb{C} \mid \Re\tau > 0\}$ to $\{\tau \in \mathbb{C} \mid \Re\tau > -\delta\}$.*

Since X, seen as a differential operator, is antiself-adjoint on its domain in $L^2(SM)$, the poles of its resolvent on the imaginary axis $i\mathbb{R}$ are of order 1 (see [Gui17a, Lemma 2.4]). Moreover, the geodesic flow of a cusp manifold is mixing (see [Moo87] for constant curvature manifolds, [DP98] in the general case) and this implies that there is a single pole at 0 (see [Gui17a, Lemma 2.5]). Actually, 0 is an embedded discrete eigenvalue of multiplicity 1 and the absolute spectrum is $i\mathbb{R}$; there is no singular continuous spectrum.

The holomorphic part of $R_\pm(\tau)$ at $\tau = 0$, denoted by $R_0^\pm$ is a well defined operator, bounded from $H^{s+C|\rho|,\rho,\rho_\perp}(SM)$ to $H^{-s-C|\rho|,\rho,\rho_\perp}(SM)$ for all $s > 0$ and $\rho \in]-d/2, d/2[$, $|\rho_\perp| \leq |\rho|$. Additionally, whenever $Xu \in H^{s+C|\rho|,\rho,\rho_\perp}(SM)$ and $\int_{SM} u d\mu = 0$, μ being the Liouville measure on SM,

$$R_0^\pm X u = u.$$

Like in the compact setting, we then define

$$\Pi := R_0 + R_0^*. \tag{6.3.1}$$

Given $f, g \in H^s(SM)$, with $s > 0$, and so that $\int f = \int g = 0$, one can prove that

$$\langle \Pi f, g \rangle := \int_{\mathbb{R}} \langle f \circ \varphi_t, g \rangle dt, \tag{6.3.2}$$

and that $\Pi \mathbf{1} = 0$. We have the following

Proposition 6.3.1. *There exists a $C > 0$ such that for any $s > 0$, $\rho \in]-d/2, d/2[$, $|\rho_\perp| \leq |\rho|$, the operator Π is bounded from $H^{s+C|\rho|,\rho,\rho_\perp}(SM)$ to $H^{-s-C|\rho|,\rho,\rho_\perp}(SM)$. It is symmetric with respect to the L^2 duality, and*

1. *$\forall f \in H^{s,\rho,\rho_\perp}(SM), X\Pi f = 0$,*
2. *$\forall f \in H^{s,\rho,\rho_\perp}(SM)$ such that $Xf \in H^{s,\rho,\rho_\perp}(SM)$, $\Pi X f = 0$.*
3. *If $f \in H^{s,\rho,\rho_\perp}(SM)$, $\langle f, \mathbf{1}\rangle_{L^2} = 0$ then : $f \in \ker \Pi$ if and only if there exists a solution $u \in H^{s,\rho,\rho_\perp}(SM)$ to the cohomological equation $Xu = f$, and u is unique modulo constants.*
4. *The operator Π is positive in the sense of quadratic forms, that is for all $s > 0$, $f \in H^{s,0,0}(SM)$, $\langle \Pi f, f\rangle_{L^2(SM)} \geq 0$.*

The operator Π will play the role of the so-called *normal* operator I^*I in the case of X-ray transform on manifolds with boundary. While Π is not a very regular operator, its action on 2-tensors is very convenient for our purposes. We let :

$$\Pi_2 := \pi_{2*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_2^*. \tag{6.3.3}$$

A priori, Π_2 is defined as an operator from $H^{s,\rho,\rho_\perp}(M, \otimes_S^2 T^*M) \to H^{-s,\rho,\rho_\perp}(M, \otimes_S^2 T^*M)$, but we will prove the

Theorem 6.3.1. *Π_2 is a $]0, d[$-L^2 admissible pseudodifferential operator of order -1. It is invertible on solenoidal tensors, in the sense that there exists another $]0, d[$-L^2-admissible operator Q_2, of order 1, such that :*

$$Q_2 \Pi_2 = \Pi_2 Q_2 = \pi_{\ker D^*},$$

where $\pi_{\ker D^*}$ is the L^2-orthogonal projection on the kernel of D^*.

The proof of this central theorem will be given in the second half of this section. We also obtain a stability estimate following the previous theorem.

Theorem 6.3.2. *There exist $s_0 > 0$ such that for all $0 < s < s_0$, there exist $\nu, C > 0$ such that :*

$$\forall f \in C^1(M, \otimes_S^2 T^*M) \text{ with } D^* f = 0, \qquad \|f\|_{H^{-s-1,0,0}} \leq C \|I_2 f\|_{\ell^\infty}^{\nu} \|f\|_{C^1}^{1-\nu}.$$

We can also consider the action of functions instead of 2-tensors :

$$\Pi_0 = \pi_{0*}(\Pi + \mathbf{1} \otimes \mathbf{1})\pi_0^*.$$

This is also a pseudo-differential operator of order -1. We will see (in Remark 6.3.1) that a similar statement as Theorem 6.3.1 holds, and so does a stability estimate for Hölder functions.

Proof. Let $f \in C^1(M, \otimes_S^2 T^*M)$ be such that $D^*f = 0$ and $\|f\|_{C^1} \leq 1$. By Theorem 6.2.1, we can write $\pi_2^* f = Xu + h$, with $\|h\|_{H^{s_0,-d/2+\delta,0}} \lesssim \|I_2 f\|_{\ell^\infty}^{\nu'\delta}$ for some $1 > \nu' > 0$. Thus for $0 < s < s_0$,

$$\|f\|_{H^{-s-1,0,0}} \lesssim \|\Pi_2 f\|_{H^{-s,0,0}} \lesssim \|\pi_{2*}\Pi h\|_{H^{-s,0,0}} \lesssim \|h\|_{H^{s,0,0}} \lesssim \|I_2 f\|_{\ell^\infty}^{\nu'\delta},$$

where the first inequality follows from Theorem 6.3.1 and the last one from Theorem 6.2.1. We then set $\nu = \nu'\delta$. □

6.3.2 Inverting the normal operator on tensors

Let us start by some preliminary arguments. Consider $f \in y^{-d/2+\epsilon}H^N$, such that $\Pi_2 f = D^*f = 0$. Then, like in Lemma 2.5.4, using the positivity of Π, we deduce that $\Pi\pi_2^* f = 0$, and thus $\pi_2^* f = Xu$ with $u \in y^{-d/2+\epsilon}H^N$. This implies that $I_2^g f = 0$. If N is large enough, we get also that $f \in y^\epsilon C^1$, and taking ϵ small enough, we can then apply the s-injectivity of the X-ray transform proved in Theorem 5.1.1, and deduce that $f = 0$.

Following this observation, it would be convenient if we could prove that the kernel of Π_2 can only contain elements of $y^{-d/2+\epsilon}H^N$. Next, we would also like to deduce from the injectivity, the fact that Π_2 is invertible ; that is, we want to prove that Π_2 is Fredholm on some spaces, with index 0. We will show that indeed it is Fredholm with constant index on a range of spaces, which includes L^2. Since Π_2 is L^2-symmetric, its index will have to be 0.

To obtain Theorem 6.3.1, it will thus suffice to build a parametrix with a good remainder. To this end, we will prove the

Lemma 6.3.2. *The normal operator Π_2 is $]0, d[$-L^2 admissible of order -1.*

This will the most technical part of the proof. Next, according to Lemma 5.2.5, $\pi_{\ker D^*}$ itself is $]0, d[$ admissible on L^2. Its principal symbol $\sigma(\pi_{\ker D^*})$ is a projector. We will find that the symbol $\sigma(\Pi_2)$ of Π_2 is elliptic on the range of $\sigma(\pi_{\ker D^*})$, in the sense that we can factorize

$$q\sigma(\Pi_2) = \sigma(\pi_{\ker D^*}),$$

with q a symbol of order 1. For Theorem 5.1.1 to apply, we would need Π_2 to be elliptic in the usual sense, that is we would need to know that a tensor in the kernel of Π_2 is actually decreasing fast enough. However, we will check that the ellipticity on the range of $\sigma(\pi_{\ker D^*})$ is sufficient to obtain the same result. Finally, it will remain to compute the indicial roots of Π_2, and check that there are none in $]0, d[$.

Local part of the operator. As suggested by (6.3.2), we first pick a cutoff χ equal to 1 in $[-t_0, t_0]$, and define

$$\Pi_{2,\chi} f = \pi_{2*} \int_{\mathbb{R}} \chi(t)(\pi_2^* f) \circ \varphi_t dt.$$

This operator commutes with local isometries in the cusp, and is properly supported. Additionally, one can check in local coordinates that it is pseudo-differential (it is the case at the bottom of the cusp, invariance by isometries guarantees that it is still the case for large y). Given $(x, \xi) \in T^*M$, we can decompose the space of tensor

$$\begin{aligned} \otimes_S^2(T_x^*M) &= \ker \sigma(D^*)(x,\xi) \oplus \operatorname{ran} \sigma(D)(x,\xi) \\ &= \ker i_\xi \oplus \operatorname{ran} \sigma j_\xi, \end{aligned}$$

where i_ξ is the contraction by $\xi^\sharp$, $\sigma j_\xi : u \mapsto \sigma(\xi \otimes u)$. We denote by $\pi_{\ker i_\xi}$ the projection on the left space, parallel to the right space. Note in particular that $\sigma(\pi_{\ker D^*}) = \pi_{\ker i_\xi}$. Then, since the principal symbol of an operator is obtained by a local computation, one gets, just like in the compact setting that the principal symbol of $\Pi_{2,\chi}$ is

$$\frac{2\pi}{B_d} |\xi|^{-1} \pi_{\ker i_\xi} \pi_{2*} \pi_2^* \pi_{\ker i_\xi},$$

where $B_d = \int_0^\pi \sin^{d+3}(\phi) d\phi$.

We conclude that $\Pi_{2,\chi}$ is a L^2 admissible operator, elliptic on $\ker i_\xi$. It remains to study the difference $\Pi_2 - \Pi_{2,\chi}$, and prove that it is a smoothing, L^2 admissible operator. Since we can write

$$\begin{aligned} \Pi_2 - \Pi_{2,\chi} = & \left[1 - \int_{\mathbb{R}} \chi\right] \mathbf{1} \otimes \mathbf{1} \\ & + \pi_{2*} \left[\int_0^{+\infty} \chi'(t) \varphi_t^* R_0^- dt \right] \pi_2^* + \pi_{2*} \left[\int_{-\infty}^0 dt \chi'(t) \varphi_t^* R_0^+ dt \right] \pi_2^*, \end{aligned}$$

we can concentrate our study on :

$$U := \int_{t_0}^{+\infty} \chi'(t) \pi_{2*} \varphi_t^* R_0^- \pi_2^* dt$$

Regularity properties. We will show in this section that U is a smoothing $]0, d[$-L^2 admissible operator. Before explaining how one can use the symmetries of the flow to prove that it is admissible, let us recall why it should be smoothing. This part of the argument is very similar to the compact case.

The space $T(SM)$ decomposes as the sum $T(SM) = \mathbb{R}X \oplus \mathbb{V} \oplus \mathbb{H}$, where $\mathbb{V} := \ker d\pi$ ($\pi : SM \to M$ being the canonical projection) is the vertical space and $\mathbb{H}$ is the horizontal space. We denote by $\mathbb{H}^*, \mathbb{V}^*$ the dual vector bundles such that

$$\mathbb{V}^*(\mathbb{V}) = 0, \mathbb{H}^*(\mathbb{H} \oplus \mathbb{R}X) = 0.$$

As soon as there are no conjugate points, the vertical bundle $\mathbb{V}$ is transverse to the Green bundles, so that we have $\mathbb{V}^* \oplus E_u^* = \mathbb{V}^* \oplus E_s^* = T(SM)$; for a proof, see [Kli74, Proposition 6]. The map $d\pi^\top : T^*M \to \mathbb{V}^*$ is an isometry and $d\pi : \mathbb{H} \oplus \mathbb{R}X \to TM$ is an isometry too. We have :

$$\mathrm{WF}(\pi_{2*}f) \subset \{(x,\xi) \mid \exists v \in S_xM,\ ((x,v), \underbrace{d\pi^\top\xi}_{\in\mathbb{V}^*}, \underbrace{0}_{\in\mathbb{H}^*}) \in \mathrm{WF}(f)\}.$$
$$\mathrm{WF}(\pi_2^*f) \subset \{((x,v), \underbrace{d\pi^\top\xi}_{\in\mathbb{V}^*}, \underbrace{0}_{\in\mathbb{H}^*}) \mid (x,\xi) \in \mathrm{WF}(f)\} \subset \mathbb{V}^*.$$

Since the curvature of the manifold is negatively pinched, there are no conjugate points. It follows that $\varphi_t(\mathbb{V}^*) \cap \mathbb{V}^* \cap \{\langle \xi, X\rangle = 0\} = \{0\}$ for all $t \neq 0$. Recall from [GW17, Theorem 3] that

$$\mathrm{WF}'(R_0^-) \subset \Delta(T^*SM) \cup \{((z,\xi),\varphi_t(z,\xi)) \mid t \geq 0,\ \langle \xi, X\rangle = 0\} \cup E_*^u \times E_*^s.$$

Since averaging along the flow is smoothing in that direction, we deduce

$$\mathrm{WF}'\left[\int \chi'(t)\varphi_t^* R_0^-\right] \subset \{((z,\xi),\varphi_t(z,\xi)) \mid t \geq t_0,\ \langle \xi, X\rangle = 0\} \cup E_*^u \times E_*^s.$$

As a consequence,

$$\mathrm{WF}'(U) = \{0\}. \tag{6.3.4}$$

All the arguments that we have exposed, and indeed, [GW17, Theorem 3], are based on propagation of singularities. We will have to come back to these more precise estimates to conclude. For the sake of simplicity, we now write $H^{s,\rho} := H^{s,\rho,\rho}$ for spaces with the same weight on the zero and non-zero modes. Following Definition 4.3.2, what we need to prove are the following properties of admissibility :

1. U is bounded from $H^{-N,\rho}$ to $H^{N,\rho}$ for all $\rho \in]-d/2, d/2[, N \in \mathbb{N}$.
2. $[\partial_\theta, U]$ is bounded from $H^{-N,d/2-\epsilon}$ to $H^{N,-d/2+\epsilon}$ for all $\epsilon > 0, N \in \mathbb{N}$.
3. There is a smoothing convolution operator $I_Z(U)$ such that $\mathcal{P}_Z U \mathcal{E}_Z - I_Z(U)$ is bounded from $e^{r(d-\epsilon)}H^{-N}(dr)$ to $e^{r\epsilon}H^N(dr)$ for all $\epsilon > 0, N \in \mathbb{N}$.

Before going on with the proof, it is convenient to recall that the scale of spaces $H^{r\mathbf{m},\rho}(SM)$ was built as

$$H^{r\mathbf{m},\rho}(SM) := \mathrm{Op}(e^{\mathbf{m}\log\langle\xi\rangle})H^{0,\rho}(SM),$$

where $\mathbf{m}$ is an order 0 symbol. It was important to impose its value on E_*^u, and E_*^s. However, in its construction, one can always impose that it is arbitrarily large or small on $\mathbb{V}^*$. In particular for any $s \in \mathbb{R}$ and $\epsilon > 0$, we can choose $\mathbf{m}$ such that

$$\pi_2^*(H^{s,\rho}(M, \otimes_S^2 T^*M)) \subset H^{\mathbf{m},\rho}(SM), \text{ and } \pi_{2*}H^{\mathbf{m},\rho}(SM) \subset H^{s-\epsilon,\rho}(M, \otimes_S^2 T^*M).$$

Let us start with property (1). In the compact case, the proof relies on the propagation of singularities estimates from [DZ16]. In [GW17], it was proved that these estimates apply almost verbatim in the case of cusp manifolds, if one uses the relevant pseudo-differential calculus. In particular, the estimates that lead to (6.3.4), which are a priori *local*, are actually uniform in over the whole manifold. While we reproduce the proof below, the reader familiar with [GL19d] will see nothing new.

We work with h-semi-classical quantization. We consider the following microlocal decomposition :

$$\pi_{2*} = \pi_{2*}A_{reg} + \pi_{2*}A_{ell} + \pi_{2*}A_{prop} + \mathcal{O}_{H^{-N,\rho}\to H^{N,\rho}}(h^N),$$

with $A_{reg,ell,prop}$, $\mathbb{R}$-L^2 admissible operators of order 0, such that A_{reg} is microlocally supported around the zero section. A_{ell} is microsupported in the region of ellipticity of the flow. And finally, A_{prop} is microsupported in a small conical neighbourhood of $\{\langle\xi, X\rangle = 0\} \cap \{|\xi| > 1\} \cap \mathbb{V}^*$.

Since

$$-X \int \chi'(t)\varphi_t^* dt = \int \chi''(t)\varphi_t^* dt,$$

we can use a parametrix construction to find that

$$A_{ell} \int \chi'(t)\varphi_t^* dt = A_{ell}^N \int \chi^{(N+1)}\varphi_t^* dt + \mathcal{O}_{H^{-N,\rho}\to H^{N,\rho}}(h^N),$$

with A^N of order $-N$. We deduce that

$$\left\| \pi_{2*} A_{ell} \int \chi'(t)\varphi_t^* R_0^- \pi_2^* u dt \right\|_{H^{N,\rho}} \leq C\|u\|_{H^{-N,\rho}}.$$

(the constant may explode as $h \to 0$). Next, since $\varphi_t(\mathrm{WF}_h(A_{prop}))$ is eventually in a neighbourhood of E_*^u, and since $\varphi_{t*}\mathbb{V}^*$ is always transverse with $\mathbb{V}^*$, uniformly as $t \to +\infty$, we deduce from the propagation of singularities [GW17, Propositions A.21, A.23] that there is $C \in \Psi^0$ whose wavefront set does not encounter $\mathbb{V}^*$, and such that for $u \in H^{r\mathbf{m},\rho}(SM)$, and $t \geq t_0$,

$$\|A_{prop}\varphi_t^* u\|_{H^{r\mathbf{m},\rho}} \leq C_t h^{-1}\|CXu\|_{H^{r\mathbf{m},\rho}} + \mathcal{O}(h^N\|u\|_{H^{-N,\rho}}).$$

(the constants are locally uniform in t). As a consequence, we get that

$$\|\pi_{2*}A_{prop}\varphi_t^* R_0^- \pi_2^* u\|_{H^{N,\rho}} \leq C_t\|C\pi_2^* u\|_{H^{r\mathbf{m},\rho}} + \mathcal{O}(h^N\|u\|_{H^{-N,\rho}}) \leq \mathcal{O}(h^N\|u\|_{H^{-N,\rho}}).$$

Finally, for fixed h, A_{reg} is bounded from $H^{-N,\rho}$ to $H^{N,\rho}$ (with norm $\sim h^{-2N}$). We conclude that

$$\|Uu\|_{H^{N,\rho}} \leq C\|u\|_{H^{-N,\rho}},$$

by taking $h > 0$ small enough. In all the arguments above, the only limitation on ρ is that we require that R_0^- is bounded on $H^{r\mathbf{m},\rho}$, hence the restriction $\rho \in]-d/2, d/2[$.

Let us now turn to the item (2). Consider a cutoff χ_1 supported in the cusp, constant for large $y > 0$. Pick $u \in C_c^\infty(SM)$, with $\int u = 0$. Then

$$\begin{aligned}[\chi_1(y)\partial_\theta, R_0^-]u &= \chi_1(y)\partial_\theta R_0^- u - R_0^-\chi_1(y)\partial_\theta u, \\ &= R_0^-[X, \chi_1(y)\partial_\theta]R_0^- u, \\ &= R_0^-(y\cos\varphi\chi_1'(y)\partial_\theta)R_0^- u\end{aligned}$$

(and the commutator vanishes on constant functions). From there, since π_2^* commutes with ∂_θ, and since the flow φ_t commutes with ∂_θ for small times, and χ' is compactly supported, if χ_1 is only supported for $y > 0$ large enough, we get

$$\begin{aligned}[\chi_1(y)\partial_\theta, U] &= \int_{t_0}^{+\infty} \chi'(t)\pi_{2*}\varphi_t^*[\chi_1(y)\partial_\theta, R_0^-]\pi_2^* dt, \\ &= \int_{t_0}^{+\infty} \chi'(t)\pi_{2*}\varphi_t^* R_0^-(y\cos\varphi\chi_1'(y)\partial_\theta)R_0^-\pi_2^* dt\end{aligned}$$

The arguments from the point (1) apply, and, using the fact that χ_1' is compactly supported, we deduce that the commutator is bounded from $H^{-N,d/2-\epsilon}$ to $H^{N,-d/2+\epsilon}$ for all $N, \epsilon > 0$.

We now prove the third item (3). Denote by $\mathbf{R}_0^-$ the inverse of $I(X)$, acting on $\mathbb{R}\times\mathbb{S}^d$, or equivalently, on functions on the full cusp that do not depend on θ. Its existence [GW17, Theorem 2, Lemma 5.5] is the foundation of the proof of [GW17, Theorem 3]. It is a convolution operator bounded on the anistropic spaces $H^{r\mathbf{m},\rho}(\mathbb{R}\times\mathbb{S}^d, e^{-rd}drd\zeta)$, for $\rho\in]-d/2,d/2[$. Let us observe that

$$\pi_{2*}\left[\mathcal{P}_Z\chi_C\varphi_t^*R_0^-\chi_C\mathcal{E}_Z-\varphi_t^*\mathbf{R}_0^-\right]\pi_2^*= \\ \pi_{2*}\varphi_t^*\mathbf{R}_0^-\left[\mathcal{P}_Z(\varphi_t^*[X,\chi_C])R_0^-\chi_C\mathcal{E}_Z+\varphi_t^*(\chi_C)\chi_C-\mathbb{1}\right]\pi_2^*.$$

Then, we observe that

$$\left[\mathcal{P}_Z(\varphi_t^*[X,\chi_C])R_0^-\chi_C\mathcal{E}_Z+\varphi_t^*(\chi_C)\chi_C-\mathbb{1}\right]\pi_2^*$$

maps $e^{(d-\epsilon)r}H^{-N}(\mathbb{R})$ to $H^{r\mathbf{m},-d/2+\epsilon}(\mathbb{R}\times\mathbb{S}^d,e^{-rd}drd\zeta)$ for all $\epsilon>0$, mapping the wavefront set to $\cup_{t>0}\varphi_t\mathbb{V}^*$. Then, we can apply the arguments from point (1) directly to $\mathbf{R}_0^-$ to conclude. The indicial operator of U is thus found to be

$$\int_{t_0}^{+\infty}\chi'(t)\pi_{2*}\varphi_t^*\mathbf{R}_0^-\pi_2^*dt.$$

This in turn implies that the indicial operator of Π_2 is (as one would hope) the Π_2 operator associated to the full cusp, restricted to the zeroth Fourier mode in θ, i.e

$$I(\Pi_2)f=\pi_{2*}\int_{\mathbb{R}}(\pi_2^*f)\circ\varphi_t dt. \tag{6.3.5}$$

This finishes the proof of Lemma 6.3.2. □

Parametrix construction for range ellipticity. So far, we have found that Π_2 is a $]0,d[$-L^2 admissible pseudo-differential operator, and that it is elliptic on $\ker\sigma(D^*)$. However, we cannot directly apply the arguments of Chapter 4 because $\ker D^*$ is not a space of sections of a fixed bundle. We will see that this is not actually a problem.

By definition of range ellipticity, we have a symbol q_0 such that

$$\mathrm{Op}(q_0)\Pi_2=\pi_{\ker D^*}+\mathcal{O}(\Psi^{-1}).$$

However, $\Pi_2=\Pi_2\pi_{\ker D^*}$, so the principal symbol of the remainder can be written $r\sigma(\pi_{\ker D^*})+\mathcal{O}(S^{-2})$. Then, we can find q_1 so that $q_1\sigma(\Pi_2)=r\sigma(\pi_{\ker D^*})$, and improve the parametrix to $\mathcal{O}(\Psi^{-2})$. By induction, we obtain a formal solution $\bar{q}\sim q_0+q_1+\dots$, for which we can build a Borel sum $q\in S^1$, and we get

$$\pi_{\ker D^*}\mathrm{Op}(q)\Pi_2=\pi_{\ker D^*}(1+R)\pi_{\ker D^*},$$

where, $\mathrm{Op}(q)$ and R are $]0,d[$-L^2 admissible, of order $1,-\infty$ respectively. In the next section, we will prove the

Lemma 6.3.3. *The indicial operator of Π_2 does not have indicial roots in $]0,d[+i\mathbb{R}$. In particular, there is an indicial resolvent $S(\Pi_2)=S_{]0,d[}(\Pi_2)$ so that $S(\Pi_2)$ is bounded from $e^{(d/2+\rho)r}H^s(\mathbb{R})$ to $e^{(d/2+\rho)r}H^{s+1}(\mathbb{R})$ for $s\in\mathbb{R}$ and $\rho\in]-d/2,d/2[$, and*

$$S(\Pi_2)I(\Pi_2)=I(\pi_{\ker D^*}).$$

Now, we follow the arguments from Section §4.3.5. We replace $\pi_{\ker D^*}\operatorname{Op}(q)$ by

$$Q = \pi_{\ker D^*}\left[\operatorname{Op}(q) + \sum_{\ell} \chi \mathcal{E}_{Z_\ell}[S(\Pi_2) - I(\pi_{\ker D^*}\operatorname{Op}(q))]\mathcal{P}_{Z_\ell}\chi\right],$$

for some cutoff function χ equal to 1 in the cusps. This is an operator such that $Q\Pi_2 = \pi_{\ker D^*}(\mathbb{1} + R)\pi_{\ker D^*}$, with R mapping $H^{-N,d/2-\epsilon}$ to $H^{N,-d/2+\epsilon}$ for all $N, \epsilon > 0$. According to the discussion at the start of Section §6.3.2, this closes the proof of Theorem 6.3.1. □

Finding the roots. It remains to prove Lemma 6.3.3. First off, since $\pi_{\ker D^*} = \mathbb{1} - D\Delta^{-1}D^*$, with $\Delta = D^*D$, we get that

$$I(\pi_{\ker D^*}) = \pi_{\ker I(D^*)},$$

this being an orthogonal projection on $e^{rd/2}L^2(\mathbb{R}, dr)$. In particular, we only need to invert $I(\Pi_2)$ on the kernel of the indicial operator of D^*. On the other hand, if we look for $S(\Pi_2)$ in the form of a Fourier multiplier, we must have

$$S(\Pi_2, \lambda)I(\Pi_2, \lambda) = \pi_{\ker I(D^*,\lambda)}.$$

Thus, we will need that for $\Re\lambda \in]0, d[$, $I(\Pi_2, \lambda)$ (which is now just a matrix) is invertible on $\ker I(D^*, \lambda)$. Denoting the inverse $\tilde{S}(\Pi_2, \lambda)$, we will consider $\tilde{S}(\Pi_2)$, the convolution operator on $\mathbb{R}$ whose Fourier multiplier is $\tilde{S}(\Pi_2, \lambda)$, as in Section §4.3.3. There may appear to be a small difficulty in the fact that so far, we have only defined $\tilde{S}(\Pi_2, \lambda)$ on $\ker I(D^*, \lambda)$; We will complete this by requiring that is just 0 on $\ker(\pi_{\ker I(D^*,\lambda)})$. The operator defined in this way will satisfy suitable bounds because $\pi_{\ker D^*}$ is itself admissible.

After these preliminary discussion, it only remains to compute the indicial family of Π_2, and prove that it is invertible. Consider a symmetric 2-tensor

$$f = a\frac{dy^2}{y^2} + \sum_i \frac{b_i}{2}\left(\frac{dy}{y}\frac{d\theta_i}{y} + \frac{d\theta_i}{y}\frac{dy}{y}\right) + \sum_{i,j} c_{i,j}\frac{d\theta_i}{y}\frac{d\theta_j}{y},$$

where $a = a_\infty y^\lambda, b_i = b^i_\infty y^\lambda, c_{i,j} = c^{i,j}_\infty y^\lambda$, c being a symmetric matrix. Then :

$$D^*f = (a(\lambda - d) + \operatorname{Tr}(c))\frac{dy}{y} + \frac{1}{2}(\lambda - (d+1))\sum_i b_i\frac{d\theta_i}{dy}$$

If $\Re(\lambda) \in]0, d[$, we get that f is a solenoidal tensor if and only if $b_i \equiv 0$ for all $i \in \{1, ..., d\}$ and :

$$a_\infty(\lambda - d) + \operatorname{Tr}(c_\infty) = 0. \tag{6.3.6}$$

From now on, we assume that these conditions hold. We now compute $\Pi\pi_2^* f$.

Given $z = (y_0, \theta_0, \phi_0, u_0)$ a point in $]0, +\infty[\times\mathbb{T}^d\times]0, \pi[\times\mathbb{S}^{d-1}$, we write $\varphi_t(z) = (y_t, \theta_t, \phi_t, u_t)$ and we have :

$$\begin{aligned}
\Pi\pi_2^* f\left(a\frac{dy^2}{y^2}\right)(y_0, \theta_0, \phi_0, u_0) &= \Pi(a_\infty y^\lambda \cos^2\phi)(y_0, \phi_0) \\
&= a_\infty \int_{-\infty}^{+\infty} y_t^\lambda \cos^2(\phi_t)dt \\
&= a_\infty\left(\frac{y_0}{\sin\phi_0}\right)^\lambda \int_{-\infty}^{+\infty} \sin^\lambda(\phi_t)(1 - \sin^2(\phi_t))dt \\
&= a_\infty\left(\frac{y_0}{\sin\phi_0}\right)^\lambda (H(\lambda) - H(\lambda+2)),
\end{aligned}$$

where $H(\lambda) := \int_{-\infty}^{+\infty} \sin^\lambda(\phi_t)dt$. This is independent of $\phi_0 \neq 0 \quad (\pi)$ and one can check that :

$$H(\lambda) = \sqrt{\pi}\frac{\Gamma(\lambda/2)}{\Gamma((\lambda+1)/2)} \tag{6.3.7}$$

Thus $H(\lambda) - H(\lambda+2) = \frac{H(\lambda)}{\lambda+1}$ and we get :

$$\Pi\pi_2^*\left(a_\infty y^\lambda \frac{dy^2}{y^2}\right) = \left(\frac{y}{\sin(\phi)}\right)^\lambda \frac{H(\lambda)}{\lambda+1} \tag{6.3.8}$$

In the same fashion :

$$\Pi\pi_2^*\left(\sum_{i,j} y^\lambda c_\infty^{i,j}\frac{d\theta_i d\theta_j}{y^2}\right) = \lambda\left(\frac{y}{\sin(\phi)}\right)^\lambda \frac{H(\lambda)}{\lambda+1}\sum_{i,j} c_\infty^{i,j}u_i u_j \tag{6.3.9}$$

Since π_2^* and π_{2*} are formally adjoint operators on the d-dimensional sphere, it is sufficient to check that :

$$\langle y^{-\lambda}\Pi\pi_2^* f, y^{-\lambda}\pi_2^* f\rangle_{L^2(\mathbb{S}^d)} \neq 0$$

Now, this is equal to :

$$\begin{aligned}
&\langle y^{-\lambda}\Pi\pi_2^* f, y^{-\lambda}\pi_2^* f\rangle_{L^2(\mathbb{S}^d)} \\
&= \frac{H(\lambda)}{\lambda+1}\int_{\mathbb{S}^d}\Bigg(|a_\infty|^2\cos^2(\phi) + \sum_{kl} a\overline{c_\infty^{kl}}u_k u_l \sin^2(\phi) + \lambda\sum_{ij}\overline{a}c_\infty^{ij}u_i u_j\cos^2(\phi) + \\
&\qquad \lambda\sum_{ijkl} c_\infty^{ij}\overline{c_\infty^{kl}}u_i u_j u_k u_l \sin^2(\phi)\Bigg)\frac{d\mu_{\mathbb{S}^d}}{\sin^\lambda(\phi)},
\end{aligned}$$

where $d\mu_{\mathbb{S}^d} = \sin^{d-1}(\phi)d\phi d\mu_{\mathbb{S}^{d-1}}(u)$ is the usual measure on the sphere. After some (non-trivial) simplifications, and using the fact that $a_\infty(\lambda-d)+\mathrm{Tr}(c_\infty) = 0$, we obtain :

$$\begin{aligned}
&\frac{1}{\mathrm{vol}(\mathbb{S}^{d-1})}\langle y^{-\lambda}\Pi\pi_2^* f, y^{-\lambda}\pi_2^* f\rangle_{L^2(\mathbb{S}^d)} \\
&= \frac{H(\lambda)H(d-\lambda)}{(\lambda+1)(d+1-\lambda)}\Bigg[|a_\infty|^2\left(1+\frac{|d-\lambda|^2}{d}+\frac{\lambda(d-\lambda)}{d}+|d-\lambda|^2\frac{\lambda(d-\lambda)}{d(d+2)}\right) \\
&\qquad +2\,\mathrm{Tr}\,|c_\infty|^2\frac{\lambda(d-\lambda)}{d(d+2)}\Bigg] \\
&= \frac{\pi}{(\lambda+1)(d+1-\lambda)}\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{d+1-\lambda}{2}\right)} \\
&\quad \times\left[|a_\infty|^2\left(1+\frac{|d-\lambda|^2}{d}+\frac{\lambda(d-\lambda)}{d}+|d-\lambda|^2\frac{\lambda(d-\lambda)}{d(d+2)}\right)+2\,\mathrm{Tr}\,|c_\infty|^2\frac{\lambda(d-\lambda)}{d(d+2)}\right]
\end{aligned} \tag{6.3.10}$$

On the strip $\{0 < \Re(\lambda) < d\}$, the cross-ratio of Γ functions is holomorphic and does not vanish (in particular, it is a positive real number on the line $\lambda = d/2 + i\mathbb{R}$). The term between parenthesis can be written in the form $A(\lambda) + \lambda(d-\lambda)B(\lambda) = -B(\lambda)\lambda^2 + \lambda d B(\lambda) + A(\lambda)$, where $A(\lambda), B(\lambda) \geq 0$. The roots of this equation must then satisfify $\lambda = d/2 \pm \sqrt{d^2/4 + A(\lambda)/B(\lambda)}$ so they are outside the strip $\{0 < \Re(\lambda) < d\}$.

Remark 6.3.1. It also has an interest on its own to compute the indicial roots of the operator Π_0 to determine on which spaces it will be invertible. Considering a function on the whole cusp $f = a_\infty y^\lambda$ for $\lambda \in \mathbb{C}$ and carrying the same sort of computations as before, one finds out that :

$$\langle y^{-\lambda}\Pi\pi_0^*(a_\infty y^\lambda), y^{-\lambda}\pi_0^*(a_\infty y^\lambda)\rangle_{L^2(\mathbb{S}^d)} = |a_\infty|^2\pi\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{d-\lambda}{2}\right)}{\Gamma\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{d-\lambda}{2}+\frac{1}{2}\right)}$$

In particular, it has no roots for $0 < \Re(\lambda) < d$, like Π_2. This may be true for tensors of higher order $m \in \mathbb{N}$ but we did not do the general computation.

6.4 Perturbing a cusp metric

6.4.1 Perturbation of the lengths

The following lemma is a Taylor expansion on the lengths :

Lemma 6.4.1. *Let (M, g) be a cusp manifold. Then, there exists a unique closed g-geodesic in eachf ree hyperbolic homotopy class. Moreover, there exists $\epsilon := \epsilon(g) > 0$ such that if and $\|g' - g\|_{C^3} \leq \epsilon$, g' has Anosov geodesic flow and there is in any free homotopy class c of a given closed geodesic γ_g for g, exactly one closed geodesic $\gamma_{g'}$ for g'. We denote the length by $L_g(c)$. Additionally, we have*

$$\frac{L_{g'}(c)}{L_g(c)} - 1 = I_2^g(g' - g) + \mathcal{O}(\|g' - g\|_{C^3}^2),$$

where the remainder is uniform in $c \in \mathcal{C}$.

Proof. We refer to [GL19c, Lemma 4.1] for a proof of this result. □

6.4.2 Reduction to solenoidal perturbations

The operator Π_2 has good analytic properties – it is elliptic and invertible – on solenoidal tensors in the spaces $H^{s,\rho_0,\rho_\perp}$ and $y^{\rho_0+d/2}C_*^s$ for $s, \rho_\perp \in \mathbb{R}, \rho_0 \in]-d/2, d/2[$. As a consequence, for one to take advantage of this, we first need to make a *solenoidal reduction*, that is find a first diffeomorphism ϕ such that $D_g^*\phi^*g' = 0$, and then apply analytic arguments to $f := \phi^*g' - g$. It turns out that the usual argument of solenoidal reduction (see Lemma B.1.7) relying on the implicit function theorem involves the Laplacian $\Delta_g = D_g^* D_g$ and works when Δ_g is an isomorphism. In our case, following Lemma 5.2.4, Δ_g is an isomorphism on the spaces $H^{s,\rho_0-d/2,\rho_\perp}, y^{\rho_0}C_*^s$ for $s, \rho_\perp \in \mathbb{R}, \rho_0 \in]\lambda_d^-, \lambda_d^+[$ and it is no longer surjective when $\rho_0 < \lambda_d^-$. This is quite a problem as we will see in few a lines.

We consider for a cutoff function χ equal to 1 in the cusps and $\rho < -1, s \geq 0$ large enough. We introduce the finite-dimensional space

$$H := \mathrm{Span}(\chi y^{-1} d\theta_i/y, \chi y^{\lambda_-^d} dy/y).$$

Let us start with the following

Lemma 6.4.2. *The operator*

$$\Delta_g : y^\rho C_*^{s+1} \oplus H \to y^\rho C_*^{s-1}(M, T^*M),$$

is an isomorphism for all $\rho < -1, s \in \mathbb{R}$.

Proof. The proof mainly relies on Lemmas 4.3.11 and 5.2.4 . First of all, it is clear that $y^\rho C_*^{s+1} \oplus H \subset C_*^{s+1}$ and Δ_g is injective on this space by Lemma 5.2.4. As to the surjectivity, we know by Lemma 4.3.11, that there exists $S \in \Psi^{-2}$ which is $(-\infty, -1)$ admissible on both L^2 and L^∞ such that

$$\Delta^{-1} = S + \chi \mathcal{E}_Z(\Pi_{\lambda_d^-} + \Pi_{-1})(\mathcal{P}_Z \chi + G), \tag{6.4.1}$$

where G maps into $e^{\rho r} H^\infty$ for all $\rho \in \mathbb{R}$. Here, the matrices $\Pi_{\lambda_d^-}, \Pi_{-1}$ are completely explicit : they are obtained from the residues at λ_d^- and -1 of the matrices $I(\Delta, \lambda)^{-1}$ computed in the proof of Lemma 5.2.4 (they can be obtained by anti-clockwise integration of $I(\Delta, \lambda)^{-1}$ on small circles surrounding the indicial roots). More precisely, in the orthonormal basis $(dy/y, d\theta_i/y)$, one has

$$\mathrm{Res}_{\lambda_-^d}(I(\Delta, \lambda)^{-1}) = \begin{pmatrix} (\lambda_d^- - \lambda_d^+)^{-1} & 0 & \cdots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & \\ 0 & & & 0 \end{pmatrix} = (\lambda_d^- - \lambda_d^+)^{-1} \langle \cdot, dy/y \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the metric on 1-forms induced by the hyperbolic metric. As a consequence, considering the formal vector bundle $E \to \mathbb{R}$, where $E = \mathrm{Span}(dy/y, d\theta_i/y)$ and given a section $f \in C_c^\infty(\mathbb{R}, E)$, one obtains

$$\Pi_{\lambda_-^d} f = \langle f, e^{-\lambda_-^d r'} dy/y \rangle_{L^2(E \to \mathbb{R}, dr)} dy/y e^{\lambda_-^d r} = \left(\int_{\mathbb{R}} \langle f(r'), e^{-\lambda_-^d r'} dy/y \rangle dr' \right) e^{\lambda_-^d r} dy/y$$

Thus, in the usual coordinates (y, θ), one can write

$$\chi \mathcal{E}_Z \Pi_{\lambda_-^d} f = \langle f, y^{-\lambda_-^d + d} dy/y \rangle_{L^2(dy d\theta / y^{d+1})} \chi y^{\lambda_-^d} dy/y \tag{6.4.2}$$

In the same fashion, one has

$$\mathrm{Res}_{-1}(I(\Delta, \lambda)^{-1}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & -2/d & & \vdots \\ \vdots & & \ddots & \\ 0 & & & -2/d \end{pmatrix},$$

and

$$\chi \mathcal{E}_Z \Pi_{-1} f = \sum_i \langle f, y^{d+1} d\theta_i / y \rangle_{L^2(dy d\theta / y^{d+1})} \chi y^{-1} d\theta_i / y. \tag{6.4.3}$$

This concludes the proof. □

We now consider a metric g' in a $y^\rho C_*^s$ neighbourhood of our cusp metric g, with $\rho < -1, s \geq 2$. Using Lemma 6.4.2, we would like to find a diffeomorphism ϕ such that $D_g^* \phi^* g' = 0$. For that, it is very likely that one would have to look for ϕ in the form $\phi := T_u \circ e_V \circ K_s$, where $e_V(x) := \exp_x(V(x))$ with[1] $V \in y^\rho C_*^{s+1}(M, TM)$, $T_u(y, \theta) := (y, \theta + \chi u \cdot \partial_\theta)$ with $u \in \mathbb{R}^d$, K_s is the flow generated by the vector field $\chi y^{\lambda_-^d + 1} \partial_y$ and χ is

1. Observe that since manifolds with cusps have pinched negative curvature, their exponential maps are covering maps. In particular, for any vector field V, it makes sense to set $e_V(x) = \exp_x(V(x))$. If $s > 0$ and V is small enough in the space $C_*^s(M, TM)$, e_V is a local diffeomorphism. Additionally, it is homotopic to the identity and thus has topological degree 1, so it is a diffeomorphism.

some cutoff function equal to 1 in the cusps and 0 outside. Indeed, in order to apply the implicit function theorem, one would have to consider $(V, u, s, g') \mapsto F(V, u, s, g') := D_g^* K_s^* e_V^* T_u^* g'$ and differentiate with respect to the triple (V, u, s), then prove that the differential is an isomorphism. But in this case, using the musical isomorphism to identify vector fields and covectors, the differential is precisely

$$\Delta : y^\rho C_*^{s+1} \oplus H \to y^\rho C_*^{s-1}$$

by Lemma 6.4.2 and this is an isomorphism. However, the subtle problem comes from the fact that $F(V, u, s, g') \notin y^\rho C_*^{s-1}$. Indeed, the pullbacks e_V^* and T_u^* preserve this space, namely if $g' = g + f, f \in y^\rho C_*^s$, then $e_V^* T_u^* g' \in y^\rho C_*^s$ (and $D_g^* e_V^* T_u^* g' \in y^\rho C_*^{s-1}$), mainly because T_u^* preserves the metric g, but this is no longer the case of K_s^*. Indeed, for such a $g' = g + f$, one can prove that $K_s^* g'$ admits a polyhomogeneous development in terms of powers $y^{k\lambda_-^d}, k \in \mathbb{N}$ and there is no particular reasons for this development to vanish. Actually, the problem is even more crooked because one can change K_s and consider another 1-parameter family $\widetilde{K}_s$ of diffeomorphisms (not a group this time) such that $\frac{d}{ds}\widetilde{K}_s|_{s=0} = y^{\lambda_-^d+1}\partial_y$ and arrange the development of $\widetilde{K}_s$ so that $F(V, u, s, g') \in y^\rho C_*^{s-1}$ which would now allow to apply the implicit function theorem. However, the same problem would still show up in the end, that is the new metric $g'' := K_s^* e_V^* T_u^* g'$ would not be decreasing enough but would only have a polyhomogeneous development in terms of powers $y^{k\lambda_-^d}, k \in \mathbb{N}$. Since we do not know how to deal with this technical issue, we have to restrict ourselves to a codimension 1 submanifold of the space of isometry classes which prevents this polyhomogeneous development to appear.

Proposition 6.4.1. *There exists a rank 1 operator $P := A(\cdot)k$, where the tensor $k \in y^{\lambda_-^d} C^\infty(M, T^*M)$ and A is a $(-\infty, \lambda_-^d)$-admissible linear form both on L^2 and on L^∞ such that the following holds. For all $s \geq 2, \rho < -1$, there exists a small $y^\rho C_*^s$-neighbourhood of g, such that for any metric g' in this neighbourhood, there exists a (unique) diffeomorphism $\phi := T_u \circ e_V$, where $V \in y^\rho C_*^{s+1}(M, TM)$, $u \in \mathbb{R}^d$ such that*

$$\phi^* g' - g \in y^\rho C_*^s \cap \ker(\mathbb{1} - P)D_g^*.$$

We call this gauge the almost solenoidal gauge.

Proof. The operator Δ_g acting on $y^{\rho'} C_*^s(M, T^*M)$ for $\rho' > \lambda_+^d, s \in \mathbb{R}$ is no longer injective (but it is still surjective). In particular, using Lemma 4.3.10, for $\lambda_+^d < \rho' < d+1$, the kernel of Δ_g on $y^{\rho'} C_*^s(M, T^*M)$ is one-dimensional, given by Span(k') for some $k' \in y^{\lambda_+^d} C^\infty(M, T^*M)$. Using Lemma 6.4.2, we write for $\rho < -1$:

$$\Delta(\mathrm{Span}(\chi y^{\lambda_-^d} dy/y)) \oplus \underbrace{\Delta(\mathrm{Span}(\chi y^{-1} d\theta_i/y) \oplus y^\rho C_*^{s+1})}_{=E} = y^\rho C_*^{s-1}$$

Given $f = \Delta_g f_0 + \Delta_g e \in y^\rho C_*^{s-1}$ with $f_0 = c\chi y^{\lambda_-^d} dy/y, c \in \mathbb{R}, e \in E$, one has :

$$\langle f, k' \rangle_{L^2} = \langle \Delta_g f_0 + \Delta_g e, k' \rangle_{L^2} = \langle \Delta_g f_0, k' \rangle_{L^2},$$

since $\langle \Delta_g e, k' \rangle_{L^2} = \langle e, \Delta_g k' \rangle_{L^2} = 0$ by duality. Since k' is non-trivial, $\langle \Delta_g \cdot, k' \rangle_{L^2}$ induces a non-trivial linear form on all the spaces $H^{s,\rho-d/2,\rho_\perp}, s \in \mathbb{R}, \rho < -1, \rho_\perp \in \mathbb{R}$ and in particular, there exists a tensor $f_0 = c\chi y^{\lambda_-^d} dy/y$ such that $\langle \Delta_g f_0, k' \rangle_{L^2} = 1$. We write $k := \Delta_g f_0 \in y^{-\infty} C^\infty$ and define $P := \langle k', \cdot \rangle_{L^2} k$ and one has $P^* = \langle k, \cdot \rangle_{L^2} k'$.

It satisfies the relation $\Delta_g P^* = 0$ on all the spaces $y^\rho C_*^s(M, T^*M)$ for $s \in \mathbb{R}, \rho > \lambda_+^d$. By duality it also satisfies $P\Delta_g = 0$ on the spaces with $\rho < \lambda_-^d, s \in \mathbb{R}$ and thus

$$(\mathbb{1} - P)\Delta_g : y^\rho C_*^{s+1} \oplus \mathrm{Span}(\chi y^{-1} d\theta_i / y) \xrightarrow{\sim} \Delta_g(y^\rho C_*^{s+1} \oplus \mathrm{Span}(\chi y^{-1} d\theta_i / y)) \quad (6.4.4)$$

is an isomorphism. In the formalism developed in Chapter 4, the operator $P \in \Psi^{-\infty}$ is a $(-\infty, \lambda_-^d)$-admissible operator both in L^2 and in L^∞ whereas $P^* \in \Psi^{-\infty}$ is (λ_+^d, ∞)-admissible.

We now consider for $\rho < -1$ the map

$$F : y^\rho C_*^{s+1}(M, TM) \times \mathbb{R}^d \times y^\rho C_*^s(M, \otimes_S^2 T^*M) \to (\mathbb{1} - P)\left(y^\rho C_*^{s-1}(M, T^*M)\right),$$

defined by $F(V, u, g') := (\mathbb{1} - P)D_g^*(e_V^* T_u^* g')$, for g' in a neighbourhood of g and V, u in a neighbourhood of 0. This is a C^1 map in both variables. Observe that

$$d_{(V,u)} T_{(0,0,g)}(W, v) = \Delta_g \left(W^\flat + \sum_i v_i \chi y^{-1} d\theta_i / y \right),$$

where $\flat$ denotes the musical isomorphism, $W \in y^\rho C_*^{s+1}(M, TM), v \in \mathbb{R}^d$, and this is an isomorphism by Lemma 6.4.2. One then concludes by the implicit function theorem. □

Remark 6.4.1. Observe that since T is C^1, the implicit function theorem also tells us that the map $g' \mapsto e_{V(g')} =: \phi(g')$ is C^1 and we thus have an estimate $\|\phi^* g' - g\|_{y^\rho C_*^s} \lesssim \|g' - g\|_{y^\rho C_*^s}$.

As a consequence, another way of formulating the previous lemma is to say that isometry classes of fast decaying metrics in a neighbourhood of g can be represented by (are in one-to-one correspondance) with almost solenoidal tensors (with respect to g). We are now going to restrict to a 1-codimensional submanifold of the space of isometry classes so that, after almost solenoidal reduction, the new metric one obtains is not only almost solenoidal but *genuinely solenoidal.* This is the content of the following

Lemma 6.4.3. *There exists a linear form $A : y^\rho C_*^s(M, \otimes_S^2 T^*M) \to \mathbb{R}$, defined for all $\rho < \lambda_-^d, s \in \mathbb{R}$ such that*

$$\ker(\mathbb{1} - P)D_g^* \cap \ker A \cap y^\rho C_*^s(M, \otimes_S^2 T^*M) = \ker D_g^* \cap y^\rho C_*^s(M, \otimes_S^2 T^*M).$$

Proof. The inclusion from the right to the left being trivial, it remains to prove the other one. For $\rho < \lambda_-^d$, assume $f \in y^\rho C_*^s(M, \otimes_S^2 T^*M) \hookrightarrow C_*^0(M, \otimes_S^2 T^*M)$ and $(\mathbb{1} - P)D^* f = 0$. Then $f = D_g p + h$ where $p \in C_*^{s+1}(M, T^*M), h \in C_*^s(M, \otimes_S^2 T^*M)$ and $D_g^* h = 0$ by standard solenoidal decomposition. Thus $D_g^* f = D_g^* D_g p$ and by (6.4.1), we get

$$p = SD_g^* f + \widetilde{A}(D_g^* f)\chi y^{\lambda_-^d} dy/y + \sum_i \widetilde{B_i}(D_g^* f)\chi y^{-1} d\theta_i / y,$$

where the linear forms $\widetilde{A}, \widetilde{B_i}$ are given respectively by (6.4.2) and (6.4.3). We set $A := \widetilde{A} D_g^*$. Assuming $f \in \cap \ker A$, one obtains $p \in y^\rho C_*^{s+1}(M, T^*M) \oplus \mathrm{Span}(\chi y^{-1} d\theta_i / y)$. But $(\mathbb{1} - P)D_g^* f = 0 = (\mathbb{1} - P)\Delta_g p$ and using (6.4.4), we get $p = 0$, that is $f = h \in \ker D_g^*$. □

As a consequence, the 1-codimensional submanifold of isometry classes on which we are going to prove the theorem is a neighbourhood of g intersected with

$$\mathcal{N}_{\mathrm{iso}} := \ker(\mathbb{1} - P)D_g^* \cap \ker A \cap y^{-N} C_*^N(M, \otimes_S^2 T^*M),$$

or, equivalently, Theorem 6.1.2 will hold in a neighbourhood around g on the submanifold

$$\mathcal{N}_{\mathrm{met}} := \left\{ \phi^* f \mid f \in \mathcal{N}_{\mathrm{iso}}, \phi = T_u \circ e_V, V \in y^{-N} C_*^{N+1}(M, TM), u \in \mathbb{R}^d \right\}. \tag{6.4.5}$$

Eventually, let us observe as a last remark that the normal operator Π_2 is still injective on the almost solenoidal gauge. Thus, it is very likely that one could carry out the interpolation argument of the following paragraph in this gauge. However, since P is only $(-\infty, \lambda_-^d)$ admissible and Π_2 is $(0, d)$ admissible (and these two intervals do not overlap !), one cannot obtain a parametrix such that $Q\Pi_2 = \pi_{\ker(\mathbb{1}-P)D^*} +$compact. For that, we would have to prove that Π_2 is actually also admissible on a slightly larger interval.

6.5 Proofs of the main Theorems

Since Theorem 6.1.1 follows directly from Theorem 6.1.2, we focus on the latter. We are given g a cusp metric, and g' another metric, such that $\|g - g'\|_{y^{-N}C^N} < \epsilon$, with $N \in \mathbb{N}$ large enough, and $\epsilon > 0$ small enough (chosen at the end). If we assume that ϵ is small enough and $g' \in \mathcal{N}_{\mathrm{met}}$, we can apply Proposition 6.4.1 and obtain a diffeomorphism ϕ such that $g'' := \phi^* g'$ is almost solenoidal, and ϕ is ϵ-close to the identity (in the topology given by Proposition 6.4.1). By construction, since $g' \in \mathcal{N}_{\mathrm{met}}$, g'' is actually genuinely solenoidal, i.e. $D_g^* g'' = 0$ by Lemma 6.4.3. We now apply a similar interpolation argument to Chapter 3. For the sake of simplicity, we now denote by $H^{s,\rho}$ the Sobolev spaces $H^{s,\rho,\rho}$, meaning that the y-weight in the zero and non-zero Fourier modes is the same. We first estimate the norm of $g'' - g$ and for that we can apply the stability estimate Theorem 6.3.2. We fix $s > 0$ arbitrarily small, then there exists $\gamma > 0$ such that,

$$\|g'' - g\|_{H^{-1-s,0}} \lesssim \|g'' - g\|_{C^1}^{1-\gamma} \|I_2^g(g'' - g)\|_{\ell^\infty}^{\gamma}.$$

From Lemma 6.4.1, we deduce that

$$\|I_2^g(g'' - g)\|_{\ell^\infty} \lesssim \|g'' - g\|_{C^3}^2 + \|L_{g'}/L_g - 1\|_{\ell^\infty}.$$

In particular, we get

$$\|g'' - g\|_{H^{-1-s,0}} \lesssim \|g'' - g\|_{C^3}^{1+\gamma} + \|g'' - g\|_{C^1}^{1-\gamma} \|L_{g'}/L_g - 1\|_{\ell^\infty}^{\gamma}.$$

Then, we use the Sobolev embedding Lemma 4.4.8 : for $r := (d+1)/2 + 3 + s$,

$$\|g'' - g\|_{C^3} \lesssim \|g'' - g\|_{H^{r,-d/2}}.$$

Next, we see $H^{r,-d/2}$ as the $\gamma/(1+\gamma)$ complex interpolation of $H^{-1-s,0}$ and $H^{N_1,-N_2}$, so that

$$\|g'' - g\|_{H^{r,-d/2}} \lesssim \|g'' - g\|_{H^{-1-s,0}}^{1/(1+\gamma)} \|g' - g_0\|_{H^{N_1,-N_2}}^{\gamma/(1+\gamma)},$$

where

$$N_1 := (1 + 1/\gamma)(3 + (d+1)/2 + s + 1 + s) - 1 - s, N_2 := (1 + 1/\gamma)d/2.$$

We deduce that

$$\|g'' - g\|_{H^{-1-s,0}} \lesssim \|g'' - g\|_{H^{-1-s,0}} \|g'' - g\|_{H^{N_1,-N_2}}^{\gamma} + \|g'' - g\|_{C^1}^{1-\gamma} \|L_{g'}/L_g - 1\|_{\ell^\infty}^{\gamma}$$

and taking $\|g'' - g\|^{\gamma}_{H^{N_1,-N_2}} \lesssim \|g' - g\|^{\gamma}_{H^{N_1,-N_2}} < \epsilon$ small enough (the first inequality follows from Remark 6.4.1), the first term on the right-hand side can get swallowed in the left-hand side, which yields :

$$\|g'' - g\|_{H^{-1-s,0}} \lesssim \|g'' - g\|^{1-\gamma}_{C^1} \|L_{g'}/L_g - 1\|^{\gamma}_{\ell^\infty}$$

Taking $N > \max(N_1, N_2 - d/2)$ and using the injection $y^{-N}C^N \hookrightarrow C^1$, we obtain the sought result.

□

Troisième partie

Boundary rigidity of non simple manifolds

Chapitre 7

X-ray transform on simple manifolds with topology

« *Et quand je n'aurais pas ce talent dont votre sourire me prouve que vous doutez, ne me resterait-il pas encore ce furieux amour de l'indépendance, qui me tiendra toujours lieu de tous les trésors (...) ?* »

Le comte de Monte-Christo,
Alexandre Dumas

This chapter is a compilation of the two articles :

- *Local marked boundary rigidity under hyperbolic trapping assumptions*, published in **Journal of Geometric Analysis**,
- *On the s-injectivity of the X-ray transform for manifolds with hyperbolic trapped set,* published in **Nonlinearity**.

Sommaire

Following the work initiated by Guillarmou [Gui17b], the present chapter studies the X-ray transform on a smooth compact connected Riemannian manifold with strictly convex boundary, no conjugate points and a non-empty trapped set K which is hyperbolic (see §7.2 for a definition), which we call simple manifold with topology. For smooth compact connected simple manifolds with topology, we prove an equivalence principle concerning the injectivity of the X-ray transform I_m on symmetric solenoidal tensors and the surjectivity of a certain operator π_{m*} on the set of solenoidal tensors. This allows us to establish the injectivity of the X-ray transform on solenoidal tensors of any order in the case of a simple surface with topology. Then, under the assumption that the X-ray transform over symmetric solenoidal 2-tensors is injective, we prove that simple manifolds with topology are *locally marked boundary rigid*. As a consequence, we obtain that simple surfaces with topology are locally marked boundary rigid, thus retrieving a recent result of Guillarmou-Mazzucchelli [GM18].

7.1 Introduction

7.1.1 Preliminaries

Geometric setting. Let us consider (M, g), a compact connected Riemannian manifold with strictly conve boundary and no conjugate points. Like in the compact setting, we denote by SM its unit tangent bundle, that is

$$SM = \{(x, v) \in TM, |v|_x = 1\},$$

and by $\pi_0 : SM \to M$, the canonical projection. The Liouville measure on SM will be denoted by $d\mu$. The incoming (-) and outcoming (+) boundaries of the unit tangent bundle of M are defined by

$$\partial_\pm SM = \{(x, v) \in TM, x \in \partial M, |v|_x = 1, \mp g_x(v, \nu) \leq 0\},$$

where ν is the outward pointing unit normal vector field to ∂M. Note in particular that $S(\partial M) = \partial_+ SM \cap \partial_- SM$, which we will denote by $\partial_0 SM$ in the following. If $i : \partial SM \to SM$ is the embedding of ∂SM into SM, we define the measure $d\mu_\nu$ on the boundary ∂SM by

$$d\mu_\nu(x, v) := |g_x(v, \nu) i^* d\mu(x, v)| \tag{7.1.1}$$

φ_t denotes the (incomplete) geodesic flow on SM and X the vector field induced on $T(SM)$ by φ_t. Given each point $(x, v) \in SM$, we define the escape time in positive (+) and negative (-) times by :

$$\begin{aligned} \ell_+(x, v) &:= \sup \{t \geq 0, \varphi_t(x, v) \in SM\} \in [0, +\infty] \\ \ell_-(x, v) &:= \inf \{t \leq 0, \varphi_t(x, v) \in SM\} \in [-\infty, 0] \end{aligned} \tag{7.1.2}$$

We say that a point (x, v) is *trapped in the future* (resp. *in the past*) if $\ell_+(x, v) = +\infty$ (resp. $\ell_-(x, v) = -\infty$). The incoming (-) and outcoming (+) tails in SM are defined by :

$$\Gamma_\mp := \{(x, v) \in SM, \ell_\pm(x, v) = \pm\infty\}$$

They consist of the sets of points which are respectively trapped in the future or the past. The trapped set K for the geodesic flow on SM is defined by :

$$K := \Gamma_+ \cap \Gamma_- = \cap_{t \in \mathbb{R}} \varphi_t(SM) \tag{7.1.3}$$

These sets are closed in SM and invariant by the geodesic flow. A manifold is said to be *non-trapping* if $K = \emptyset$. The aim of the present chapter is precisely to study the case $K \neq \emptyset$, which we will assume to hold from now on.

It is convenient to embed the manifold M into a strictly larger manifold M_e, such that M_e satisfies the same properties : it is smooth, has strictly convex boundary and no conjugate points (see [Gui17b, Section 2.1 and Section 2.3]). This can be done so that the longest connected geodesic ray in $SM_e \setminus SM^\circ$ has its length bounded by some constant $L < +\infty$. Moreover, for some technical reasons which will appear later, the extended metric is chosen without non-trivial Killing tensor fields (see the following paragraph for a definition), which is a generic condition (see [PZ16, Proposition 3.2]). The trapped set of M_e is the same as the trapped set of M and the sets $\Gamma_\pm$ are naturally extended to SM_e. In the following, for $t \in \mathbb{R}$, φ_t will actually denote the extension of $\varphi_t|_{SM}$ to SM_e.

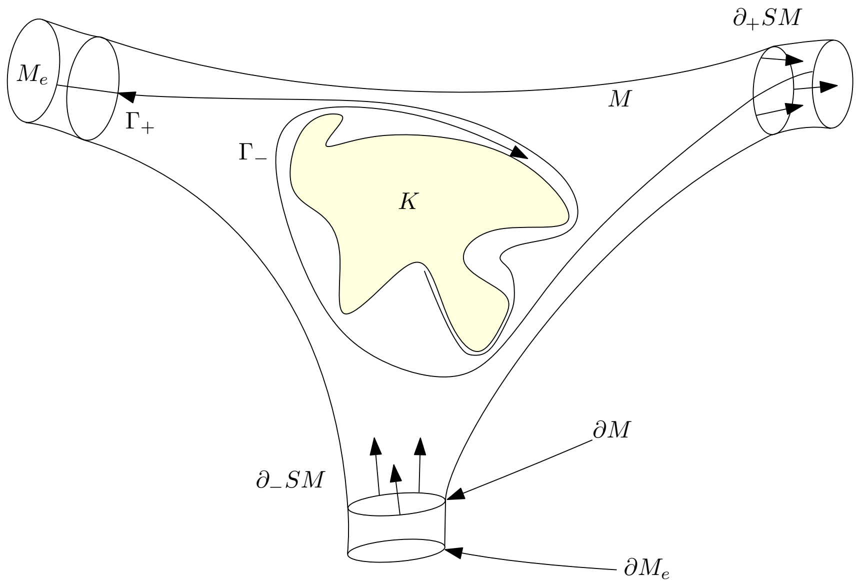

FIGURE 7.1 – The manifold M embedded in M_e

From now, we assume that the trapped set K of the manifold (M, g) is hyperbolic, that is there exists some constants $C > 0$ and $\nu > 0$ such that for all $z \in K$, there is a continuous flow-invariant splitting

$$T_z(SM) = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{7.1.4}$$

where $E_s(z)$ (resp. $E_u(z)$) is the *stable* (resp. *unstable*) vector space in z, which satisfy

$$\begin{aligned} |d\varphi_t(z) \cdot \xi|_{\varphi_t(z)} \leq Ce^{-\nu t}|\xi|_z, \quad \forall t > 0, \xi \in E_s(z) \\ |d\varphi_t(z) \cdot \xi|_{\varphi_t(z)} \leq Ce^{-\nu |t|}|\xi|_z, \quad \forall t < 0, \xi \in E_u(z) \end{aligned} \tag{7.1.5}$$

The norm, here, is given in terms of the Sasaki metric. We have the usual definitions of *stable* and *unstable manifolds*.

Definition 7.1.1. For each $z \in K$, we define the *global stable* and *unstable manifolds* $W_s(z), W_u(z)$ by :

$$\begin{aligned} W_s(z) = \{z' \in SM_e^\circ, d(\varphi_t(z), \varphi_t(z')) \to_{t \to +\infty} 0\} \\ W_u(z) = \{z' \in SM_e^\circ, d(\varphi_t(z), \varphi_t(z')) \to_{t \to +\infty} 0\} \end{aligned}$$

For $\varepsilon > 0$ small enough, we define the *local stable* and *unstable manifolds* $W_s^\varepsilon(z) \subset W_s(z), W_u^\varepsilon(z) \subset W_u(z)$ by :

$$W_s^\varepsilon(z) = \{z' \in W_s(z), \forall t \geq 0, d(\varphi_t(z), \varphi_t(z')) \leq \varepsilon\}$$
$$W_u^\varepsilon(z) = \{z' \in W_u(z), \forall t \geq 0, d(\varphi_{-t}(z), \varphi_{-t}(z')) \leq \varepsilon\}$$

Eventually, we define :

$$W_s(K) = \cup_{z\in K} W_s(z), \quad W_u(K) = \cup_{z\in K} W_u(z)$$

Let us now mention some properties of these sets, and relate them to the tails $\Gamma_\pm$. First, we have :

$$T_z W_s^\varepsilon(z) = E_s(z), \quad T_z W_u^\varepsilon(z) = E_u(z)$$

Since the trapped set K is hyperbolic, we also have (see [Gui17b, Lemma 2.2]) the equalities :

$$\Gamma_- = W_s(K), \quad \Gamma_+ = W_u(K)$$

Given $z_0 \in K$, the stable (resp. unstable) space of the decomposition (8.2.11) can be extended to points $z \in W_s^\varepsilon(z_0)$ (resp. $W_u^\varepsilon(z_0)$) by $E_-(z) := T_z W_s^\varepsilon(z_0)$ (resp. $E_+(z) := T_z W_u^\varepsilon(z_0)$). In particular, note that $E_-(z) = E_s(z), E_+(z) = E_u(z)$ for $z \in K$. These subbundles can once again be extended by propagating them by the flow to subbundles $E_\pm \subset T_{\Gamma_\pm} SM_e$ over $\Gamma_\pm$. Let $T_K^* SM$ denote the restriction of the cotangent bundle of SM to K. The flow-invariant splitting (8.2.11) of the tangent space between stable, unstable and flow directions admits a dual splitting which is also invariant by the flow and defined as $T_z^*(SM) = E_0^*(z) \oplus E_s^*(z) \oplus E_u^*(z)$, for $z \in K$, with :

$$E_u^*(E_u \oplus \mathbb{R}X) = 0, \quad E_s^*(E_s \oplus \mathbb{R}X) = 0, \quad E_0^*(E_u \oplus E_s) = 0 \tag{7.1.6}$$

Now, this splitting naturally extends to the tails $\Gamma_\pm$ by defining the flow-invariant subbundles $E_\pm^* \subset T_{\Gamma_\pm}^* SM_e$ by :

$$E_\pm^*(E_\pm \oplus \mathbb{R}X) = 0, \tag{7.1.7}$$

over $\Gamma_\pm$. In particular, $E_-^*(z) = E_s^*(z), E_+^*(z) = E_u^*(z)$ for $z \in K$. These sets can be seen as conormal bundles to $\Gamma_\pm$. They will be used in order to describe the wavefront set of the operator Π (see §7.2.1).

X-ray transform. We can now define the X-ray transform :

Definition 7.1.2. The X-ray transform is the map $I : C_c^\infty(SM \setminus \Gamma_-) \to C_c^\infty(\partial_- SM \setminus \Gamma_-)$ defined by :

$$If(x, v) := \int_0^{+\infty} f(\varphi_t(x, v))dt$$

Note that since f has compact support in the open set $SM \setminus \Gamma_-$, we know that the exit time of any $(x, v) \in SM \setminus \Gamma_-$ is uniformly bounded, so the integral is actually computed over a compact set. We introduce the non-escaping mass function :

Definition 7.1.3. Let $\mathcal{T}_+(t) = \{z \in SM, \varphi_s(z) \in SM, \forall s \in [0, t]\ \}$. We define the non-escaping mass function V by :

$$\forall t \geq 0, \qquad V(t) = \mu(\mathcal{T}_+(t)) \tag{7.1.8}$$

We define the *escape rate* $Q \leq 0$ which measures the exponential rate of decay of the non-escaping mass function V :

$$Q = \limsup_{t \to +\infty} t^{-1} \log V(t) \tag{7.1.9}$$

In particular, it is possible to prove that if K is hyperbolic, the following properties hold (see [Gui17b, Proposition 2.4]) :

Proposition 7.1.1. *1.* $\mu(\Gamma_- \cup \Gamma_+) = 0$,

2. $\tilde{\mu}(\Gamma_\pm \cap \partial_\pm SM) = 0$, *where* $\tilde{\mu}$ *is the measure on* ∂SM *induced by the Sasaki metric,*

3. $Q < 0$

Note that usually, K has Hausdorff dimension $\dim_H(K) \in [1, 2(n+1)-1)$, where $n+1 = \dim(M)$. An immediate consequence of the previous Proposition is that there exists a constant $\delta > 0$ such that $V(t) = O(e^{-\delta t})$. It is interesting to extend the X-ray transform to larger sets of function like $L^p(SM)$ spaces for some $p \geq 1$. This will be done more precisely in §7.2.1 but let us mention, as for the introduction, the

Proposition 7.1.2. *1. If* $\mu(K) = 0$ *(and no other assumptions are made on* K*), then* $I : L^1(SM) \to L^1(\partial_- SM, d\mu_\nu)$ *is bounded.*

2. If there exists a $p \in (2, +\infty]$*, such that*

$$\int_1^{+\infty} t^{\frac{p}{p-2}} V(t) dt < \infty, \tag{7.1.10}$$

then $I : L^p(SM) \to L^2(\partial_- SM, d\mu_\nu)$ *is bounded.*

Note that both conditions are satisfied if K is hyperbolic (this stems from Proposition 7.1.1). The proof of the first item is very standard and relies on Santaló's formula [San52] :

Lemma 7.1.1. *If* $\mu(K) = 0$ *and* $f \in L^1(SM)$*, then :*

$$\int_{SM} f d\mu = \int_{\partial_- SM} \int_0^{\ell_+(x,v)} f(\varphi_t(x,v)) dt d\mu_\nu(x,v)$$

The second item in Proposition 7.1.2 is established in [Gui17b, Lemma 5.1], using Cavalieri's principle. From this, we can define a formal adjoint $I^* : C_c^\infty(\partial_- SM^\circ \setminus \Gamma_-) \to C^\infty(SM \setminus \Gamma_-)$ to the X-ray transform by the formula

$$I^* u(x,v) = u(\varphi_{\ell_-(x,v)}(x,v)), \tag{7.1.11}$$

for the L^2 inner scalar products induced by the Liouville measure $d\mu$ on SM and by the measure $d\mu_\nu$ on $\partial_-(SM)$, that is $\langle If, u \rangle_{L^2(\partial_- SM, d\mu_\nu)} = \langle f, I^* u \rangle_{L^2(SM, d\mu)}$, for $f \in C_c^\infty(SM \setminus \Gamma_-), u \in C_c^\infty(\partial_- SM \setminus \Gamma_-)$. By the previous Proposition, it naturally extends to a bounded operator $I^* : L^2(\partial_- SM, d\mu_\nu) \to L^{p'}(SM)$, where p' is the conjugate exponent to p (it satisfies the equality $1/p + 1/p' = 1$).

X-ray transform on tensors. Juste like in the closed setting (see Chapter 2), from this definition of the X-ray transform on functions on SM, we can derive the definition of the X-ray transform for symmetric m-tensors.

Definition 7.1.4. Let $p > 2$ and p' denote its dual exponent such that $1/p + 1/p' = 1$. The X-ray transform for symmetric m-tensors is defined by

$$I_m := I \circ \pi_m^* : L^p(M, \otimes_S^m T^*M) \to L^2(\partial_- SM, d\mu_\nu) \qquad (7.1.12)$$

It is a bounded operator, as well as its adjoint

$$I_m^* = \pi_{m*} \circ I^* : L^2(\partial_- SM, d\mu_\nu) \to L^{p'}(M, \otimes_S^m T^*M) \qquad (7.1.13)$$

Here, the L^p-space, for $p \geq 1$, (resp. Sobolev space for $s \geq 0$) of symmetric m-tensors thus consists of tensors whose coordinate functions are all in $L^p(M)$ (resp. $H^s(M)$). An equivalent way to define $H^s(M, \otimes_S^m T^*M)$ (which will be used in Section 7.2.2) is to consider tensors u such that $(\mathbb{1} - \Delta)^{s/2} u \in L^2(M, \otimes_S^m T^*M)$, where $\Delta = D^*D$ is the Dirichlet Laplacian [1] on M. It is easy to check that $\pi_m^* : L^p(M, \otimes_S^m T^*M) \to L^p(SM)$ is bounded (resp. $\pi_m^* : H^s(M, \otimes_S^m T^*M) \to H^s(SM)$).

The derivative D and divergence D^* of symmetric tensors are defined like in the compact setting (see Appendix B). A *Killing tensor field* $v \in C^\infty(M, \otimes_S^m T^*M)$ is such that $Dv = 0$. The *trivial* Killing tensor fields are the ones obtained for m even by $c \cdot \sigma(\otimes^{m/2} g)$ for some constant c. Like in the compact setting, if $f \in H^s(M, \otimes_S^m T^*M)$ for some $s \geq 0$, there exists a unique decomposition of the tensor f such that

$$f = f^s + Dp, \qquad D^* f^s = 0, \quad p|_{\partial M} = 0, \qquad (7.1.14)$$

where $f^s \in H^s(M, \otimes_S^m T^*M), p \in H^{s+1}(M, \otimes_S^{m-1} T^*M)$ (see [Sha94, Theorem 3.3.2] for a proof of this result). f^s is called the *solenoidal part* of the tensor whereas Dp is called the *potential part*. Moreover, this decomposition holds in the smooth class and extends to any distribution $f \in H^{-s}(M, \otimes_S^m T^*M)$, $s \geq 0$, as long as it has compact support within M° (see the arguments given in the proof of Lemma 7.2.3 for instance). We will say that I_m is injective over solenoidal tensors, or in short *s-injective*, if it is injective when restricted to

$$C^\infty_{\text{sol}}(M, \otimes_S^m T^*M) := C^\infty(M, \otimes_S^m T^*M) \cap \ker D^*$$

This definition stems from the fact that given $p \in C^\infty(M, \otimes_S^{m-1} T^*M)$ such that $p|_{\partial M} = 0$, one always has $I_m(Dp) = 0$, using the formula $X\pi_m^* = \pi_{m+1}^* D$. Thus it is morally impossible to recover the potential part of a tensor f in the kernel of I_m.

Remark 7.1.1. All these definitions also apply to M_e, the extension of M. In the following, an index e on an application will mean that it is considered on the manifold M_e. The lower indices *inv*, *comp*, *sol* attached to a set of functions or distributions will respectively mean that we consider *invariant* functions (or distributions) with respect to the geodesic flow, *compactly supported* functions (or distributions) within a prescribed open set, *solenoidal* tensors (or tensorial distributions).

1. This is an elliptic differential operator with zero kernel and cokernel satisfying the Lopatinskii's transmission condition (see [Sha94, Theorem 3.3.2]). It can thus be used in order to define the scale of Sobolev spaces.

Normal operator. Eventually, we define the normal operator $\Pi_m := I_m^* I_m = \pi_{m*}\Pi\pi_m^*$, for $m \geq 0$. The following result asserts that Π_m is a pseudodifferential operator of order -1 (this mainly follows from the absence of conjugate points), which is elliptic on $\ker D^*$. The proof is the same as the one given in Chapter 2. It will be at the core of our arguments in §7.5.1.

Proposition 7.1.3 ([Gui17b], Proposition 5.9)**.** *Under the assumption that (M, g) has no conjugate points and a hyperbolic trapped set, Π_m is a pseudodifferential operator of order -1 on the bundle $\otimes_S^m T^*M^\circ$ which is elliptic on $\ker D^*$ in the sense that there exists pseudodifferential operators Q, S, R of respective order $1, -2, -\infty$ on M° such that :*

$$Q\Pi_m = \mathbb{1}_{M^\circ} + DSD^* + R$$

We will sometimes use this Proposition by adding appropriate cutoff functions : it is actually the way it is stated in [Gui17b].

7.1.2 S-injectivity of the X-ray transform

We now consider simple manifolds with topology. In particular, this means that the two items of Proposition 7.1.2 are satisfied, and the X-ray transform at least makes sense as a map $I_m : L^p(M, \otimes_S^m T^*M) \rightarrow L^2(\partial_- SM, d\mu_\nu)$, for any $p > 2$. One of the main results of this chapter is the s-injectivity of the X-ray transform for symmetric m-tensors in dimension 2 :

Theorem 7.1.1. *Let (M, g) be a compact connected simple surface with topology. Then I_m is s-injective for any $m \geq 0$.*

As mentioned previously, this result was proved in any dimension by Guillarmou [Gui17b] for $m = 0, 1$, and $m > 1$ under the additional assumption that the sectional curvatures of the metric are non-positive. We are here able to relax the hypothesis on the curvature. As stated in the introduction, we also prove an equivalence principle in the spirit of Paternain-Zhou [PZ16] relating the injectivity of the X-ray transform on smooth symmetric solenoidal m-tensors and the existence of invariant functions by the geodesic flow, with prescribed pushforward on the set of solenoidal symmetric m-tensors. This is the analogue of Lemma 2.5.8 which deals with the closed case.

Theorem 7.1.2. *Let (M, g) be a compact connected simple manifold with topology. Then the three following assertions are equivalent :*

1. *I_m is injective on $C^\infty_{\text{sol}}(M, \otimes_S^m T^*M)$,*
2. *For any $f \in C^\infty_{\text{sol}}(M, \otimes_S^m T^*M)$, there exists a $w \in \cap_{p<+\infty} L^p(SM)$ such that $Xw = 0$ and $\pi_{m*}w = f$,*
3. *For any $u \in L^2_{\text{sol}}(M, \otimes_S^m T^*M)$, there exists $w \in H^{-1}(SM_e)$ such that $Xw = 0$ and $\pi_{m*}w = u$ on M.*

In the case of a simple surface with topology, we are able to prove the second item, which in turn implies Theorem 7.1.1 :

Theorem 7.1.3. *Let (M, g) be a simple surface with topology. Then for any $f \in C^\infty_{\text{sol}}(M, \otimes_S^m T^*M)$, there exists $w \in \cap_{p<+\infty} L^p(SM)$ such that $Xw = 0$ and $\pi_{m*}w = f$.*

Eventually, a corollary of Theorem 7.1.1 is a deformation rigidity result relative to the lens data, which completes [Gui17b, Theorem 4]. The lens data with respect to the metric g is the pair $(\sigma_g, \ell_+^g)|_{\partial_- SM}$, where ℓ_+^g is the exit time function and $\sigma_g : (x,v) \mapsto \varphi_{\ell_+^g(x,v)}(x,v)$ is the scattering map. We refer to the introduction of [Gui17b], or the lecture notes [Pat] for further details.

Corollary 7.1.1. *Assume that M is a smooth compact surface with boundary equipped with a smooth 1-parameter family of simple metrics $(g_s)_{s\in(-1,1)}$ satisfying the assumptions of Theorem 3.2.1 which are lens equivalent (i.e. the lens data agree). Then, there exists a smooth family of diffeomorphisms $(\phi_s)_{s\in(-1,1)}$ such that $\phi_s^* g_s = g_0$ and $\phi_s|_{\partial M} = \mathbb{1}$.*

The proof directly stems from the injectivity of the X-ray transform over solenoidal 2-tensors of Theorem 7.1.1 (see [Gui17b, Section 5.3] for the implication). So far, this had been an open statement for $m \geq 2$ (the two cases $m = 0$ and $m = 1$ being adressed by Guillarmou [Gui17b]). Let us briefly mention that the s-injectivity is known to hold in the case of an open manifold for

- simple surfaces (thus $K = \emptyset$) and $m = 0$ by Mukhometov [Muk77], $m = 1$ by Anikonov-Romanov [AR97], any order $m \in \mathbb{N}$ by Paternain-Salo-Uhlmann [PSU13],
- simple manifolds with non positive curvature and $m \in \mathbb{N}$ by Croke-Sharafutdinov [CS98],
- under a certain foliation assumption on the metric (which can allow conjugate points but no topology) and $m = 0$ by Uhlmann-Vasy [UV16], $m = 1, 2$ by Stefanov-Uhlmann-Vasy [SUV17] and $m \geq 3$ by De Hoop-Uhlmann-Zhai [dUZ18],
- simple manifolds with topology and $m = 0, 1$ by Guillarmou [Gui17b] and $m \geq 2$ in non-positive curvature by Guillarmou [Gui17b] too.

The interest of the X-ray transform is manifold and this notion has been extensively studied in the literature, but most of the articles assume a non-trapping condition. In particular, this operator naturally arises as the differential of the marked boundary distance function as we will see in this chapter. We refer to the surveys of Paternain-Salo-Uhlmann [PSU14b] and Ilmavirta-Monard [IM18] for an overview of the subject.

7.1.3 Marked boundary rigidity

The *marked boundary distance function* is defined as the map

$$d_g : \{(x, y, [\gamma]), (x, y) \in \partial M \times \partial M, [\gamma] \in \mathcal{P}_{x,y}\} \to \mathbb{R}_+$$

which associates to x and y on the boundary and a homotopy class

$$[\gamma] \ \in \mathcal{P}_{x,y} := \{[\gamma], \gamma \text{ is a curve joining } x \text{ to } y\},$$

the distance between x and y computed as the infimum over the piecewise C^1-curves joining x to y in the homotopy class of $[\gamma]$. This map generalizes the classical notion of *boundary distance* to the case of a manifold with topology. It can be seen as an analogue of the *marked length spectrum* in the case of a closed Riemannian manifold (see Chapter 2).

In the case of a manifold with strictly convex boundary and no conjugate points (and without any assumption on the curvature), there exists a unique geodesic in each homotopy class of curves joining x to y which realizes the distance (see [GM18, Lemma 2.2]). As a consequence, given $[\gamma] \in \mathcal{P}_{x,y}$, $d_g(x, y, [\gamma])$ is nothing but the length of this unique geodesic in the class $[\gamma]$. Given g', another metric with strictly convex boundary and no conjugate points, we will say that their marked boundary distance *agree* if $d_g = d_{g'}$. Note that one can also lift this distance to the universal cover $\widetilde{M}$ of M. Then, there exists a unique geodesic joining any pair of points on the boundary of $\widetilde{M}$ and the marked boundary distances agree if and only if the two boundary distances $d_{\widetilde{g}}$ and $d_{\widetilde{g}'}$ agree.

It is conjectured that under suitable assumptions on the metric, this marked boundary distance determines the metric up to a natural obstruction, in the sense that if g' is another metric with same marked boundary distance function, then there exists a diffeomorphism $\phi : M \to M$ such that $\phi|_{\partial M} = \mathbb{1}$ and $\phi^* g' = g$. When this occurs, we say that (M, g) is *marked boundary rigid*.

In the case of a *simple* manifold, i.e. a manifold with strictly convex boundary and such that the exponential map is a diffeomorphism at all points (such manifolds are topological balls without trapping and conjugate points), this conjecture was first stated by Michel [Mic82] in 1981, and later proved by Pestov-Uhlmann [PU05] in 2002, in the two-dimensional case. It is still an open question in higher dimensions but Stefanov-Uhlmann-Vasy [SUV17] proved the rigidity of a wide range of simple (and also non-simple actually) manifolds satisfying a foliation assumption.

There is actually a long history of results regarding the boundary rigidity question on simple manifolds. Let us mention the contributions of Gromov [Gro83], for regions of $\mathbb{R}^n$, the original paper of Michel [Mic82] for subdomains of the open hemisphere and the Besson-Courtois-Gallot theorem [BCG95], which implies the boundary rigidity for regions of $\mathbb{H}^n$ (see also the survey of Croke [Cro04]). Still in the simple setting, the *local boundary rigidity* was studied by Croke-Dairbekov-Sharafutdinov in [CDS00], by Stefanov-Uhlmann in [SU04] and positive results were obtained. More recently, Burgo-Ivanov [BI10] proved the local boundary rigidity for metrics close enough to the euclidean metric. But very few papers deal with manifolds with trapping. In that case, the first general results where obtained by Guillarmou-Mazzucchelli [GM18] for surfaces, where the local marked boundary rigidity was established under suitable assumptions. One of the main results of this chapter is the following marked boundary rigidity result for manifolds of non-positive curvature, which is a local version of Michel's conjecture.

Theorem 7.1.4. *Let (M, g) be a compact connected $(n+1)$-dimensional manifold with strictly convex boundary and negative curvature. We set $N := \lfloor \frac{n+2}{2} \rfloor + 1$. Then (M, g) is locally marked boundary rigid in the sense that : for any $\alpha > 0$ arbitrarily small, there exists $\varepsilon > 0$ such that for any metric g' with same marked boundary distance as g and such that $\|g' - g\|_{C^{N,\alpha}} < \varepsilon$, there exists a $C^{N+1,\alpha}$-diffeomorphism $\phi : M \to M$, such that $\phi|_{\partial M} = \mathbb{1}$ and $\phi^* g' = g$. If g' is smooth, then ϕ is smooth.*

We stress that the marked boundary distance is the natural object to consider insofar as one can construct examples of surfaces satisfying the assumptions of Theorem 7.1.4 with same boundary distance but different marked boundary distances which are not isometric. Indeed, consider a negatively-curved surface (M, g) whose strictly convex boundary has a single component. We can always choose such a surface so that the distance between two points on the boundary is realized by minimizing geodesics which only visit a neighborhood of this boundary. Thus, any small perturbation of the metric away from the boundary will still provide the same boundary distance function but the metrics will no longer be isometric.

Let us eventually mention that the problem of boundary rigidity is closely related to the *lens rigidity* question, that is the reconstruction of the metric g from the knowledge of the scattering map and the exit time function. This question has been extensively in the literature. Among other contributions, we mention that of Stefanov-Uhlmann [SU09], who prove a *local lens rigidity* result on a non-simple manifold (without the assumption on convexity and with a possible trapped set), which is somehow in the spirit of our article.

Our proof can be interpreted as a non-trivial inverse function theorem, like in [CDS00] or [SU04]. Indeed, as mentioned earlier, it can be showed that the linearized version of the marked boundary distance problem is equivalent to the s-injectivity of the X-ray transform I_2. The problem here is non-linear, but still local, which allows us to recover some of the features of the linearized problem. The key argument here is a quadratic control of the X-ray transform of the difference of the two metrics $f := g' - g$ (see Lemma 8.3.1) stemming from a Taylor expansion of the marked boundary distance. We do not choose a normal gauge to make the metrics coincide on the boundary but rather impose a solenoidal gauge (this is made possible thanks to an essential lemma in [CDS00]). A finer control on the regularity of the distributions which are at stake in the last paragraph is also important. This is crucial to apply interpolation estimates to conclude in the end. The analysis is made possible by technical tools introduced in both papers of Guillarmou [Gui17a] and [Gui17b], which are based on recent and powerful analytical techniques developed in the framework of hyperbolic dynamical systems (and detailed in Chapter 2) and adapted to the open setting.

If g' is another metric satisfying $\|g'-g\|_{C^2} < \varepsilon$, then (M, g') is a simple manifold with topology too [2] (see [GM18, Proposition 2.1]). In the following, we will always assume that g' is close enough to g in the C^2 topology so that it satisfies these assumptions. We introduce $N = \lfloor \frac{n+2}{2} \rfloor + 1 \geq 2$.

Theorem 7.1.5. *Let (M, g) be a compact connected simple $(n+1)$-dimensional manifold with topology If I_2^e is s-injective on some extension M_e of M (as detailed in §7.1.1), then (M, g) is locally marked boundary rigid in the sense that : for any $\alpha > 0$ arbitrarily small, there exists $\varepsilon > 0$ such that for any metric g' with same marked boundary distance as g and such that $\|g' - g\|_{C^{N,\alpha}} < \varepsilon$, there exists a smooth diffeomorphism $\phi : M \to M$, such that $\phi|_{\partial M} = \mathbb{1}$ and $\phi^* g' = g$.*

In particular, under the assumption that the curvature of (M, g) is non-positive, it was proved in [Gui17b] that I_m is s-injective for any $m \geq 0$, and thus $m = 2$ in particular. This yields the following

Corollary 7.1.2. *Assume (M, g) satisfies the assumptions of Theorem 7.1.5 and has non-positive curvature. Then it is locally marked boundary rigid.*

As a consequence of Theorems 7.1.1 and 7.1.5, we recover the following result, which was already proved in [GM18] using a different approach.

Corollary 7.1.3. *Assume (M, g) is a simple surface with topology. Then it is locally marked boundary rigid.*

2. Note that the proposition is stated in dimension 2, but the proof is actually independent of the dimension.

7.2 The resolvent of the geodesic vector field

Just like in the closed setting developed in the first chapters of this thesis, one of the main ideas at the root of the recent developments in inverse problems on manifolds with boundary the past few years has been to link the X-ray transform I to the resolvent of the operator X (seen as a differential operator), when acting on some anisotropic Sobolev spaces adapted to the hyperbolic decomposition (see [Gui17b, Section 4] for instance). We define for $\Re(\lambda) \gg 0$ the resolvents

$$R_{\pm}(\lambda) : C^{\infty}_{\mathrm{comp}}(SM^{\circ} \setminus \Gamma_{\pm}) \to C^{\infty}(SM)$$

by the formulas :

$$R_{+}(\lambda)f(z) = \int_0^{\infty} e^{-\lambda t} f(\varphi_t(z))dt, \quad R_{-}(\lambda)f(z) = -\int_{-\infty}^{0} e^{\lambda t} f(\varphi_t(z))dt \tag{7.2.1}$$

They satisfy the relations $(-X \pm \lambda)R_{\pm}(\lambda)f = f$. Just like in the closed setting, following the work of Dyatlov-Guillarmou [DG16], the operators $R_{\pm}(\lambda)$ can be meromorphically extended to the whole complex plane when acting on suitable anisotropic spaces. This result has to be understood as the transposition of the meromorphic extension results of Faure-Sjöstrand [FS11] and Dyatlov-Zworski [DZ16], initially proved in the closed setting, to the open setting. Although the results of [DG16] are similar in spirit to those of [FS11, DZ16], they are more technical, mainly because manifolds with boundary inevitably bring new complications which do not change the nature of the results but involve more computations. As a consequence, we will not be as exhaustive as we were in Chapter 2 and refer the reader to that chapter for results of microlocal nature.

Once again, like in the compact setting, we will be interested in the value at 0 of the meromorphic extension of $R_{\pm}(\lambda)$. For $f \in C^{\infty}_{\mathrm{comp}}(SM^{\circ} \setminus (\Gamma_+ \cup \Gamma_-))$, we define operator

$$\Pi f := (R_+(0) - R_-(0))f.$$

We have the fundamental

Lemma 7.2.1. *For $f \in C^{\infty}_{\mathrm{comp}}(SM^{\circ} \setminus (\Gamma_+ \cup \Gamma_-))$,*

$$\Pi f = I^* I f.$$

The proof is a straightforward computation and left as an exercise to the reader. The operator Π is called the normal operator. Note that there is no pole at 0 in this case. These operators can also be defined on the manifold M_e and we will add an index e (Π^e for instance) in order not to confuse them.

7.2.1 The operators I_m, I_m^* and Π

Action on L^p spaces. The idea is now to extend the operator Π to larger sets of functions (like L^p spaces) and to deduce from this the action of I and I^* on these sets.

Proposition 7.2.1. *Let $1 < p \leq +\infty$, then :*

$$\Pi : L^p(SM) \to \cap_{q<p} L^q(SM),$$
$$I : L^p(SM) \to \cap_{q<p} L^q(\partial_- SM, d\mu_\nu),$$
$$I^* : L^p(\partial_- SM, d\mu_\nu) \to \cap_{q<p} L^q(SM)$$

are bounded.

Proof. If K is hyperbolic, then $\ell_+ \in L^p(SM)$, for any $p \in [1, +\infty)$. Indeed, one has $\mu(\{\ell_+ > T\}) = V(T)$ and by Cavalieri's principle :

$$\|\ell_+\|^p_{L^p(SM)} = \int_0^{+\infty} t^{p-1}\mu(\{\ell_+ > T\})\, dt = \int_0^{+\infty} t^{p-1}V(t)dt < +\infty,$$

since $V(t) = O(e^{-\delta t})$.

For $f \in C^\infty_{\text{comp}}(SM^\circ \setminus \Gamma_-)$, let us write $u(x, v) = R_+(0)f(x, v) = \int_0^{+\infty} f(\varphi_t(x, v))dt$. We consider $1 \leq q < p$. We have, using Jensen's inequality :

$$\begin{aligned}
\|u\|^q_{L^q(SM)} &= \int_{SM} \left| \int_0^{\ell_+(z)} f(\varphi_t(z))dt \right|^q d\mu(z) \\
&\leq \int_{SM} |\ell_+(z)|^{q-1} \int_0^{+\infty} \mathbf{1}(\varphi_t(z) \in SM)|f(\varphi_t(z))|^q dt d\mu(z) \\
&= \int_0^{+\infty} \int_{\mathcal{U}_t} |\ell_+(z)|^{q-1}|f(\varphi_t(z))|^q d\mu(z)dt,
\end{aligned}$$

where $\mathcal{U}_t = \{\ell_+(z) > t\}$, by applying Fubini in the last equality. For a fixed $t \geq 0$, we make the change of variable in the second integral $y = \varphi_t(z)$ and since the Liouville measure is preserved by the geodesic flow, we obtain :

$$\begin{aligned}
\|u\|^q_{L^q(SM)} &\leq \int_0^{+\infty} \int_{\varphi_t(\mathcal{U}_t)} |\ell_+(\varphi_{-t}(y))|^{q-1}|f(y)|^q dt d\mu(y) \\
&= \int_{SM} |f(y)|^q \int_0^{+\infty} \mathbf{1}(y \in \varphi_t(\mathcal{U}_t))(\ell_+(y) + t)^{q-1} dt d\mu(y)
\end{aligned}$$

But $y \in \varphi_t(\mathcal{U}_t) \cap SM$ if and only if $\varphi_t(y) \in SM$, that is if and only if $|\ell_-(y)| > t$. In other words, $\varphi_t(\mathcal{U}_t) \cap SM = \{|\ell_-(y)| > t\}$. Thus :

$$\begin{aligned}
\|u\|^q_{L^q(SM)} &\leq \int_{SM} |f(y)|^q \int_0^{|\ell_-(y)|} (\ell_+(y) + t)^{q-1} dt d\mu(y) \\
&\leq C \int_{SM} |f(y)|^q \left(\ell_+(y) + |\ell_-(y)|\right)^q d\mu(y) \leq C\|f\|^q_{L^p(SM)},
\end{aligned}$$

using Hölder in the last inequality, and where $C > 0$ is a constant depending on p and q. We cannot recover the L^p-norm of f insofar as the functions ℓ_+, ℓ_- are not L^∞. By density of $C^\infty_{\text{comp}}(SM^\circ \setminus \Gamma_-)$ in $L^p(SM)$, this proves that

$$R_+(0) : L^p(SM) \to \cap_{q<p} L^q(SM)$$

extends as a bounded operator. The same arguments prove that

$$R_-(0) : L^p(SM) \to \cap_{q<p} L^q(SM)$$

extends as a bounded operator and thus $\Pi : L^p(SM) \to \cap_{q<p} L^q(SM)$ is bounded. Of course, the same arguments show that $\Pi^e : L^p(SM_e) \to \cap_{q<p} L^q(SM_e)$ is bounded. We extend f by 0 outside SM to obtain a function on SM_e (still denoted f). Now, we have for some $\varepsilon > 0$ small enough :

$$\begin{aligned}\varepsilon\|If\|^q_{L^q(\partial_-SM,d\mu_\nu)} &= \int_{\partial_-SM} \varepsilon|If(x,v)|^q d\mu_\nu(x,v)\\ &= \int_{\partial_-SM}\int_0^\varepsilon |\tilde{I}^*If(\varphi_t(x,v))|^q dt d\mu_\nu(x,v)\\ &= \int_{SM} |\Pi^e f|^q\, \mathbf{1}_{A_\varepsilon}(x,v)d\mu(x,v) \leq \|\Pi^e f\|^q_{L^q(SM_e)},\end{aligned}$$

where $\tilde{I}^*u(\varphi_t(x,v)) = u(x,v)$, for $(x,v)\in\partial_-SM, t\in[0,\varepsilon]$, and

$$A_\varepsilon = \{\varphi_t(x,v)\in SM_e, (x,v)\in\partial_-SM, 0\leq t\leq\varepsilon\}$$

Thus, using the boundedness of Π^e and the fact that $f\equiv 0$ on $M_e\setminus M$, we get that $I : L^p(SM)\to\cap_{q<p}L^q(\partial_-SM,d\mu_\nu)$ is bounded and by a duality argument $I^* : L^p(\partial_-SM,d\mu_\nu)\to\cap_{q<p}L^q(SM)$ is bounded too.

□

Action on some Sobolev spaces. Recall that $\pi_0 : SM\to M$ denotes the projection on the manifold. There exists a decomposition of the tangent space to the unit tangent bundle over M :

$$T(SM) = \mathbb{H}\oplus\mathbb{V}$$

which is orthogonal for the Sasaki metric (see §7.4.1 for the case of a surface), where $\mathbb{V} = \ker d\pi_0$, $\mathbb{H} = \ker\mathcal{K}$ and $\mathcal{K}$ is the connection map, defined such that $\mathcal{K}(\zeta)\in T_{\pi_0(\zeta)}M$ is the only vector such that the local geodesic $t\mapsto\gamma(t)\in SM$ starting from $(\pi_0(\zeta),\mathcal{K}(\zeta))$ satisfies $\dot\gamma(0)=\zeta$ (see [Pat99] for a reference). We define the dual spaces $\mathbb{H}^*$ and $\mathbb{V}^*$ such that $\mathbb{H}^*(\mathbb{H})=0, \mathbb{V}^*(\mathbb{V})=0$. We recall the that given $u\in C^{-\infty}(M,\otimes^m_S T^*M)$, we have $\mathrm{WF}(\pi_m^*u)\subset\mathbb{V}^*$ (see Lemma 2.5.1). We define

$$H^1_0(\partial_-SM,d\mu_\nu) := \left\{u\in H^1(\partial_-SM,d\mu_\nu), u|_{\partial_0SM}=0\right\}$$

Its dual for the natural L^2-scalar product given by the measure $d\mu_\nu$ is $H^{-1}(\partial_-SM,d\mu_\nu)$. Let us recall that given $u\in C^{-\infty}(SM)$, its H^s-wavefront set is defined for $s\in\mathbb{R}$ by :

$$\mathrm{WF}_s(u)^c = \left\{(z,\xi)\in T^*(SM), \exists A\in\Psi^0 \text{ elliptic at } (z,\xi) \text{ such that}, Au\in H^s_{\mathrm{loc}}\right\}$$

Eventually, we will denote by $p:(x,\xi)\mapsto\langle\xi,X(x)\rangle$ the principal symbol of $\frac{1}{i}X$ and by $\Sigma := p^{-1}(\{0\})$ its characteristic set.

Proposition 7.2.2. *Let $u\in H^{-1}_{\mathrm{comp}}(M^\circ,\otimes^m_S T^*M^\circ)$. Then $\Pi\pi_m^*u\in H^{-1}(SM)$ and $I_m u\in H^{-1}(\partial_-SM,d\mu_\nu)$. The same result holds for M_e.*

The proof is based on classical propagation of singularities (for which we refer to [Ler, Theorem 4.3.1] for instance) and more recent propagation estimates with radial sources/sinks in open manifolds due to Dyatlov-Guillarmou [DG16, Lemma 3.7].

Proof. Since $\Pi = R_+(0)-R_-(0)$, we will actually prove that both $R_\pm(0)\pi_m^*$ satisfy the proposition. We will only deal with $R_-(0)\pi_m^*$ since the operator $R_+(0)\pi_m^*$ can be handled in the same fashion. Consider $u\in H^{-1}_{\mathrm{comp}}(M^\circ,\otimes^m_S T^*M^\circ)$. We have $\pi_m^*u\in H^{-1}_{\mathrm{comp}}(SM^\circ)$ and $\mathrm{WF}(\pi_m^*u)\subset\mathbb{V}^*$. The wavefront set of the Schwartz kernel of $R_-(0)$ is described in [DG16] :

$$\mathrm{WF}'(R_-(0))\subset N^*\Delta(SM^\circ\times SM^\circ)\cup\Omega_+\cup(E_+^*\times E_-^*),$$

where

$$N^*\Delta(SM^\circ \times SM^\circ) = \{(x, \xi, x, -\xi) \in T^*(SM^\circ \times SM^\circ)\},$$

denotes the conormal to the diagonal and

$$\Omega_+ = \left\{(\varphi_t(z), (d\varphi_t(z))^{-\top}(\xi), z, -\xi) \in T^*(SM^\circ \times SM^\circ), t \geq 0, (z, \xi) \in \Sigma\right\},$$

with $d\varphi_t(z)^{-\top}$ denoting the inverse transpose. Thus, by the rules of composition for the wavefront sets (see [Hö03, Chapter 8] for a reference) and since there are no conjugate points, $R_-(0)f$ is well-defined as a distribution, as long as $\mathrm{WF}(f) \cap E_-^* = \emptyset$. This is the case for $\pi_m^* u$ because over $\Gamma_\pm$ the decomposition $T(SM_e) = \mathbb{R}X \oplus \mathbb{V} \oplus E_\pm$ holds (see [Kli74, Proposition 6]) and thus $\mathbb{V}^* \cap E_\pm^* = \{0\}$. Furthermore,

$$\mathrm{WF}(R_-(0)\pi_m^* u) \subset \mathbb{V}^* \cup B_+ \cup E_+^* \tag{7.2.2}$$

where

$$B_+ := \left\{(\varphi_t(z), (d\varphi_t(z))^{-\top}(\xi)) \in T^*(SM^\circ), t \geq 0, (z, \xi) \in \mathbb{V}^* \cap \Sigma\right\}$$

is the forward propagation of $\mathbb{V}^* \cap \Sigma$ by the Hamiltonian flow in the characteristic set. Note that $XR_-(0)\pi_m^* u = -\pi_m^* u$ and by ellipticity of X outside the characteristic set Σ, one has $\mathrm{WF}_{-1}(R_-(0)\pi_m^* u) \cap \Sigma^c = \emptyset$, that is $R_-(0)\pi_m^* u$ is microlocally in H^{-1} outside Σ.

Given a point $z \notin \Gamma_+$, we know that there exists a finite time $T > 0$ such that $\varphi_{-T}(z) \in \partial_- SM$. But since u was taken with compact support in M°, we know that there exists a whole neighborhood of $\partial_- SM$ where $R_-(0)\pi_m^* u$ vanishes (and thus is H^{-1} locally). By classical propagation of singularity, since $XR_-(0)\pi_m^* u = -\pi_m^* u$ is H^{-1} on SM, we deduce that $R_-(0)\pi_m^* u$ is locally H^{-1} at z.

The points left to study are the $z \in \Gamma_+$. Let us prove that $R_-(0)\pi_m^* u$ is microlocally H^{-1} on B_+. Given $(z, \xi) \in B_+$, there exists by definition a finite time $T \geq 0$ such that $(\varphi_{-T}(z), (d\varphi_{-T}(z))^{-\top}(\xi)) \in B_-$ (where B_- is the backward propagation of $\mathbb{V}^* \cap \Sigma$ by the Hamiltonian flow, defined analogously as B_+ but for strictly negative time ; the absence of conjugate points implies that $B_- \cap B_+ = \emptyset$ [3]). But by (7.2.2), $R_-(0)\pi_m^* u$ is microlocally in H^{-1} on B_- (it is smooth actually) and $XR_-(0)\pi_m^* u = -\pi_m^* u$ is in H^{-1}, thus it is in particular H^{-1} along the trajectory $\left\{(\varphi_{-s}(z), (d\varphi_{-s}(z))^{-\top}(\xi)), s \in [0, T]\right\}$ so by classical propagation of singularities, $R_-(0)\pi_m^* u$ is microlocally H^{-1} at (z, ξ) (regularity propagates forward and backwards since the principal symbol is real).

As a consequence, $\mathrm{WF}_{-1}(R_-(0)\pi_m^* u) \subset E_+^*$. To conclude, we will use the result of propagation of estimates for a radial sink as it is formulated in [DG16, Lemma 3.7]. We embed the outer manifold M_e into N, a smooth closed manifold and extend smoothly the metric g and the vector field X (see [DG16, Section 2]). We extend $R_-(0)\pi_M^* u$ by 0 outside SM. We consider $A, B, B_1 \in \Psi^0(SN)$ such that (see Figure 7.2) :

- $\mathrm{WF}(A)$ is contained in a conic neighborhood of $E_u^* = E_+^*|_K$ and A is elliptic on a (smaller) conic neighborhood of E_u^*,
- $\mathrm{ell}(B)$ contains a whole neighborhood of $\pi^{-1}(K)$ (larger than that chosen for A), except a conic vicinity of E_+^*, and $\mathrm{WF}(B) \cap E_+^* = \emptyset$ (in other words B is elliptic over a punctured neighborhood in the fibers over K),
- $\mathrm{ell}(B_1)$ is contained in SM° and contains $\mathrm{WF}(A)$ and $\mathrm{WF}(B)$.

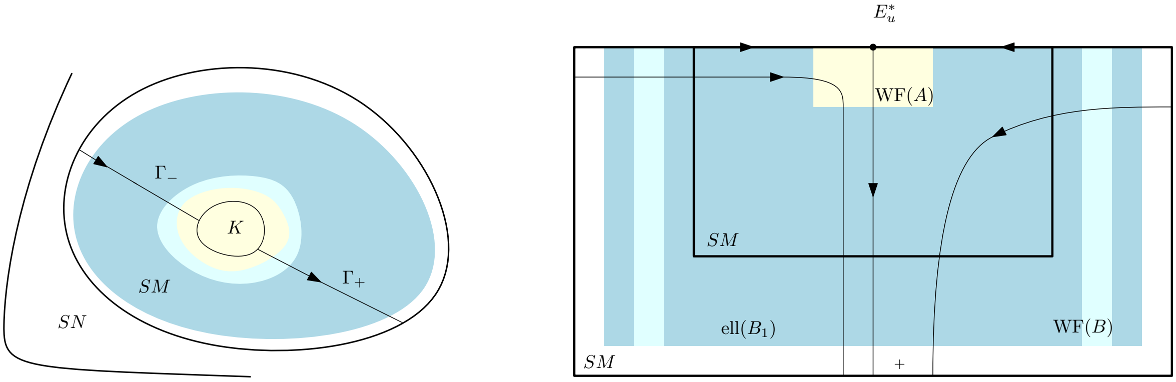

FIGURE 7.2 – In yellow, light blue, darker blue : (resp.) $\mathrm{WF}(A), \mathrm{WF}(B), \mathrm{ell}(B_1)$. Left : The projection of the previous sets on the base SM. Right : Vertical lines represent the dynamics in the physical space SM, horizontal lines represent the dynamics in the cotangent space $T^*(SM)$.

Moreover, we take these operators so that they do not "see" the exterior manifold SN, in the sense that their Schwartz kernel is supported in $SM^\circ \times SM^\circ$. Actually, once one is able to construct three operators satisfying the three previous items, it is sufficient to truncate their Schwartz kernel so that they satisfy this condition of support. These operators satisfy [DG16, Lemma 3.7] where $L := E_u^*$ is the sink. Indeed, if $(z,\xi) \in \mathrm{WF}(A)$, then by [DG16, Lemma 2.11] :

- if $z \notin \Gamma_+$, then there exists a finite time $T \geq 0$ such that $\varphi_{-T}(z) \in \mathrm{ell}(B)$ (in the past, the point physically escapes from a neighborhood of K and falls in a region where B is elliptic),
- if $z \in \Gamma_+, \xi \notin E_+^*$, $d\varphi_t^{-\top}(\xi) \to_{t\to-\infty} E_-^*$ which is contained in $\mathrm{ell}(B)$ (in the past, z goes to K while in phase space, the covector ξ goes to E_-^* and falls in a region of ellipticity of B),
- if $z \in \Gamma_+, \xi \in E_+^*$, $(\varphi_t(z), d\varphi_t^{-\top}(\xi)) \to_{t\to-\infty} L = E_u^*$ (and these points stay in $\mathrm{ell}(B_1)$).

Note that [DG16, Lemma 3.7] is satisfied for any $s < 0$ (thus in particular for $s = -1$) as mentioned in [DG16, Lemma 4.2] because X is formally skew-adjoint. Moreover, by construction, $BR_-(0)\pi_m^* u$ is H^{-1} because we already know that $R_-(0)\pi_m^* u$ is microlocally H^{-1} away from E_+^* and $\mathrm{WF}(B) \cap E_+^* = \emptyset$. By [DG16, Lemma 3.7], there exists a constant $C > 0$ (independent of u) and an integer $N \geq 2$ such that :

$$\begin{aligned}
&\|AR_-(0)\pi_m^* u\|_{H^{-1}(SN)} \\
&= \|Aw\|_{H^{-1}(SN)} \\
&\lesssim \left(\|Bw\|_{H^{-1}(SN)} + \|B_1 X w\|_{H^{-1}(SN)} + \|w\|_{H^{-N}(SN)}\|\right) \\
&\lesssim \left(\|BR_-(0)\pi_m^* u\|_{H^{-1}(SM)} + \|B_1 \pi_m^* u\|_{H^{-1}(SM)} + \|R_-(0)\pi_m^* u\|_{H^{-N}(SM)}\right)
\end{aligned}$$

As a consequence, by the choice of A, $R_-(0)\pi_m^* u$ is microlocally H^{-1} on E_+^* in a neighborhood of K and classical propagation of singularities implies that this holds on SM. Thus $R_-(0)\pi_m^* u \in H^{-1}$.

3. Assume $B_- \cap B_+ \neq \emptyset$. Then, there exists $t_0 \geq 0, t_1 > 0, (z,\xi), (y,\eta) \in \mathbb{V}^* \cap \Sigma$, such that $(\varphi_{t_0}(z), (d\varphi_{t_0}(z))^{-\top}(\xi)) = (\varphi_{-t_1}(y), (d\varphi_{-t_1}(y))^{-\top}(\eta))$, so $y = \varphi_t(z)$ (with $t := t_0 + t_1 > 0$), $\eta = (d\varphi_t(z))^{-\top}(\xi)$, the latter equality being contradicted by the absence of conjugate points.

To prove the last part of the proposition, it is sufficient to establish that $\Pi\pi_m^* u \in H^{-1}(SM)$ restricts on the boundary $\partial_- SM$. The restriction makes sense as long as

$$\mathrm{WF}(\Pi\pi_m^* u) \cap N^*(\partial_- SM) = \emptyset,$$

Remark that, since u has compact support in M°, $R_\pm(0)\pi_m^* u \equiv 0$ in a vicinity of $\partial_0 SM$, so there is no singular support in a vicinity of $\partial_0 SM$. Moreover, since $X\Pi\pi_m^* u = 0$, we know that $\mathrm{WF}(\Pi\pi_M^* u) \subset \Sigma$. But if $\xi \in N^*(\partial_- SM)$ is not 0, one has $\langle \xi, X\rangle \neq 0$ (since X intersects transversally the boundary away from $\partial_0 SM$ by convexity) and thus $\xi \notin \Sigma$ by construction, so $\xi \notin \mathrm{WF}(\Pi\pi_M^* u)$.

□

Remark 7.2.1. Note that any other regularity H^{-s} for some $s > 0$ could have been chosen instead of H^{-1}.

7.2.2 Some lemmas of surjectivity

The two following lemmas are stated by Paternain-Zhou [PZ16, Lemmas 4.2, 4.3]. We detail the proof of the second lemma which morally follows that of Dairbekov-Uhlmann [DU10, Lemma 2.2].

Lemma 7.2.2. *Assume I_m is s-injective. Then,*

$$P := r_M \Pi_m^e : H^{-1}_{\mathrm{comp}}(M_e^\circ, \otimes_S^m T^* M_e^\circ) \to L^2_{\mathrm{sol}}(M, \otimes_S^m T^* M)$$

is surjective.

Lemma 7.2.3. *Assume I_m is s-injective. Then*

$$P : C^\infty_{\mathrm{comp}}(M_e^\circ, \otimes_S^m T^* M_e^\circ) \to C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^* M)$$

is surjective.

Let r_M denote the operator of restriction to the manifold M and E_0 the operator of extension by 0 outside M. Note that if $u \in H^s_{\mathrm{sol}}(M, \otimes_S^m T^* M)$ (for some $s < 1/2$), then $E_0 u \in H^s_{\mathrm{comp}}(M_e^\circ, \otimes_S^m T^* M_e^\circ)$ is not necessarily solenoidal as $D^* E_0 u$ may have some support in ∂M. Let $E : H^N_{\mathrm{sol}}(M, \otimes_S^m T^* M) \to L^2_{\mathrm{sol,\ comp}}(M_e^\circ, \otimes_S^m T^* M_e^\circ)$ be the operator of extension of [PZ16, Proposition 3.4], where $N \geq 2$ is an integer and $E(C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^* M)) \subset C^\infty_{\mathrm{sol,\ comp}}(M_e^\circ, \otimes_S^m T^* M_e^\circ)$ (this is made possible by the absence of non-trivial Killing tensor fields). For the sake of simplicity, we will write $C^\infty_{\mathrm{sol}}(M)$ instead of $C^\infty_{\mathrm{sol}}(M, \otimes_S^m T^* M)$ in the proof.

Proof of Lemma 7.2.3. We first prove that P has closed range and finite codimension. By [Gui17b, Proposition 5.9], we know that Π_m^e is elliptic of order -1 on $\ker D^*$ in the sense that there exists Q, S, R, pseudo-differential operators on M_e° of respective order $1, -2, -\infty$ such that

$$\Pi_m^e Q = \mathbb{1}_{M_e^\circ} + DSD^* + R, \qquad (7.2.3)$$

Note that we can always assume that Q is properly supported in M_e° since any pseudodifferential operator can be splitted as the sum of a properly supported ΨDO and a smooth ΨDO (see [Hö03, Proposition 18.1.22]). We stress the fact that these operators (defined on M_e°) will be applied to functions with compact support in M_e°. As a consequence, we have for $f \in C^\infty_{\mathrm{sol}}(M)$ that

$$PQEf = f + r_M REf$$

Since R is of order $-\infty$ (it is smoothing), it is compact on $H^N_{\text{sol}}(M)$ and so is $r_M RE$ (for $N \geq 0$). Thus, $A := \mathbb{1}_M + r_M RE = PQE : H^N_{\text{sol}}(M) \to H^N_{\text{sol}}(M)$ has closed range and finite codimension (it is Fredholm). This implies that $A : C^\infty_{\text{sol}}(M) \to C^\infty_{\text{sol}}(M)$ has closed range and finite codimension.

The inclusion relation

$$\begin{aligned} PQE(C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)) &\subset P(C^\infty_{\text{comp}}(M_e^\circ, \otimes^m_S T^*M_e^\circ)) \\ &\subset C^\infty_{\text{sol}}(M, \otimes^m_S T^*M), \end{aligned}$$

proves that the intermediate space is closed with finite codimension in $C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)$. It is now sufficient to prove that $P^* : (C^\infty_{\text{sol}}(M, \otimes^m_S T^*M))' \to (C^\infty_{\text{comp}}(M_e^\circ, \otimes^m_S T^*M_e^\circ))'$ is injective.

As mentioned in (7.1.14), there is a natural decomposition of tensors into $C^\infty(M) = C^\infty_{\text{sol}}(M) \oplus C^\infty_{\text{pot}}(M)$ which is orthogonal for the L^2-scalar product. Any continuous functional on $C^\infty_{\text{sol}}(M)$ extends as a continuous functional on $C^\infty(M)$ which vanishes on $C^\infty_{\text{pot}}(M)$ (and vice-versa). In other words, there is a canonical identification of the dual to $C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)$ with the sub-space of distributions

$$C^{-\infty}_{\text{sol, }0}(M_e^\circ) := \left\{ u \in C^{-\infty}_{\text{comp}}(M_e^\circ), \text{supp }(u) \subset M, \text{ and } \forall f \in C^\infty_{\text{pot}}(M), \langle u, \bar{E}f \rangle = 0 \right\},$$

where $\bar{E}f$ is any smooth extension with compact support of f.

Assume that $P^* f = 0$ for some continuous functional f on $C^\infty_{\text{sol}}(M)$, that is $\langle f, Pu \rangle = 0 = \langle E_0 f, \Pi^e_m u \rangle$, for all $u \in C^\infty_{\text{comp}}(M_e^\circ)$. Here $E_0 f \in C^{-\infty}_{\text{sol, }0}(M_e^\circ)$ is the distribution on the exterior manifold identified with f. One has $E_0 f \in H^{-N}_{\text{comp}}(M_e^\circ)$ for some N large enough which gives that $\langle \Pi^e_m E_0 f, u \rangle = 0$, for all $u \in C^\infty_{\text{comp}}(M_e^\circ)$, that is $\Pi^e_m E_0 f = 0$.

We can still make sense of the decomposition $E_0 f = q + Dp_0$, where we have $p_0 := \Delta^{-1} D^* E_0 f \in H^{-N+1}(M_e, \otimes^{m-1}_S T^*M_e)$ (with $\Delta := D^*D$ the Dirichlet Laplacian for m-tensors on M_e, see [DS10]) and $q := E_0 f - Dp_0 \in H^{-N}_{\text{sol}}(M_e, \otimes^m_S T^*M_e)$ (in the sense that $D^*q = 0$ in the sense of distributions). One has $\Pi^e_m(E_0 f - Dp_0) = \Pi^e_m(q) = 0$. By ellipticity of Δ, p_0 has singular support contained in ∂M (and the same holds for Dp_0). Moreover, from $q = -Dp_0$ on $M_e \setminus M$, we see that q is smooth on $M_e \setminus M$ and since it is solenoidal on M_e and in the kernel of Π^e_m, it is smooth on M_e° (this stems from the ellipticity of Π^e_m (7.2.3)), so q is smooth on M_e.

Since[4] $Dp_0 = -q$ on $M_e \setminus M$ and q is smooth on M_e, one can find a smooth tensor p_1 defined on M_e such that $p_1 = p_0$ and $Dp_1 = -q$ on $M_e \setminus M$. Then $Dp_1 + q$ is smooth, supported in M and $\Pi_m(Dp_1 + q) = 0$. By s-injectivity of the X-ray transform, we obtain $Dp_1 + q = Dp_2$ on M for some smooth tensor p_2 supported in M such that $p_2|_{\partial M} = 0$ (and all its derivatives vanish on the boundary since $Dp_1 + q$ vanish to infinite order on ∂M). Since $Dp_1 + q = 0$ on $M_e \setminus M$, we get $Dp_1 + q = DE_0 p_2$ on M_e so $E_0 f = q + Dp_0 = D(p_0 + E_0 p_2 - p_1) = Dp$, where $p := p_0 + E_0 p_2 - p_1$.

We have $E_0 f = Dp$ and $E_0 f = 0$ on $M_e \setminus M$, $p|_{\partial M_e} = 0$. By unique continuation, we obtain that $p = 0$ in $M_e \setminus M$. Now, by ellipticity, one can also find (other) pseudo-differential operators Q, S, R on M_e° of respective order $1, -2, -\infty$, such that :

$$Q\Pi^e_m = \mathbb{1}_{M_e^\circ} + DSD^* + R,$$

where S is a parametrix of D^*D. Since $E_0 f = Dp$ has compact support in M_e°, we obtain :

$$Q\Pi^e_m E_0 f = 0 = Q\Pi^e_m Dp = Dp + DSD^*Dp + Rp = 2Dp + \text{ smooth terms}$$

4. The argument given in this paragraph was communicated to us by one of the referees.

This implies that $E_0 f = Dp$ is smooth on M_e (and actually p is smooth by ellipticity of D). Therefore :

$$\langle f, f\rangle_{L^2(M)} = \langle f, Dp\rangle = 0,$$

where the equality holds because $p|_{\partial M} = 0$ and, by assumption, f vanishes on such potential tensors. Thus $f = 0$ and P is surjective. □

7.3 Proof of the equivalence theorem

We can now complete the proof of Theorem 7.1.2.

Proof of Theorem 7.1.2. (1) $\implies$ (2) We assume that I_m is injective on the space $C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$. According to Lemma 7.2.3, we know that given $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, there exists $u \in C^\infty_{\mathrm{comp}}(M_e^\circ, \otimes^m_S T^*M_e^\circ)$ such that $r_M \Pi^e_m u = r_M {I^e_m}^* I^e_m u = r_M {I^e_m}^* \tilde\varphi = f$, where $\tilde\varphi = I^e_m u \in \cap_{p<\infty} L^p(\partial_- SM_e, d\mu_\nu)$ by Proposition 7.2.1. We want to prove that $\varphi := (I^{e*}\tilde\varphi)\,|_{\partial_- SM} \in L^p(\partial_- SM, d\mu_\nu)$. Note that by construction $I^*_m \varphi = f$. Since there exists a minimal time $\tau > 0$ for a point $(x, v) \in \partial_- SM$ to reach $\partial_- SM_e$ (in negative time), we obtain :

$$\begin{aligned}
\|\varphi\|^p_{L^p(\partial_- SM, d\mu_\nu)} &= \int_{\partial_- SM} |I^{e*}\tilde\varphi|^p(x, v)\, d\mu_\nu(x, v) \\
&= \int_{\partial_- SM} \frac{1}{\ell^e_-(x, v)} \int_0^{\ell^e_-(x,v)} |I^{e*}\tilde\varphi|^p(\varphi_t(x, v))\, dt\, d\mu_\nu(x, v) \\
&\leq \tau^{-1} \int_{\partial_- SM} \int_0^{\ell^e_-(x,v)} |I^{e*}\tilde\varphi|^p(\varphi_t(x, v))\, dt\, d\mu_\nu(x, v) \\
&= \tau^{-1} \int_A |I^{e*}\tilde\varphi|^p(x, v)\, d\mu(x, v) \leq \tau^{-1} \int_{SM_e} |I^{e*}\tilde\varphi|^p(x, v)\, d\mu(x, v) < \infty
\end{aligned},$$

where $A := \cup_{t\geq 0} \varphi_{-t}(\partial_- SM)$ By Proposition 7.2.1, $w := I^*\varphi \in \cap_{p<\infty} L^p(SM)$ and $\pi_{m*} w = f$.

(2) $\implies$ (1) Let us assume that $I_m f = I\pi^*_m f = 0$, for some $f \in C^\infty_{\mathrm{sol}}(M, \otimes^m_S T^*M)$. We can apply the Livcic theorem in our context : by [Gui17b, Proposition 5.5], there exists a function $h \in C^\infty(SM)$ such that $h|_{\partial SM} = 0$ and $\pi^*_m f = Xh$. Now, by hypothesis, π_{m*} is surjective, so there exists an invariant $w \in \cap_{p<\infty} L^p(SM)$ such that $f = \pi_{m*} w$, with $Xw = 0$. We thus claim that

$$0 = \langle Xw, h\rangle = -\langle w, Xh\rangle = -\langle w, \pi^*_m f\rangle = -\langle \pi_{m*} w, f\rangle = -\|f\|^2, \qquad (7.3.1)$$

which would conclude the proof of this point. All we have to justify is the second equality since the others are immediate. This can be done using an approximation lemma. We extend w by flow-invariance to SM_e and still denote it $w \in L^2(SM_e)$. We consider a test function $\chi \in C^\infty_{\mathrm{comp}}(SM_e^\circ)$ such that $\chi \equiv 1$ on SM. By [DZ, Lemma E. 47], there exists a sequence $(w_k)_{k\in\mathbb{N}}$ of smooth functions in SM_e° such that $\chi w_k \to \chi w$ in $L^2(SM_e^\circ)$ and $\chi X w_k \to \chi X w = 0$ in $L^2(SM_e^\circ)$ too. In particular, one has both convergences in $L^2(SM)$ without the test function. Now (7.3.1) is satisfied for each w_k, $k \in \mathbb{N}$, since h vanishes on the boundary ∂SM and passing to the limit as $k \to \infty$, we get the sought result.

(1) $\iff$ (3) If I_m is s-injective, then the operator P in Lemma 7.2.2 is surjective : if $u \in L^2_{\mathrm{sol}}(M, \otimes^m_S T^*M)$, there exists a $v \in H^{-1}_{\mathrm{comp}}(M_e^\circ, \otimes^m_S T^*M_e^\circ)$ such that $Pv = r_M \pi_{m*} \Pi^e \pi_m^* v = u$. We set $w := \Pi^e \pi_m^* v \in H^{-1}(SM_e)$ (according to Proposition 7.2.2). Then it is clear that $Xw = 0$ and $\pi_{m*} w = u$ on M. To prove the converse, it is sufficient to repeat the previous proof of (2) $\implies$ (1). $\square$

7.4 Surjectivity of π_{m*} for a surface

We now assume that M is two-dimensional and satisfies the assumptions of Theorem 3.2.1.

7.4.1 Geometry of a surface

In local isothermal coordinates (x, y, θ), we denote by V the vertical vector field $\partial/\partial\theta$. There exists a third vector field $X_\perp$ such that the family $\{X, X_\perp, V\}$ forms an orthonormal basis of $T(SM)$ with respect to the Sasaki metric. The functional space $L^2(SM)$ decomposes as the orthogonal sum

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} C^\infty(M, H_k),$$

where each $C^\infty(M, H_k)$ is the fiberwise eigenspace of $-iV$ corresponding to the eigenvalue k. A function $u \in L^2(SM)$ can be decomposed into $u = \sum_{k \in \mathbb{Z}} u_k$, where $u_k \in C^\infty(M, H_k)$. In particular, in the local isothermal coordinates, one has :

$$u_k(x, y, \theta) = \left(\frac{1}{2\pi} \int_0^{2\pi} u(x, y, t) e^{-ikt} dt \right) e^{ik\theta}$$

This decomposition extends to distributions in $C^{-\infty}(SM)$. Indeed, if $u \in C^{-\infty}(SM)$, we set for $\varphi \in C^\infty(SM)$,

$$\langle u_k, \varphi \rangle := \langle u, \varphi_{-k} \rangle$$

In particular, if $u_k \in C^\infty(M, H_k)$, then $\pi_k^* \pi_{k*} u_k = c_k u_k$ for some constant $c_k \neq 0$. There exist two fundamental differential operators $\eta_\pm : C^\infty(M, H_k) \to C^\infty(M, H_{k\pm1})$ acting on the spaces $C^\infty(M, H_k)$, defined by $\eta_\pm := \frac{1}{2}(X \mp iX_\perp)$ (see [GK80a]) and the formal adjoint of η_+ is $-\eta_-$.

Thanks to the explicit expression of the vector fields X and $X_\perp$ in isothermal coordinates (x, θ), one can compute explicitly $\eta_\pm u$ for $u_k \in C^\infty(M, \Omega_k)$. If $u_k(x, y, \theta) = \tilde{u_k}(x, y) e^{ik\theta}$ in local isothermal coordinates, then one has

$$\eta_-(u) = e^{-(k+1)\lambda} \bar{\partial}(\tilde{u_k} e^{k\lambda}) e^{i(k-1)\theta}, \tag{7.4.1}$$

$$\eta_+(u) = e^{(k-1)\lambda} \partial(\tilde{u_k} e^{-k\lambda}) e^{i(k+1)\theta}, \tag{7.4.2}$$

where λ is the factor of conformity with the euclidean metric, $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

We denote by κ the canonical line bundle, that is the complex line bundle generated by the complex-valued 1-form dz in local holomorphic coordinates. A smooth $u_k \in C^\infty(M, \Omega_k)$ can be identified with a section of $\kappa^{\otimes k}$ according to the mapping $u_k \mapsto \tilde{u_k} e^{k\lambda} (dz)^{\otimes k}$, written in local holomorphic coordinates, where $u_k(z, \theta) = \tilde{u_k}(z) e^{ik\theta}$ (see [PSU13, Section 2] for more details).

7.4.2 Proof of Theorem 7.1.3

Like in [Gui17a], we introduce the Szegö projector in the fibers using the classical Fourier decomposition :

$$S : C^\infty(SM_e) \to C^\infty(SM_e), \qquad S(u) = \sum_{k\geq 1} u_k$$

This operator extends as a self-adjoint bounded operator on $L^2(SM_e)$ and as a bounded operator on $H^s(SM_e)$ for all $s \in \mathbb{R}$. By duality, it extends continuously to $C^{-\infty}(SM_e)$ using the L^2-pairing, according to the formula $\langle S(u), v\rangle = \langle u, S(v)\rangle$, for $u \in C^{-\infty}(SM_e)$ and $v \in C^\infty(SM_e)$.

The Hilbert transform is defined as :

$$H : C^\infty(SM_e) \to C^\infty(SM_e), \qquad H(u) = -i \sum_{k\in\mathbb{Z}} \mathrm{sgn}(k) u_k,$$

with the convention that $\mathrm{sgn}(0) = 0$. It extends as a bounded skew-adjoint operator on $L^2(SM_e)$ and thus defines by duality a continuous operator on $C^{-\infty}(SM_e)$, using the L^2-pairing $\langle H(u), v\rangle = -\langle u, H(v)\rangle$, for $u \in C^{-\infty}(SM_e), v \in C^\infty(SM_e)$. In particular, the Szegö projector can be rewritten using the Hilbert transform, according to the formula :

$$S(u) = \frac{1}{2}\left((\mathbb{1} + iH)(u) - u_0\right), \tag{7.4.3}$$

for $u \in C^{-\infty}(SM_e)$ (where $u_0 = \frac{1}{2\pi}\pi_0^*(\pi_{0*}u)$). We have the following commutation relation (see [Gui17a] for instance), valid for $u \in C^{-\infty}(SM)$ in the sense of distributions :

Lemma 7.4.1. $XSu = SXu - \eta_+ u_0 + \eta_- u_1$

We can now prove a similar result to [Gui17b, Proposition 5.10] :

Lemma 7.4.2. *Under the assumptions of Theorem 7.1.3, given $f_1 \in C^\infty(M, T^*M)$ satisfying $D^* f_1 = 0$, there exists $w \in \cap_{p<\infty} L^p(SM_e)$ such that $Xw = 0$ in SM_e° and $\pi_{1*}w = f_1$ in M. Moreover, we can take w odd i.e. without even frequencies in its Fourier decomposition.*

Proof. The first part of the statement is an immediate consequence of Theorem 7.1.2 and the s-injectivity of I_1^e [Gui17b, Theorem 5]. The second part comes from the fact that if $w \in C^{-\infty}(SM)$ satisfies $Xw = 0$, then $Xw_{\mathrm{odd}} = Xw_{\mathrm{even}} = 0$. Moreover, $\pi_{1*}w$ only depends on w_1 and w_{-1} (for $f \in C^\infty(M, T^*M)$, $\langle \pi_{1*}w, f\rangle = \langle w, \pi_1^* f\rangle = \langle w_{-1} + w_1, \pi_1^* f\rangle$ since $\pi_1^* f \in \Omega_{-1} \oplus \Omega_1$), which implies that $\pi_{1*}w = \pi_{1*}w_{\mathrm{odd}}$. As a consequence, we can take w_{odd} and the result still holds. The regularity $w_{\mathrm{odd}} \in \cap_{p<\infty} L^p(SM_e)$ is a consequence of the fact that $w_{\mathrm{odd}} = \frac{1}{2}(\mathbb{1} - A^*)w \in L^p(SM)$ if $w \in L^p(SM)$, where A is the antipodal map in the fibers (it preserves the Liouville measure). $\square$

Lemma 7.4.3. *H extends as a bounded operator $H : L^p(SM) \to L^p(SM)$, for any $p \in (1, +\infty)$.*

Proof. First, let us note that given $p \geq 1$ and $u \in L^p(SM)$, we have that $x \mapsto \|u\|^p_{L^p(S_xM)} = \int_{S_xM} |u|^p dS_x$ is almost-everywhere defined and finite, and by integration over the fibers :

$$\|u\|^p_{L^p(SM)} = \int_{SM} |u|^p d\mu = \int_M \int_{S_xM} |u|^p dS_x \mathrm{dvol}(x) = \int_M \|u\|^p_{L^p(S_xM)} \mathrm{dvol}(x)$$

Since H acts separately on each fiber, we are reduced to proving the lemma on the circle $\mathbb{S}^1$ endowed with a smooth measure $d\theta$. Now, it is clear that $H : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is bounded. The hard point, here, is to prove that $H : L^1(\mathbb{S}^1) \to L^{1,w}(\mathbb{S}^1)$ (the weak L^1-space) is bounded too. This is a classical fact in harmonic analysis for which we refer to [Tao]. Assuming this claim, we obtain by Marcinkiewicz interpolation theorem the boundedness of $H : L^p(\mathbb{S}^1) \to L^p(\mathbb{S}^1)$ for any $p \in (1, 2]$ and since H is formally skew-adjoint, this also provides its boundedness on $L^p(\mathbb{S}^1)$ for $p \geq 2$ by duality. □

We prove that for a w like in Lemma 7.4.2, $S(w)$ makes sense as a function on SM_e. More precisely :

Lemma 7.4.4. *S extends as a bounded operator $S : L^p(SM) \to L^p(SM)$, for any $p \in (1, +\infty)$.*

Proof. Using (7.4.3), we can write for $w \in C^\infty(SM)$, $S(w) = \frac{1}{2}(w + iH(w) - w_0) = \frac{1}{2}(\mathbb{1} + iH - \frac{1}{2\pi}\pi_0^*\pi_{0*})w$. Now, H is a bounded operator $L^p(SM) \to L^p(SM)$, for any $p \in (1, +\infty)$, and as mentioned in §7.1.1, $\pi_0^*\pi_{0*} : L^p(SM) \to L^p(SM)$ is bounded for any $p \in (1, +\infty)$. □

Lemma 7.4.4 shows that if $w(1), ..., w(m) \in \cap_{p<\infty} L^p(SM)$, then

$$S(w(1))...S(w(m)) \in \cap_{p<\infty} L^p(SM)$$

is well-defined. We can now prove Theorem 7.1.3 :

$$\pi_{m*} : \cap_{p<\infty} L^p_{\text{inv}}(SM) \to C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)$$

is surjective for a surface. According to [PZ16, Lemma 7.2], the proof actually boils down to the

Lemma 7.4.5. *Assume $a_m \in \Omega_m$ satisfies $\eta_- a_m = 0$. Then there exists a function $\omega \in \cap_{p<\infty} L^p(SM)$ such that $X\omega = 0$ and $\pi_{m*}\omega = \pi_{m*}a_m$.*

Proof. This relies on the fact that the canonical line bundle κ for a smooth compact surface with boundary is holomorphically trivial, that is, there exists a nowhere vanishing holomorphic section α (see [For81, Theorem 30.3] for a reference). As a consequence, $\kappa^{\otimes m}$ is trivial too, with non-vanishing section $\alpha^{\otimes m}$ and the element of $\kappa^{\otimes m}$ canonically associated to a_m (according to the mapping introduced in the previous Section) is of the form $v\alpha^{\otimes m}$ for some smooth complex-valued v. But according to the expression (7.4.1), if $a_m \in \Omega_m$ satisfies $\eta_- a_m = 0$ then $\bar{\partial}(\tilde{a_m} e^{m\lambda}) = 0$ which yields that v is holomorphic. Thus, we can write locally $\tilde{a_m} e^{m\lambda}(dz)^{\otimes m} = (v\alpha) \otimes \alpha^{\otimes(m-1)}$ and all the factors of the product are holomorphic.

In other words, $a_m = f(1)...f(m)$, where each $f(i) \in \Omega_1$ satisfies $\eta_- f(i) = 0$. Now, according to Lemma 7.4.2, we can find, for each $1 \leq i \leq m$, a $w(i) \in \cap_{p<\infty} L^p(SM_e)$ such that $Xw(i) = 0$ in SM_e°, $w(i)$ is odd and $\pi_{1*}w(i) = \pi_{1*}f(i)$ in M. Indeed, $\pi_{1*}f(i)$ is in $C^\infty(M, T^*M)$ and one has

$$\pi_0^*\left(D^*(\pi_{1*}f(i))\right) = \eta_+(f(i))_{-1} + \eta_-(f(i))_1 = 0,$$

since $f(i) = (f(i))_1 \in \Omega_1$ satisfies $\eta_- f(i) = 0$. So $D^*(\pi_{1*}f(i)) = 0$ and the hypothesis of the Lemma 7.4.2 are satisfied.

Note in particular that since $w(i) \in L^2(SM)$, the equality $\pi_{1*}w(i) = \pi_{1*}f(i)$ also provides

$$\pi_1^*\pi_{1*}w(i) = c_1(w(i)_1 + w(i)_{-1}) = \pi_1^*\pi_{1*}f(i) = c_1 f(i)_1,$$

that is $w(i)_1 = f(i)_1 \in \Omega_1$ and $w(i)_{-1} = 0$. Thus, each $w(i)$ satisfies $\eta_-(w(i))_1 = \eta_- f(i) = 0$ and $\eta_+(w(i))_0 = 0$ insofar as it is odd. As a consequence, applying the commutation relation stated in Lemma 7.4.1, we obtain

$$XS(w(i)) = S\left(Xw(i)\right) = 0$$

and $\pi_{1*}(S(w(i))) = \pi_{1*}(w(i)) = \pi_{1*}f(i)$.

Thus, we can define $\omega := S(w(1))...S(w(m)) \in \cap_{p<+\infty} L^p(SM)$ and it satisfies $X\omega = 0$ on SM. By construction, we have $\omega_m = f(1)...f(m) = a_m \in \Omega_m$ and $\omega_l = 0$ for $l < m$ on M. We conclude that $\pi_{m*}\omega = \pi_{m*}a_m$ on M.

□

Remark 7.4.1. The proof relies on the fact that we are here able to find sufficiently regular invariant distributions $w \in \cap_{p<\infty} L^p(SM_e)$ such that, given $f_1 \in C^\infty_{\text{sol}}(M, T^*M)$, we have $\pi_{1*}w = f_1$, and that $\cap_{p<\infty} L^p(SM_e)$ is an algebra. Had we not been able to obtain such a regularity, one could have skirted this issue by analyzing the kernel of the Szegö projector (see [Gui17a, Lemma 3.10]) and proving that the multiplication $S(w)S(v)$ at least makes sense as a distribution, using [Hö3, Theorem 8.2.10].

7.5 Proof of the local marked boundary rigidity

We now use the s-injectivity of the X-ray transform I_2 proved in Theorem 7.1.1 in order to prove results of local rigidity, namely Theorems 7.1.4 and 7.1.5.

7.5.1 Technical tools

Let us fix some $\varepsilon > 0$ so that any metric g' in an ε-neighborhood of g (with respect to the C^2 topology) is simple with topology. We also assume that I_2^e is s-injective on M_e.

Reduction of the problem. The metric g is solenoidal with respect to itself since $D^*g = -\operatorname{Tr}_{12}(\nabla g) = 0$ ($\nabla g = 0$ since ∇ is the Levi-Civita connection). Like in the closed setting (see Lemma B.1.7), any metric in a vicinity of g is actually isometric to a solenoidal metric (with respect to g). We recall that $N = \lfloor \frac{n+1}{2} \rfloor + 1$.

Proposition 7.5.1 ([CDS00], Theorem 2.1)**.** *Let $N \geq 2, \alpha \in (0,1)$. There exists a $C^{N,\alpha}$-neighborhood W of g such that for any $g' \in W$, there exists a $C^{N+1,\alpha}$-diffeomorphism $\phi : M \to M$ preserving the boundary, such that $g'' = \phi^* g'$ is solenoidal with respect to the metric g. Moreover, if W is chosen small enough, one can guarantee that $\|g'' - g\|_{C^N} < \varepsilon$.*

We can thus reduce ourselves to the case where g' is solenoidal with respect to the metric g. We introduce $f := g' - g$, which is, by construction, C^N, solenoidal and satisfies $\|f\|_{C^N} < \varepsilon$. Our goal is to prove that $f \equiv 0$. We define $g_\tau := g + \tau f$ for $0 \leq \tau \leq 1$. As mentioned earlier, since f is small enough, each of these metrics have strictly convex boundary, a hyperbolic trapped set and no conjugate points. From now on, we assume that $d_g = d_{g'}$.

Lemma 7.5.1. *$I_2(f) \geq 0$ almost everywhere.*

Proof. Let $\widetilde{M}$ denote the universal cover of M. We lift all the objects to the universal cover and denote them by $\widetilde{\cdot}$. We consider $(p,\xi) \in \partial_-S\widetilde{M} \setminus \widetilde{\Gamma}_-$ and denote by $q \in \widetilde{M}$ the endpoint of the geodesic generated by (p,ξ). By [GM18, Lemma 2.2], we know that for each $\tau \in [0,1]$, there exists a unique g_τ-geodesic $\gamma_\tau : [0,1] \to \widetilde{M}$ with endpoints p and q. Note that γ_τ depends smoothly on τ[5]. We introduce the energy $E(\tau) := \int_0^1 \widetilde{g}_\tau(\dot{\gamma_\tau}(s), \dot{\gamma_\tau}(s))ds$. The arguments of [CDS00, Proposition 3.1] apply here as well : they prove that E is a C^2 function on $[0,1]$ which is concave. Moreover, since the boundary distance of $\widetilde{g}$ and $\widetilde{g}'$ agree, one has $E(0) = E(1)$. This implies that $E'(0) \geq 0$, but one can see that $E'(0) = \widetilde{I}_2(\widetilde{f})(p,\xi)$. Eventually, since $\partial_-S\widetilde{M} \cap \widetilde{\Gamma}_-$ has zero measure (with respect to $d\widetilde{\mu}_\nu$) by Proposition 7.1.1, we obtain the result on the universal cover and projecting $\widetilde{f}$ on the base, we obtain the sought result. □

Notice that, since $\pi_2^* g \equiv 1$ on SM, one has for some constant $c_2 > 0$:

$$\begin{aligned}\langle g, f\rangle_{L^2(\otimes_S^2 T^*M)} &= c_2 \langle \pi_2^* g, \pi_2^* f\rangle_{L^2(SM)} \\ &= c_2 \int_{SM} \pi_2^* f(x,v) d\mu(x,v) \\ &= c_2 \int_{\partial_- SM} I_2(f)(x,v) d\mu_\nu(x,v),\end{aligned}$$

where the last equality follows from Santaló's formula. But since $I_2(f) \geq 0$ almost everywhere, one gets :

$$\langle g, f\rangle_{L^2(\otimes_S^2 T^*M)} = c_2 \int_{\partial_- SM} I_2(f)(x,v) d\mu_\nu(x,v) = c_2 \|I_2(f)\|_{L^1(\partial_- SM)}$$

We will now prove an estimate on the L^1-norm of $I_2(f)$ which is crucial in our proof. It is based on the equality of the volume of g and g', which is a consequence of the fact that their marked boundary distance functions coincide because ϕ is isotopic to the identity. Indeed, one can first construct a diffeomorphism $\psi : M \to M$ such that $\psi|_{\partial M} = \mathbb{1}$ and both $g_0 := \psi^* g$ and g' coincide at all points of ∂M (it is a well-known fact for simple metrics and was proved in [GM18, Lemma 2.3] in our case). Note that $\mathrm{vol}(g_0) = \mathrm{vol}(g)$ and that the marked boundary distance function of g_0 and g' still coincide. By [GM18, Lemma 2.4], this implies that the metrics g_0 and g' have same lens data, which, in turn, implies the equality of the two volumes by Santalo's formula (see [GM18, Lemma 2.5]).

Lemma 7.5.2. *There exists a constant $C > 0$, such that :*

$$\|I_2(f)\|_{L^1(\partial_- SM)} \leq C \|f\|^2_{L^2(M, \otimes_S^2 T^*M)}$$

Proof. Consider a finite atlas (U_i, φ_i) on M and a partition of unity $\sum_i \chi_i = 1$ subordinated to this atlas, i.e. such that $\mathrm{supp}(\chi_i) \subset U_i$. One has for $\tau \in [0,1]$:

$$\mathrm{vol}(g_\tau) = \sum_i \int_{\varphi_i(U_i)} \chi_i \circ \varphi_i^{-1} \sqrt{\det(g_\tau(x))} dx,$$

5. Indeed, $\tau \mapsto g_\tau$ depends smoothly on τ, so $\xi_\tau := \left(\exp_p^{g_\tau}\right)^{-1}(q)$ depends smoothly on τ. Thus $(t,\tau) \mapsto \varphi_t^{g_\tau}(p,\xi_\tau)$ is smooth in both variables and by the implicit function theorem, the length $\ell_+^{g_\tau}(p,\xi_\tau)$ is smooth in τ. Thus, the reparametrized geodesic γ_τ depends smoothly on τ.

where dx denotes the Lebesgue measure and $g_\tau(x)$ the matrix representing the metric in coordinates. In [CDS00], Proposition 4.1, it is proved that for $\|f\|_{C^0} < \varepsilon$ (which is our case), one has pointwise :

$$\sqrt{\det(g_\tau(x))} \geq \sqrt{\det(g(x))} \left(1 + \frac{1}{2}\tau\langle g(x), f(x)\rangle_g - \frac{1}{4}\tau^2\ |f(x)|_g^2 - C\varepsilon\tau^3|f(x)|_g^2\right),$$

where the inner products are computed with respect to the metric (see Appendix B). Inserting this into the previous integral, we obtain :

$$\mathrm{vol}(g_\tau) \geq \mathrm{vol}(g) + \frac{1}{2}\tau\langle g, f\rangle_{L^2} - \frac{1}{4}\tau^2\|f\|_{L^2}^2 - C\varepsilon\tau^3\|f\|_{L^2}^2$$

Taking $\tau = 1$ and using the fact that $\mathrm{vol}(g') = \mathrm{vol}(g)$, we obtain the sought result. □

Remark 7.5.1. If (M, g) were a simple manifold, then a well-known Taylor expansion (see [SU04, Section 9] for instance) shows that for $x, y \in \partial M$, one has :

$$d_{g'}(x, y) = d_g(x, y) + \frac{1}{2}I_2(f)(x, y) + R_g(f)(x, y),$$

where $I_2(f)(x, y)$ stands for the X-ray transform with respect to g along the unique geodesic joining x to y, $R_g(f)$ is a remainder satisfying :

$$|R_g(f)(x, y)| \ \lesssim |x - y| \cdot \|f\|_{C^1(M)}^2$$

As a consequence, if the two boundary distances agree, one immediately gets that

$$\|I_2(f)\|_{L^\infty(\partial_- SM)} \lesssim \|f\|_{C^1(M)}^2$$

In our case, because of the trapping issues, $I_2(f)$ is not L^∞ and such an estimate is hopeless. This is why we have to content ourselves with L^1/L^2 estimates in Lemma 8.3.1 (and this will be sufficient in the end) but the idea that linearizing the problem brings an inequality with a square is unchanged.

Functional estimates. Given a tensor f defined on M, $E_0 f$ denotes its extension by 0 to M_e, whereas $r_M f$ denotes the restriction to M of a tensor defined on M_e. If $f \in H^{1/4}(M, \otimes_S^2 T^*M)$, then $E_0 f \in H^{1/4}(M_e, \otimes_S^2 T^*M_e)$ (see [Tay11a, Corollary 5.5]) and we can decompose the extension $E_0 f$ into $E_0 f = q + Dp$, where $q \in H^{1/4}_{\mathrm{sol}}(M_e, \otimes_S^2 T^*M_e)$ and $p \in H^{5/4}(M_e, \otimes_S^2 T^*M_e)$, with $p|_{\partial M_e} = 0$.

Lemma 7.5.3. *For any $r \geq 0$, there exists a constant $C > 0$ such that if $f \in H^{1/4}_{\mathrm{sol}}(M, \otimes_S^2 T^*M)$:*

$$\|f\|_{H^{-r}(M, \otimes_S^2 T^*M)} \leq C\|q\|_{H^{-r}(M, \otimes_S^2 T^*M)}$$

Actually, this lemma is valid not just for 1/4 but for any $0 < s < 1/2$. We chose to take a specific s in order to simplify the notations, and because it will be applied for a much regular f which will therefore be in $H^{1/4}$. Note that, from now on, in order to simplify the notations, we will sometimes write $\|T\|_{H^s(M)}$ in short, instead of $\|T\|_{H^s(M, \otimes_S^2 T^*M)}$.

Proof. We argue by contradiction. Assume we can find a sequence of elements $f_n \in H^{1/4}_{\text{sol}}(M, \otimes^2_S T^*M)$ such that :

$$\|f_n\|_{H^{-r}(M,\otimes^2_S T^*M)} > n\|q_n\|_{H^{-r}(M,\otimes^2_S T^*M)}$$

We can always assume that $\|f_n\|_{H^{1/4}(M)} = 1$ and thus :

$$\|q_n\|_{H^{-r}(M)} \leq \frac{1}{n}\|f_n\|_{H^{-r}(M)} \lesssim \frac{1}{n}\|f_n\|_{H^{1/4}(M)} \to 0$$

Now, by compactness, we can extract subsequences so that :

$$\begin{gathered} f_n \rightharpoonup f \in H^{1/4}_{\text{sol}}(M, \otimes^2_S T^*M) \\ f_n \to f \ \text{ in } L^2(M, \otimes^2_S T^*M) \\ \\ p_n \rightharpoonup p \in H^{5/4}(M_e, \otimes^2_S T^*M_e) \\ p_n \to p \ \text{ in } H^1(M_e, \otimes^2_S T^*M_e) \\ \\ q_n \rightharpoonup q \in H^{1/4}_{\text{sol}}(M_e, \otimes^2_S T^*M_e) \\ q_n \to q \ \text{ in } L^2(M_e, \otimes^2_S T^*M_e) \end{gathered}$$

Remark that the decomposition $E_0 f_n = q_n + Dp_n$ implies, when passing to the limit in L^2, that $E_0 f = q + Dp$. Since $\|q_n\|_{H^{-r}(M)} \to 0$, we have that $q \equiv 0$ in M. In $M_e \setminus M$, we have $q = -Dp$. Thus :

$$0 = \langle D^* q, p\rangle_{L^2(M_e)} = \langle q, Dp\rangle_{L^2(M_e)} = \langle q, Dp\rangle_{L^2(M_e \setminus M)} = -\|q\|^2_{L^2(M_e\setminus M)},$$

that is $q \equiv 0$. As a consequence, in $M_e \setminus M^\circ$, $E_0 f = 0 = Dp$ and $p|_{\partial M_e} = 0$, so $p \equiv 0$ in $M_e \setminus M^\circ$ by unique continuation. Since $p \in H^{5/4}$, by the trace theorem, we obtain that $p|_{\partial M} = 0$ (in $H^{3/4}(\partial M)$). Since f is solenoidal, $D^* f = 0$, and

$$0 = \langle D^* f, p\rangle_{L^2(M)} = \langle D^* Dp, p\rangle_{L^2(M)} = \|Dp\|^2_{L^2(M)}$$

Therefore, $p \equiv 0$ and, in particular, in M, we get that $f = 0$ which is contradicted by the fact that $\|f_n\|_{H^{1/4}(M)} = 1$. ☐

We recall that I^e_2 is assumed to be injective. Let us mention that if a tensor $u \in C^\infty(M_e, \otimes^2_S T^*M_e)$ is in the kernel of Π^e_2, then :

$$0 = \langle \Pi^e_2 u, u\rangle = \langle I^{e*}_2 I^e_2 u, u\rangle = \|I^e_2 u\|^2,$$

that is $I^e_2 u = 0$. This will be used in the following lemma :

Lemma 7.5.4. *Under the assumption that I^e_2 is injective, for any $r \geq 0$, there exists a constant $C > 0$ such that if $f \in H^{1/4}_{\text{sol}}(M, \otimes^2_S T^*M)$, then :*

$$\|f\|_{H^{-r-1}(M,\otimes^2_S T^*M)} \leq C\|\Pi^e_2 E_0 f\|_{H^{-r}(M_e,\otimes^2_S T^*M_e)}$$

Proof. Let χ be a smooth positive function supported within M°_e such that $\chi \equiv 1$ in a vicinity of M. We know by [Gui17b], that there exists pseudodifferential operators Q, S, R with respective order $1, -2, -\infty$ on M°_e such that :

$$Q\chi\Pi^e_2\chi = \chi^2 + D\chi S\chi D^* + R$$

Let us decompose $E_0 f = q + Dp$, where $q \in H^{1/4}_{\text{sol}}(M_e, \otimes^2_S T^*M_e)$ and Dp is the potential part given by $p := \Delta^{-1} D^* E_0 f$, $\Delta = D^*D$ being the Laplacian with Dirichlet conditions. Remark that $\chi E_0 f = E_0 f$, and

$$\begin{aligned} r_M Q \chi \Pi_2^e(E_0 f) &= r_M Q \chi \Pi_2^e(\chi E_0 f) \\ &= r_M Q \chi \Pi_2^e \chi(q) + \underbrace{r_M Q \chi \Pi_2^e D(\chi p)}_{=0} + r_M Q \chi \Pi_2^e [\chi, D](p) \\ &= r_M(q) + r_M R(q) + r_M Q \chi \Pi_2^e [\chi, D] \Delta^{-1} D^* E_0(f) \end{aligned}$$

Note that $[\chi, D]$ is a differential operator supported in the annulus $\{\nabla \chi \neq 0\}$. In particular, $r_M T := r_M Q \chi \Pi_2^e [\chi, D] \Delta^{-1} D^* E_0 : H^{-r-1} \to H^{-r-1}$ is a well-defined compact operator on M. Using the previous lemma, we obtain :

$$\begin{aligned} \|f\|_{H^{-r-1}(M)} &\lesssim \|q\|_{H^{-r-1}(M)} \\ &\lesssim \|r_M Q \chi \Pi_2^e E_0 f\|_{H^{-r-1}(M)} + \|r_M R q\|_{H^{-r-1}(M)} + \|r_M T f\|_{H^{-r-1}(M)} \\ &\lesssim \|\Pi_2^e E_0 f\|_{H^{-r}(M_e)} + \|r_M R q\|_{H^{-r-1}(M)} + \|r_M T f\|_{H^{-r-1}(M)} \end{aligned}$$

In other words, there exists a constant $C > 0$ such that :

$$\|f\|_{H^{-r-1}(M)} \leq C(\|\Pi_2^e E_0 f\|_{H^{-r}(M_e)} + \|r_M R q\|_{H^{-r-1}(M)} + \|r_M T f\|_{H^{-r-1}(M)}) \quad (7.5.1)$$

The rest of the proof now boils down to a standard argument of functional analysis. Assume by contradiction that we can find a sequence of elements $f_n \in H^{1/4}(M, \otimes^2_S T^*M)$ such that

$$\|f_n\|_{H^{-r-1}(M, \otimes^2_S T^*M)} > n \|\Pi_2^e E_0 f_n\|_{H^{-r}(M_e, \otimes^2_S T^*M_e)}$$

We can always assume that $\|f_n\|_{H^{-r-1}} = 1$ and thus $\|\Pi_2^e E_0 f_n\|_{H^{-r}} \to 0$. By construction, $\|q_n\|_{H^{-r-1}} \lesssim \|f_n\|_{H^{-r-1}} = 1$, i.e. (q_n) is bounded in H^{-r-1}. Moreover, since $r_M R$ and $r_M T$ are compact, we know that up to a subsequence $r_M R q_n \to v_1, r_M T f_n \to v_2$, with $v_1, v_2 \in H^{-r-1}(M, \otimes^2_S T^*M)$. As a consequence, $(r_M R q_n)_{n \geq 0}, (r_M T f_n)_{n \geq 0}$ are Cauchy sequences and applying (7.5.1) with $f_n - f_m$, we obtain that $(f_n)_{n \geq 0}$ is a Cauchy sequence too. It thus converges to an element $f \in H^{-r-1}_{\text{sol}}(M, \otimes^2_S T^*M)$ which satisfies $\Pi_2^e E_0 f = 0$. But we claim that $\Pi_2^e E_0$ is injective on $H^{-r-1}_{\text{sol}}(M, \otimes^2_S T^*M)$. Assuming this claim, this implies that $f = 0$, which contradicts the fact that $\|f_n\|_{H^{-r-1}} = 1$.

Let us now prove the injectivity. Assume $\Pi_2^e E_0 f = 0$ for some $f \in H^{-r-1}_{\text{sol}}(M, \otimes^2_S T^*M)$. Since $E_0 f$ has compact support within M_e°, we can still make sense of the decomposition $E_0 f = q + Dp$, where $p := \Delta^{-1} D^* E_0 f \in H^{-r}(M_e, \otimes^2_S T^*M_e)$, $\Delta := D^*D$ is the Laplacian with Dirichlet conditions and $q := E_0 f - Dp \in H^{-r-1}_{\text{sol}}(M_e, \otimes^2_S T^*M_e)$ (in the sense that $D^*q = 0$ in the sense of distributions). By ellipticity of Δ, p has singular support contained in ∂M (since $\Delta p = D^* E_0 f$), and the same holds for Dp. Moreover :

$$\Pi_2^e(E_0 f) = 0 = \Pi_2^e(q) + \Pi_2^e(Dp) = \Pi_2^e(q)$$

From $q = -Dp$ on $M_e \setminus M$, we see that q is smooth on $M_e \setminus M$ and since it is solenoidal on M_e and in the kernel of Π_2^e, it is smooth on M_e° (this stems from the ellipticity of Π_2^e on $\ker D^*$). As a consequence, $q \in C^\infty_{\text{sol}}(M_e, \otimes^2_S T^*M_e) \cap \ker I_2^e$ and thus $q = 0$ by s-injectivity of the X-ray transform. We have $E_0 f = Dp$ and $E_0 f = 0$ on $M_e \setminus M$, $p|_{\partial M_e} = 0$. By unique continuation, we obtain that $p = 0$ in $M_e \setminus M$. Now, by ellipticity, one can also find pseudo-differential operators Q, S, R on M_e° of respective order $1, -2, -\infty$, such that :

$$Q \Pi_2^e = \mathbb{1}_{M_e^\circ} + DSD^* + R,$$

where S is a parametrix of D^*D. Since $E_0 f = Dp$ has compact support in M_e°, we obtain :

$$\begin{aligned} Q\Pi_2^e E_0 f &= 0 \\ &= Q\Pi_2^e Dp \\ &= Dp + DSD^*Dp + Rp \\ &= 2Dp + \text{ smooth terms} \end{aligned}$$

This implies that $E_0 f = Dp$ is smooth on M_e, vanishes on ∂M. Therefore :

$$\langle f, f \rangle_{L^2(M)} = \langle f, Dp \rangle = \langle D^* f, p \rangle = 0,$$

that is $f \equiv 0$. □

For $s \in \mathbb{R}$, we define $H^s_{\text{inv}}(SM)$ to be the set of $u \in H^s(SM)$ such that $Xu = 0$ (in the sense of distributions if $s < 1$). The following lemma will allow us some gain in the "battle" of exponents in the proof of the Theorem.

Lemma 7.5.5. *For all $s \in \mathbb{R}$, $m \geq 0$,*

$$\pi_{m*} : H^s_{inv}(SM) \to H^{s+1/2}(M, \otimes_S^m T^* M)$$

is bounded (and the same result holds for M_e).

Proof. We fix $s \in \mathbb{R}$. The idea is to see π_{m*} as an averaging operator in order to apply Gérard-Golse's result of regularity ([GG92, Theorem 2.1]). In local coordinates, given $f \in C^\infty(SM)$, one has (see [PZ16, Section 2] for instance) :

$$\pi_{m*} f(x)_{i_1 \ldots i_m} = g_{i_1 j_1}(x) \ldots g_{i_m j_m}(x) \int_{S_x M} f(x, \xi) \xi^J dS_x(\xi),$$

where $\xi^J = \xi^{j_1} \ldots \xi^{j_m}$. It is thus sufficient to prove that the $H^{s+1/2}$-norm of each of these coordinates is controlled by the H^s-norm of f. Since (M, g) is smooth, it is actually sufficient to control the $H^{s+1/2}$-norm of the integral. Note that

$$\|X(f\xi^J)\|_{H^s(SM)} \lesssim \|f\|_{H^s(SM)} + \|Xf\|_{H^s(SM)} \tag{7.5.2}$$

Since X satisfies the transversality assumption of [GG92, Theorem 2.1], we conclude that $u : x \mapsto \int_{S_x M} f(x, \xi) \xi^J dS_x(\xi)$ is in $H^{s+1/2}(M)$. By (7.5.2), we also know that its $H^{s+1/2}$-norm is controlled by

$$\|u\|_{H^{s+1/2}(SM)} \lesssim \|f\|_{H^s(SM)} + \|Xf\|_{H^s(SM)}. \tag{7.5.3}$$

Now, if $f \in H^s_{\text{inv}}(SM)$, there exists by [DZ, Lemma E.47] a sequence of smooth functions $f_n \in C^\infty(SM)$ such that $f_n \to f, Xf_n \to Xf = 0$ in $H^s(SM)$. We obtain the sought result by passing to the limit in (7.5.3). □

We will apply this lemma with $m = 2$. Eventually, the following lemma is stated for Sobolev spaces in [PZ16, Lemma 6.2], but the same result holds for Lebesgue spaces. The proof relies on the fact that, by construction of the extension M_e, there exists a maximum time $L < +\infty$ for a point in $\partial_- SM_e$ to either exit SM_e or to hit $\partial_- SM$.

Lemma 7.5.6. *Let $1 \leq p < +\infty$. There exists a constant $C > 0$ such that if $f \in L^1(M, \otimes_S^2 T^* M)$ is a section such that $I_2(f) \in L^p(\partial_- SM)$ and $E_0 f$ denotes its extension by 0 to M_e, one has :*

$$\|I_2^e(E_0 f)\|_{L^p(\partial_- SM_e)} \leq C \|I_2(f)\|_{L^p(\partial_- SM)} \tag{7.5.4}$$

7.5.2 End of the proof

We now have all the ingredients to conclude the proof of Theorem 7.1.5. Note that there are arbitrary choices made as to the functional spaces considered. The bounds we obtain are clearly not optimal, but this is of no harm as to the content of the theorem. In particular, we are limited by the Sobolev injection used in the proof, which depends on the dimension : this is why we loose regularity in the theorem as the dimension increases. We recall that $n+1$ is the dimension of M.

Proof of the Theorem. We already know by Lemma 8.3.1 that

$$\|I_2(f)\|_{L^1(\partial_-SM)} \lesssim \|f\|^2_{L^2(M,\otimes^2_S T^*M)}$$

We recall that $N = \lfloor \frac{n+2}{2} \rfloor + 1 > \frac{n+2}{2}$. We fix $q \in (1,2)$ close to 1 and set $s = (n+1)\left(\frac{1}{q} - \frac{1}{2}\right)$, the exponent of the Sobolev injection $L^q \hookrightarrow H^{-s}$. Interpolating L^2 between the Sobolev spaces $H^{-s-1/2}$ and H^N, we obtain for $\gamma = \frac{N}{s+1/2+N}$:

$$\|I_2(f)\|_{L^1(\partial_-SM)} \lesssim \|f\|^2_{L^2} \lesssim \|f\|^{2\gamma}_{H^{-s-1/2}} \|f\|^{2(1-\gamma)}_{H^N} \lesssim \|f\|^{2\gamma}_{H^{-s-1/2}} \|f\|^{2(1-\gamma)}_{C^N}$$

Moreover, by Lemma 8.3.3, we have that for $p > 1$ large enough and for $\delta > 0$ as small as wanted, $\|I_2(f)\|_{L^p(\partial_-SM)} \lesssim \|f\|_{L^{p+\delta}(M,\otimes^2_S T^*M)} \lesssim \|f\|_{L^\infty(M,\otimes^2_S T^*M)}$. By interpolation, we obtain that :

$$\begin{aligned} \|I_2(f)\|_{L^{q+\delta}(\partial_-SM)} &\lesssim \|I_2(f)\|^{\theta}_{L^1} \|I_2(f)\|^{1-\theta}_{L^p} \\ &\lesssim \|f\|^{2\theta}_{L^2} \|f\|^{1-\theta}_{L^\infty} \\ &\lesssim \|f\|^{2\gamma\theta}_{H^{-s-1/2}} \|f\|^{2(1-\gamma)\theta}_{C^N} \|f\|^{1-\theta}_{L^\infty}, \end{aligned}$$

where $\theta \in [0,1]$ satisfies

$$\frac{1}{q+\delta} = \theta + \frac{1-\theta}{p} \tag{7.5.5}$$

As a consequence, we obtain :

$$\begin{aligned} \|f\|_{H^{-s-1/2}} &\lesssim \|\Pi_2^e E_0 f\|_{H^{-s+1/2}} && \text{by Lemma 7.5.4} \\ &\lesssim \|I^{e*} I_2^e E_0 f\|_{H^{-s}} && \text{by Lemma 7.5.5} \\ &\lesssim \|I^{e*} I_2^e E_0 f\|_{L^q} && \text{by Sobolev injection } L^q \hookrightarrow H^{-s} \\ &\lesssim \|I_2^e E_0 f\|_{L^{q+\delta}} && \text{by Proposition 7.2.1} \\ &\lesssim \|I_2 f\|_{L^{q+\delta}} && \text{by Lemma 7.5.6} \\ &\lesssim \|f\|^{2\gamma\theta}_{H^{-s-1/2}} \|f\|^{2(1-\gamma)\theta}_{C^N} \|f\|^{1-\theta}_{L^\infty} \end{aligned}$$

Remark that we can choose q as close we want to 1, thus s close enough to $(n+1)/2$ and θ close enough to $1/q$. In the limit $q = 1, s = (n+1)/2, \hat{\theta} = 1/q, \hat{\gamma} = \frac{N}{(n+1)/2+1/2+N}$, we have :

$$2\hat{\gamma}\hat{\theta} = \frac{2N}{(n+1)/2+1/2+N} > 1,$$

since $N = \lfloor \frac{n+2}{2} \rfloor + 1 > \frac{n+1}{2}$. As a consequence, we can always make some choice of constants q, p, δ which guarantees that $2\gamma\theta > 1$. Now, if f were not zero, one would obtain :

$$C \leq \|f\|^{2\gamma\theta-1}_{H^{-s-1/2}} \|f\|^{2(1-\gamma)\theta}_{C^N} \|f\|^{1-\theta}_{L^\infty} \leq C' \varepsilon^{\theta},$$

for some constants C and C', independent of f, and we get a contradiction, provided ε is chosen small enough at the beginning.

□

Chapitre 8

Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces

> « *Je hais les voyages et les explorateurs. Et voici que je m'apprête à raconter mes expéditions. Mais que de temps pour m'y résoudre !* »
>
> *Tristes Tropiques*, Claude Lévi-Strauss

This chapter contains the article *Boundary rigidity of negatively-curved asymptotically hyperbolic surfaces*, published in **Commentarii Mathematici Helvetici**.

Sommaire

In the spirit of Otal [Ota90] and Croke [Cro90], we prove that a negatively-curved asymptotically hyperbolic surface is *marked boundary distance rigid*, where the distance between two points on the boundary at infinity is defined by a renormalized quantity.

8.1 Introduction

8.1.1 Main result

We consider $\overline{M}$ a smooth compact connected $(n+1)$-dimensional manifold with boundary. We say that $\rho : \overline{M} \to \mathbb{R}_+$ is a *boundary defining function* on $\overline{M}$ if it is smooth and satisfies $\rho = 0$ on ∂M, $d\rho \neq 0$ on ∂M and $\rho > 0$ on M. Let us fix such a function ρ. A metric g on M is said to be *asymptotically hyperbolic* if

1. the metric $\overline{g} = \rho^2 g$ extends to a smooth metric on $\overline{M}$,
2. $|d\rho|_{\rho^2 g} = 1$ on ∂M.

Note that these two properties are independent of the choice of ρ because any other boundary function ρ_0 can be written $\rho_0 = e^f \rho$ and $\overline{g_0} = \rho_0^2 g = e^{2f}\ \rho^2 g$ also extends smoothly on ∂M and satisfies on the boundary :

$$\left|d(e^f \rho)\right|_{e^{2f}\rho^2 g} = e^{-f}\left|e^f d\rho\right|_{\rho^2 g} = |d\rho|_{\rho^2 g} = 1$$

However, the extension of the metric $\rho^2 g$ on the boundary, that is $\rho^2 g|_{\partial M}$, is not independent of the choice of ρ but its conformal class is. This conformal class of metrics on ∂M is called the *conformal infinity*. In the rest of the paper, $\overline{M}$ will be two-dimensional, so ∂M will be one-dimensional and in this case, all the metrics are conformally equivalent. As a consequence, this statement is rather pointless but it takes another interest if the manifold has dimension superior or equal to three.

Such a manifold admits a canonical product structure in a neighborhood of the boundary ∂M (see [Gra00] for instance) that is, given a metric h_0 on ∂M (in the conformal class $[\rho^2 g|_{T\partial M}]$), there exists a smooth set of coordinates (ρ, y) on $\overline{M}$ (where ρ is a boundary defining function) such that $|d\rho|_{\rho^2 g} = 1$ in a neighborhood of ∂M and $\rho^2 g|_{T\partial M} = h_0$. The function ρ is uniquely determined by h_0 in a neighborhood of ∂M. Moreover, on a collar neighborhood near ∂M, the metric has the form

$$g = \frac{d\rho^2 + h_\rho}{\rho^2}, \quad \text{on } (0,\varepsilon) \times \partial M, \tag{8.1.1}$$

for some $\varepsilon > 0$ and where h_ρ is a smooth family of metrics on ∂M. From this expression, one can prove that the sectional curvatures of (M, g) all converge towards -1 as ρ goes to 0.

The manifold M is not compact and the length of a geodesic $\alpha(x, x')$ joining two points x and x' on the boundary at infinity is clearly not finite. However, in [GGSU17], a *renormalized length* $L(\alpha(x, x'))$ for a geodesic $\alpha(x, x')$ is introduced, which roughly consists in the constant term in the asymptotic development of the length of $\alpha_\varepsilon(x, x') := \alpha(x, x') \cap \{\rho \geq \varepsilon\}$ as ε goes to 0. This yields a new object characterized by the asymptotically hyperbolic manifold (M, g) and one can actually wonder, as usual in inverse problem theory, up to what extent it conversely determines (M, g). Notice that the renormalized length is not independent of the choice of the boundary defining function ρ, and thus, neither of the choice of the conformal representative h_0 in the conformal infinity.

From now on, we further assume that M has dimension 2 and is negatively-curved. If M is simply connected, then it is a well-known fact that there exists a unique geodesic between any pair of points $(x, x') \in \partial M \times \partial M \setminus \mathrm{diag}$, where diag is the diagonal in $\partial M \times \partial M$. The *renormalized boundary distance* is defined as :

$$D : \partial M \times \partial M \setminus \mathrm{diag} \to \mathbb{R}, \qquad D(x, x') = L(\alpha(x, x')),$$

where $L(\alpha(x, x'))$ denotes the renormalized length of the unique geodesic joining x to x'. In the terminology of [GGSU17], such surfaces are called *simple* : this definition naturally extends the notion of a simple manifold (compact manifold with boundary such that the exponential map is a diffeomorphism at each point) to the non-compact setting.

More generally, we will deal with the case of negatively-curved surfaces with topology. Then, the natural object one has to consider is the marked boundary *renormalized distance*. In this case, given two points $(x, x') \in \partial M \times \partial M \setminus \mathrm{diag}$, there exists a unique geodesic in each homotopy class $[\gamma] \in \mathcal{P}_{x,x'}$ of curves joining x to x' ($\mathcal{P}_{x,x'}$ being the set of homotopy classes). We define

$$\mathcal{D} := \{(x, x', [\gamma]), (x, x') \in \partial M \times \partial M \setminus \mathrm{diag}, [\gamma] \in \mathcal{P}_{x,x'}\},$$

and introduce the renormalized marked boundary distance D as :

$$D : \mathcal{D} \to \mathbb{R}, \qquad D(x, x', [\gamma]) = L(\alpha(x, x', [\gamma])), \tag{8.1.2}$$

where $\alpha(x, x', [\gamma])$ is the unique geodesic in $[\gamma]$ joining x to x' and L the renormalized length. Our main result is the following :

Theorem 8.1.1. *Assume (M, g_1) and (M, g_2) are two asymptotically hyperbolic surfaces with negative curvature. We suppose that g_1 and g_2 admit the same renormalized boundary distances, i.e. $D_1 = D_2$ for some choices h_1 and h_2 in the conformal infinity. Then, there exists a smooth diffeomorphism $\Phi : \overline{M} \to \overline{M}$ such that $\Phi^* g_2 = g_1$ on M and $\Phi|_{\partial M} = Id$.*

Notice that if $\Phi : \overline{M} \to \overline{M}$ is a diffeomorphism preserving the boundary, then $L_g = L_{\Phi^* g}$, where both renormalized lengths are computed with respect to the same representative in the conformal infinity. In other words, the previous theorem asserts that the action of the group of diffeomorphisms preserving the boundary is the only obstruction to the injectivity of the map $g \mapsto L_g$.

This result can be seen as an analogue of [GM18, Theorem 2] for the case of asymptotically hyperbolic surfaces. It is new even in the simply connected case, where the marked boundary distance is simply the ordinary renormalized boundary distance. It is very likely that one can relax the assumption in Theorem 8.1.1 so that only one of the two metrics has negative curvature (but still a hyperbolic trapped set). In the usual terminology, Theorem 8.1.1 roughly says that an asymptotically hyperbolic surface with negative curvature is *marked boundary distance rigid* among the class of surfaces having negative curvature.

This result follows in spirit the ones proved independently by Otal [Ota90] and Croke [Cro90] establishing that two negatively-curved closed surfaces with same marked length spectrum are isometric. More recently, Guillarmou and Mazzucchelli [GM18] extended Otal's proof to the case of two surfaces with strictly convex boundary without conjugate points and a trapped set of zero Liouville measure, one being of negative curvature. In both cases, the central object of interest is the *Liouville current* η, which is the natural projection of the Liouville measure μ (initially defined on the unit tangent bundle SM) on the set of geodesics $\mathcal{G}$ of the manifold. Our arguments follow in principle the layout of proof of these articles, but we need to address new issues caused by the loss of the compactness assumption. The crucial step in our proof to deal with the infinite ends of the manifold is a version of Otal's lemma (see [Ota90, Lemma 8]) with a stability estimate (Proposition 8.4.1). To the best of our knowledge, this bound had never been stated before in the literature. As far as we know, this is also the first boundary rigidity result obtained in a non-compact setting. Let us eventually mention that boundary rigidity questions appear naturally in the physics literature concerning the AdS/CFT duality and holography (see [PR04, CLMS15]).

8.1.2 Outline of the proof

In Section §8.2, we introduce the notion of renormalized length for a geodesic. We heavily rely on the cautious study made in [GGSU17] of the geodesic flow near the boundary at infinity. In Section §8.2.3, we recall the definition of the Liouville current η on the space of geodesics of the universal cover $\widetilde{M}$ and prove that if the renormalized marked lengths agree, then the Liouville currents agree, juste like in the compact setting.

Section §8.3 is devoted to the construction of an application of deviation κ. Like in [Ota90], we introduce *the angle of deviation* f between the two metrics on the universal cover $\widetilde{M}$. The idea is to make use of Gauss-Bonnet formula, in order to prove that this angle is the identity. This requires to introduce an *average angle of deviation*. Since we are in a non-compact setting, technical issues arise from the fact that the volume is infinite. In particular, we need to consider its average (denoted by Θ_ε) on compact domains $\{\rho \geq \varepsilon\}$ parametrized by ε and to study their limit as $\varepsilon \to 0$.

Because of the possible existence of a *trapped set*, we are unable to prove a priori that the averages Θ_ε are C^1 (or at least uniformly Lipschitz), which would truly simplify the proof. A cautious analysis of the derivative of the angle of deviation f is needed to deal with these technical complications. Combined with a version of Otal's lemma with an estimate (see Proposition 8.4.1), this allows to conclude that the average angle of deviation is the identity in the limit $\varepsilon \to 0$, which itself implies that the angle of deviation f is the identity. We then conclude the proof by constructing a natural application Φ which is an isometry between (M, g_1) and (M, g_2). Eventually, a last difficulty comes from the fact that it is not immediate that the isometry obtained is C^∞ down to the boundary of $\overline{M}$.

If the reader is familiar with Otal's proof [Ota90], he will morally see the same features appear, but the novelty here is that we are able to deal with the asymptotic ends of the manifold. The price we have to pay is that this requires to compute tedious estimates in the limit $\varepsilon \to 0$.

8.2 Geometric preliminaries

This section is not specific to the two-dimensional case, so we state it in full generality. (M, g) is only assumed to be an $(n+1)$-dimensional asymptotically hyperbolic manifold. In our setting, it will be more convenient to work on the unit cotangent bundle rather than on the unit tangent bundle, using the construction of Melrose [**?**] of b-bundles.

8.2.1 Geometry on the unit cotangent bundle

The b-cotangent bundle. The unit cotangent bundle is defined by

$$S^*M := \left\{(x,\xi) \in T^*M \mid x \in M, \xi \in T_x^*M, |\xi|_g^2 = 1\right\}, \tag{8.2.1}$$

and we denote by $\pi : S^*M \to M$ the projection on the base. The geodesic flow $(\varphi_t)_{t\in\mathbb{R}}$ is induced by the Hamiltonian vector field X, obtained from the Hamiltonian $H(x,\xi) = \frac{1}{2}|\xi|_g^2$. We will denote by $\flat : TM \to T^*M$ the Legendre transform between these two vector bundles, that is $v \mapsto g(v,\cdot)$, and by $\sharp : T^*M \to TM$ its inverse. We stress that we will often drop the notation of these isomorphisms and identify (without mentioning it) a vector with its dual covector.

There exists a canonical splitting of $T(S^*M)$ according to :

$$T(S^*M) = \mathbb{H} \oplus \mathbb{V}, \tag{8.2.2}$$

where $\mathbb{V} := \ker d\pi$ is the vertical bundle and $\mathbb{H} := \ker \mathcal{K}$ is the horizontal bundle. $\mathcal{K}$ is the connection map, defined for $(x,\xi) \in S^*M$, $Z \in T_{(x,\xi)}(S^*M)$, by $\mathcal{K}(Z) = \nabla_{\dot{x}}\xi^\sharp(0) \in T_xM$, where $t \mapsto z(t) = (x(t),\xi(t)) \in S^*M$ is any curve such that $z(0) = (x,\xi)$ and $\dot{z}(0) = Z$ (see [Pat99] for a reference). The metric g on M induces a natural metric G on S^*M, called the *Sasaki metric* and defined by :

$$G(Z,Z') := g(d\pi(Z), d\pi(Z')) + g(\mathcal{K}(Z),\mathcal{K}(Z')) \tag{8.2.3}$$

Recall from [Mel93] that the *b-tangent bundle* ${}^bT\overline{M} \to \overline{M}$ is defined to be the smooth vector bundle whose sections are vectors fields tangent to ∂M. Let V be a smooth vector field on M. If $(\rho, y_1, ..., y_n)$ denotes smooth local coordinates in a neighborhood of ∂M, we can write

$$V = a\partial_\rho + \sum_i b_i \partial_{y_i},$$

for some smooth functions a, b_i. If V vanishes on the boundary, then $a|_{\partial M} = 0$, and we can write $a = \rho\alpha$ for some smooth function α. In other words, in coordinates, $(\rho\partial_\rho, \partial_{y_i})$ is a local frame for ${}^bT\overline{M}$. Now, $\rho\partial_\rho$ is well defined on ∂M, independently of the choice of coordinates in a neighborhood of ∂M. Indeed, if (ρ', y') denotes another choice of coordinates, then one can write $\rho' = \rho A(\rho,y), y_i' = Y_i(\rho,y)$ for some smooth functions (such that $A(0,0) > 0$) and one has

$$\rho\partial_\rho = \left(1 + \frac{\rho}{A}\right)\rho'\partial_{\rho'} + \frac{\rho'}{A}\sum_j \partial_\rho(Y_j)\partial_{y_j'},$$

that is, both elements $\rho\partial_\rho$ and $\rho'\partial_{\rho'}$ agree on the boundary as elements of ${}^bT\overline{M}|_{\partial M}$.

The *b-cotangent bundle* ${}^bT^*\overline{M}$ is the vector bundle of linear forms on ${}^bT\overline{M}$. In coordinates, $(\rho^{-1}d\rho, dy_i)$ is a local frame of ${}^bT^*\overline{M}$ and $\rho^{-1}d\rho$ on ∂M (the covector associated to $\rho\partial_\rho$) is independent of any choice of coordinates (and of the metric g). From the coordinates $(\rho, y, \xi = \xi_0 d\rho + \sum_i \eta_i dy_i)$ on $T^*\overline{M}$, we introduce on ${}^bT^*\overline{M}$ the smooth coordinates $(x,\xi) = (\rho, y, \overline{\xi_0}, \eta)$, where $\xi_0 = \overline{\xi_0}\rho^{-1}$, that is $\xi = \overline{\xi_0}\rho^{-1}d\rho + \sum_i \eta_i dy_i$. In particular, we see from the previous discussion that the function $\xi \mapsto \overline{\xi}_0$ on ${}^bT^*\overline{M}|_{\partial M}$ is intrinsic to the manifold, as well as the two subsets $\left\{\overline{\xi}_0 = \pm 1\right\}$ of ${}^bT^*\overline{M}|_{\partial M}$ (they do not depend on the choice of coordinate (ρ, y), not even on the metric g).

Note that given $\xi = \overline{\xi_0}\rho^{-1}d\rho + \sum_i \eta_i dy_i \in {}^bT^*\overline{M}$, one has :

$$|\xi|_g^2 = \overline{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2,$$

where, here, h_ρ actually denotes the dual metric on $T^*\partial M$. We denote by :

$$\overline{S^*M} = \left\{(x,\xi) \in {}^bT^*\overline{M}, |\xi|_g^2 = 1\right\}$$

One has for $x \in \overline{M}$:

$$\overline{S^*_xM} = \left\{(x,\xi) \in {}^bT^*\overline{M}, \overline{\xi}_0^2 + \rho^2|\eta|_{h_\rho}^2 = 1\right\}$$

As a consequence, there is a splitting :

$$\overline{S^*M} = S^*M \sqcup \partial_-S^*M \sqcup \partial_+S^*M,$$

where $\partial_\pm S^*M = \left\{(x,\xi), x \in \partial M, \overline{\xi_0} = \mp 1\right\}$ (which are independent of any choice). We see ∂_-S^*M (resp. ∂_+S^*M) as the *incoming* (resp. *outcoming*) boundary.

Lemma 8.2.1. *[GGSU17, Lemma 2.1] There exists a smooth vector field $\overline{X}$ on $\overline{S^*M}$ which is transverse to the boundary $\partial\overline{S^*M} = \partial_-S^*M \sqcup \partial_+S^*M$ and satisfies $X = \rho\overline{X}$ on S^*M. Moreover, for $x \in \overline{M}$ sufficiently close to ∂M, in suitable local coordinates as before, we have $\overline{X} = \partial_\rho + \rho Y$, for some smooth vector field Y on $\overline{S^*M}$.*

The flow on $\overline{S^*M}$ induced by $\overline{X}$ will be denoted by $\overline{\varphi}_\tau$. For $z \in S^*M$ and $\tau > 0$ such that $\overline{\varphi}_s(z)$ is defined for $s \in [0,\tau]$, one has $\varphi_t(z) = \overline{\varphi}_\tau(z)$, where

$$t(\tau,z) = \int_0^\tau \frac{1}{\rho(\overline{\varphi}_s(z))}ds. \tag{8.2.4}$$

Trapped set. The results of the following paragraph can be found in [GGSU17, Section 2.1]. We recall them for the sake of clarity. For $\varepsilon > 0$ small enough, the compact surfaces $M_\varepsilon := M \cap \{\rho \geq \varepsilon\}$ have strictly convex boundary with respect to the geodesic flow.

Lemma 8.2.2. *[GGSU17, Lemma 2.3] There exists $\varepsilon > 0$ small enough so that for each $(x,\xi) \in S^*M$ with $\rho(x) < \varepsilon$, $\xi = \xi_0 d\rho + \sum_{i=1}^{n-1}\xi_i dy_i$ and $\xi_0 \leq 0$, the flow trajectory $\varphi_t(x,\xi)$ converges to some point $z_+ \in \partial_+S^*M$ with rate $\mathcal{O}(e^{-t})$ as $t \to +\infty$ and $\rho(\varphi_t(x,\xi)) \leq \rho(x,\xi)$ for all $t \geq 0$. The same result holds with $\xi_0 \geq 0$ and negative time, with limit point $z_- \in \partial_-S^*M$.*

We define the *tails* $\Gamma_\pm$: they consist of the points in S^*M which are respectively trapped in the past or in the future :

$$S^*M \setminus \Gamma_\mp := \{z \in S^*M, \rho(\varphi_t(z))_{t\to\pm\infty} \to 0\} \tag{8.2.5}$$

The *trapped set* K is defined by :

$$K := \Gamma_+ \cap \Gamma_- \tag{8.2.6}$$

In particular, in negative curvature, the trapped set has zero Liouville measure. We can define the exit and enter maps $B_\pm : S^*M \setminus \Gamma_\mp \to \partial_\pm S^*M$ such that

$$B_\pm(z) := \lim_{t\to\pm\infty} \varphi_t(z) \tag{8.2.7}$$

These are smooth, well-defined maps and they extend smoothly to $\overline{S^*M} \setminus \overline{\Gamma_\mp}$, where $\overline{\Gamma_\mp}$ is the closure of $\Gamma_\mp$ in $\overline{S^*M}$ (see [GGSU17, Corollary 2.5]). There also exist smooth functions $\tau_\pm : \overline{S^*M} \setminus \overline{\Gamma_\mp} \to \mathbb{R}_\pm$ defined such that :

$$\overline{\varphi}_{\tau_\pm(z)}(z) = B_\pm(z) \in \partial_\pm S^*M \tag{8.2.8}$$

Using the vector field $\overline{X}$, another way of describing the sets $\overline{\Gamma_\pm}$ is

$$\overline{\Gamma_\pm} = \left\{ z \in \overline{S^*M}, \tau_\mp(z) = \pm\infty \right\} \tag{8.2.9}$$

The *scattering map* is the smooth map $\sigma : \partial_- S^*M \setminus \overline{\Gamma_-} \to \partial_+ S^*M \setminus \overline{\Gamma_+}$ defined by :

$$\sigma(z) := B_+(z) = \overline{\varphi}_{\tau_+(z)}(z) \tag{8.2.10}$$

Hyperbolic splitting in negative curvature. In this section, (M, g) has dimension 2 and negative curvature $\kappa < 0$. Since the curvature at infinity converges towards -1, we know that κ is pinched between two constants $-k_0^2 \leq \kappa < -k_1^2 < 0$. It is a classical fact that the geodesic flow on such a surface is Anosov (see [Ebe72, Kli74]) in the sense that there exists some constants $C > 0$ and $\nu > 0$ (depending on the metric g) such that for all $z = (x, \xi) \in S^*M$, there is a continuous flow-invariant splitting

$$T_z(S^*M) = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{8.2.11}$$

where $E_s(z)$ (resp. $E_u(z)$) is the *stable* (resp. *unstable*) vector space in z, which satisfy

$$\begin{aligned} |d\varphi_t(z) \cdot Z|_G \leq Ce^{-\nu t}|Z|_G, \quad \forall t > 0, Z \in E_s(z) \\ |d\varphi_t(z) \cdot Z|_G \leq Ce^{-\nu |t|}|Z|_G, \quad \forall t < 0, Z \in E_u(z) \end{aligned} \tag{8.2.12}$$

The norm, here, is given in terms of the Sasaki metric. The bundles $z \mapsto E_u(z), E_s(z)$ are Hölder-continuous everywhere on S^*M. Moreover, the differential of the geodesic flow is governed uniformly by an exponential growth (see [Rug07, Chapter 3]) in the sense that there exists (other) constants $C, k > 0$ such that :

$$|d\varphi_t(z) \cdot Z|_G \leq Ce^{kt}|Z|_G, \quad \forall t > 0, \forall Z \in T_z(S^*M) \tag{8.2.13}$$

Let us now fix $\varepsilon > 0$ small enough and consider $M_\varepsilon := M \cap \{\rho \geq \varepsilon\}$. Like in [Gui17b], we define the *non-escaping mass function* $V_\varepsilon(T)$ on the domain M_ε by

$$V_\varepsilon(T) := \mu\left(\{z \in S^*M_\varepsilon \mid \forall s \in [0, T], \varphi_s(z) \in S^*M_\varepsilon\}\right).$$

Since the trapping set is hyperbolic, there exists a constant $Q < 0$ such that $Q := \limsup_{T \to +\infty} \log(V_\varepsilon(T))/T$. Note that this constant is independent of ε (see [Gui17b, Proposition 2.4]). In the rest of this paragraph, we fix some $\varepsilon_0 > 0$ small enough. For $0 < \varepsilon < \varepsilon_0$, we want to link explicitly the decay of the non-escaping mass function V_ε to V_{ε_0}.

Lemma 8.2.3. *Let $\delta \in (Q, 0)$. There exists a constant $C > 0$ and an integer $N_0 \in \mathbb{N} \setminus \{0\}$, such that for all $T \geq -N_0 \log(\varepsilon)$:*

$$V_\varepsilon(T) \leq C\varepsilon^{-(1+4\delta)}e^{-\delta T}.$$

Proof. For $(x, \xi) \notin \Gamma_-$ we denote by $\ell_{\varepsilon,+}(x, \xi)$ the exit time of the manifold M_ε, that is the maximum time such that : $\forall t \in [0, \ell_{\varepsilon,+}(x, \xi)], \varphi_t(x, \xi) \in S^*M_\varepsilon$. By Santaló's formula, we can express $V_\varepsilon(T)$ as :

$$V_\varepsilon(T) = \int_{\partial_- S^*M_\varepsilon} (\ell_{\varepsilon,+}(x, \xi) - T)_+ \, d\mu_{\nu,\varepsilon}$$

where $x_+ = \sup(x, 0)$, $d\mu_{\nu,\varepsilon}(x,\xi) = |g(\xi,\nu)| i^*_{\partial S^* M_\varepsilon}(d\mu)$ [1], ν is the unit covector conormal to the boundary, $i^*_{\partial S^* M_\varepsilon}(d\mu)$ is the restriction of the Liouville measure to the boundary (the measure induced by the Sasaki metric restricted to $\partial S^* M_\varepsilon$). There exists a maximum time T^*_ε, such that given any $(x,\xi) \in \partial_+ S^* M_{\varepsilon_0}$, $\varphi_{T_\varepsilon}(x,\xi)$ has exited the manifold M_ε. One can bound this time T^*_ε by $\log(C\varepsilon_0/\varepsilon)$, where $C > 0$ is some constant independent of (x,ξ) and ε (see the proof of [GGSU17, Lemma 2.3]). We introduce $T_\varepsilon := -2\log(\varepsilon) > T^*_\varepsilon$ for ε small enough. As a consequence, for $T \geq 2T_\varepsilon$, one has :

$$V_\varepsilon(T) \leq \int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} \left(\ell_{\varepsilon_0,+}(\psi_\varepsilon(x,\xi)) - (T - 2T_\varepsilon)\right)_+ d\mu_{\nu,\varepsilon},$$

where $\psi_\varepsilon^{-1} : \partial_- S^* M_{\varepsilon_0} \to \psi_\varepsilon^{-1}(\partial_- S^* M_{\varepsilon_0}) =: D_\varepsilon \subset \partial_- S^* M_\varepsilon$ is the diffeomorphism which flows backwards (by $\overline{\varphi}_\tau$) a point $(x,\xi) \in \partial_- S^* M_{\varepsilon_0}$ to the boundary $\partial_- S^* M_\varepsilon$ (see Figure 8.1).

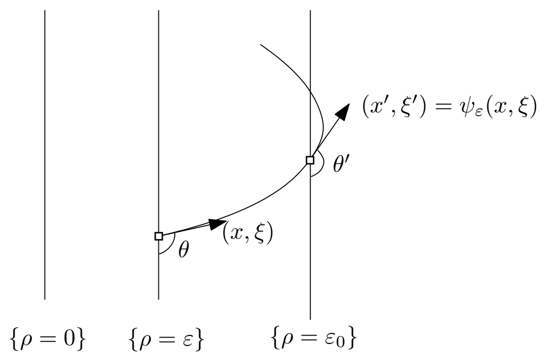

FIGURE 8.1 – The diffeomorphism ψ_ε in the proof of Lemma 8.2.3

The dependence of ψ_ε^{-1} on ε is smooth down to $\varepsilon = 0$: this follows from the implicit function theorem. In the local product coordinates (ρ, y), one can write $d\mu_{\nu,\varepsilon} = 1/\varepsilon \sin(\theta) h(\varepsilon, y) dy d\theta$, where $[0,\pi] \ni \theta \mapsto \xi(\theta)$ parametrizes the cosphere fiber, h is a smooth non-vanishing function down to $\varepsilon = 0$. The point (x,ξ) corresponds to (y,θ) in these coordinates and we write $(y',\theta') = \psi_\varepsilon(y,\theta)$. If T is large enough, for the integrand not to vanish, one has to require that the angle $\theta'(\psi_\varepsilon(y,\theta))$ is uniformly contained in a compact interval of $]0,\pi[$. In other words, if we fix some constant $c > 0$, there exists an integer $N_0 \geq 2$ large enough (independent of ε) such that for $T \geq -N_0 \log(\varepsilon)$, if $\theta'(\psi_\varepsilon(y,\theta)) \in [0,c] \cup [\pi - c, \pi]$, it will satisfy $\left(\ell_{\varepsilon_0,+}(\psi_\varepsilon(y,\theta)) - (T - 2T_\varepsilon)\right)_+ = 0$. We can now make a change of variable in the previous integral by setting $(y',\theta') = \psi_\varepsilon(y,\theta)$. Since the dependence of ψ_ε^{-1} is smooth in ε (down to $\varepsilon = 0$) and $[0,\varepsilon_0] \times \{\rho = \varepsilon_0\}$ is compact, $|\det(\psi_\varepsilon^{-1}(y',\theta'))|$ is bounded independently of (y',θ') and ε. We get for

1. The metric g here actually denotes the dual metric to g which is usually written g^{-1}. As mentioned in the introduction, we do not employ this notation in order to keep the reading affordable.

$T \geq -N_0 \log(\varepsilon)$:

$$\begin{aligned}
&\int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} \left(\ell_{\varepsilon_0,+}(\psi_\varepsilon(x,\xi)) - (T - 2T_\varepsilon)\right)_+ d\mu_{\nu,\varepsilon} \\
&\quad = \int_{\partial_- S^* M_\varepsilon \cap D_\varepsilon} \left(\ell_{\varepsilon_0,+}(\psi_\varepsilon(y,\theta)) - (T - 2T_\varepsilon)\right)_+ \sin(\theta) h(\varepsilon, y) \frac{dy d\theta}{\varepsilon} \\
&\quad = \int_{\partial_- S^* M_{\varepsilon_0}} \left(\ell_{\varepsilon_0,+}(y',\theta') - (T - 2T_\varepsilon)\right)_+ \sin\left(\theta(\psi_\varepsilon^{-1}(y',\theta'))\right) \\
&\qquad\qquad h\left(\varepsilon, y(\psi_\varepsilon^{-1}(y',\theta'))\right) |\det(\psi_\varepsilon^{-1}(y',\theta'))| \frac{d\theta' dy'}{\varepsilon} \\
&\quad \leq C \int_{\partial_- S^* M_{\varepsilon_0}} \left(\ell_{\varepsilon_0,+}(y',\theta') - (T - 2T_\varepsilon)\right)_+ \frac{d\theta' dy'}{\varepsilon} \\
&\quad \leq C\varepsilon^{-1} \int_{\partial_- S^* M_{\varepsilon_0,+}} \left(\ell_{\varepsilon_0}(y',\theta') - (T - 2T_\varepsilon)\right)_+ h(\varepsilon_0, y) \sin(\theta') \frac{d\theta' dy'}{\varepsilon_0} \\
&\quad \leq C\varepsilon^{-1} V_{\varepsilon_0}(T - 2T_\varepsilon),
\end{aligned}$$

for some constant $C > 0$ (which may be different from one line to another) and where the penultimate inequality follows from the uniform bound on the angle (i.e. $\sin(\theta') \in [\sin(c), 1]$). But we know that for any $\delta \in (Q, 0)$, there exists an (other) constant $C > 0$ such that for all $T \geq 0$, $V_{\varepsilon_0}(T) \leq Ce^{-\delta T}$. Thus, for $T \geq -N_0 \log(\varepsilon)$

$$V_\varepsilon(T) \leq C\varepsilon^{-1} e^{-\delta(T - 2T_\varepsilon)} \leq C\varepsilon^{-(1+4\delta)} e^{-\delta T}$$

□

8.2.2 The renormalized length

Definition. This paragraph supplies with the definitions of [GGSU17, Section 4.1]. Let $\alpha(x, x')$ be a geodesic in M joining two distinct points at infinity $x, x' \in \partial M$. For the sake of simplicity, we will only write α in this paragraph, instead of $\alpha(x, x')$. The renormalized length of the geodesic α is the real number defined by :

$$L(\alpha) := \lim_{\varepsilon \to 0} \ell\left(\alpha \cap \{\rho \geq \varepsilon\}\right) + 2\log(\varepsilon), \tag{8.2.14}$$

where $\ell(\cdot)$ denotes the Riemannian length. This limit exists and is finite by [GGSU17, Lemma 4.1].

Note that there is *a priori* no canonical choice of the renormalized length L insofar as it depends on the choice of the boundary defining function ρ. One can actually prove that if $\hat{\rho} = e^\omega \rho$ is another choice, then (see [GGSU17, Equation (4.2)]) :

$$\hat{L}(\alpha(x, x')) - L(\alpha(x, x')) = \omega(x) + \omega(x').$$

Remark 8.2.1. As a consequence, if two defining functions induce the same representative for the conformal infinity, then they induce the same renormalized lengths. Thus, if $\psi : \overline{M} \to \overline{M}$ is a diffeomorphism which preserves the boundary, $\rho \circ \psi$ and ρ induce the same representative for the conformal infinity, so $L_g = L_{\psi^* g}$, where both renormalized lengths are computed with respect to ρ.

An example : the hyperbolic disk. Let us consider the hyperbolic disk $(\mathbb{D}, \frac{4|dz|^2}{(1-|z|^2)^2})$. The set of geodesics on $\mathbb{D}$ can be naturally identified with $\partial\mathbb{D} \times \partial\mathbb{D} \setminus \text{diag}$ insofar as there exists a unique geodesic joining to points on the ideal boundary. There is a natural choice for the boundary defining function which is given by $\rho(z) := \frac{1}{2}(1-|z|^2)$.

Proposition 8.2.1. *Let $\xi, \zeta \in \partial\mathbb{D}$ be two points on the boundary. Then :*

$$L(\xi, \zeta) = 2\log(|\xi - \zeta|) \tag{8.2.15}$$

Proof. We denote by α the geodesic joining ξ to ζ. For $\varepsilon > 0$, we denote by p_ε and q_ε the points of intersection of α with $\{\rho = \varepsilon\}$ in a respective neighborhood of ξ and ζ. We have :

$$d(p_\varepsilon, q_\varepsilon) = \log\left(\frac{|\xi - q_\varepsilon||\zeta - p_\varepsilon|}{|\xi - p_\varepsilon||\zeta - q_\varepsilon|}\right) = 2\log(|\xi - q_\varepsilon|) - 2\log(|\xi - p_\varepsilon|),$$

by symmetry. As $\varepsilon \to 0$, $|\xi - q_\varepsilon| \to |\xi - \zeta|$ and, using elementary arguments of geometry, one can prove that $|\xi - p_\varepsilon| = \varepsilon(1 + o(1))$. Thus :

$$d(p_\varepsilon, q_\varepsilon) = 2\log(|\xi - \zeta|) - 2\log(\varepsilon) - 2\log(1 + o(1))$$

□

Remark 8.2.2. In the model of the hyperbolic plane $(\mathbb{H}, \frac{dx^2+dy^2}{y^2})$, if one takes the boundary defining function $\rho(x, y) = y$, then given two points ξ, ζ on the real line, one can check that :

$$L(\xi, \zeta) = 2\log(|\xi - \zeta|)$$

We see in particular that the renormalized length is not a proper "length" according to the usual terminology insofar as it can be negative, and we even have $L(\xi, \zeta) \to -\infty$ as $\xi \to \zeta$. This is not specific to the hyperbolic disk and can be proved in the general frame. Moreover, we see from the expression (8.2.15) that the renormalized length is not invariant by the isometries of the disk

$$\gamma : z \mapsto e^{i\theta}\frac{z + c}{cz + 1},$$

if $c \neq 0$, but :

$$\begin{aligned} L(\gamma(\xi), \gamma(\zeta)) &= 2\log(|\gamma(\xi) - \gamma(\zeta)|) \\ &= 2\log(|\xi - \zeta||\gamma'(\xi)|^{1/2}|\gamma'(\zeta)|^{1/2}) \\ &= L(\xi, \zeta) + \log(|\gamma'(\xi)||\gamma'(\zeta)|) \end{aligned}$$

Action of isometries on the renormalized length. Let γ be an isometry on (M, g), then γ acts smoothly on the compactification $\overline{M}$ (see the arguments given in §8.5 for instance).

Lemma 8.2.4. *Let α be a geodesic joining two points $x, x' \in \partial M$. We have :*

$$L(\gamma \circ \alpha) = L(\alpha) + n^{-1}\log(|d\gamma_x||d\gamma_{x'}|),$$

where $|d\gamma_x|$ is the Jacobian of $\gamma|_{\partial M}$ in x with respect to the metric h, $n + 1$ being the dimension of M.

Proof. We denote by $z = (x, \xi)$ the point in $\partial_- S^*M$ generating α. Assume for the sake of simplicity that α is a half-line joining $x \in \partial M$ to a point in the interior M. Let $x_\varepsilon := \alpha \cap \{\rho = \varepsilon\}$ and $\alpha_\varepsilon := \alpha \cap \{\rho \geq \varepsilon\}$. We define $\varepsilon' := \rho(\gamma(x_\varepsilon))$. We have :

$$\ell(\alpha_\varepsilon) + \log(\varepsilon) = (\ell(\gamma(\alpha_\varepsilon)) + \log(\varepsilon')) - \log(\varepsilon'/\varepsilon)$$

As $\varepsilon \to 0$, the left-hand side converges to $L(\alpha)$ whereas the term between parenthesis on the right-hand side goes to $L(\gamma(\alpha))$, so all is left to compute is the limit of ε'/ε as $\varepsilon \to 0$. We write $\varepsilon' = \rho(\gamma(\pi(\overline{\varphi}_{\tau_\varepsilon}(z))))$, where τ_ε is defined to be the unique time such that $\rho(\overline{\varphi}_{\tau_\varepsilon}(z)) = \varepsilon$. By the implicit function theorem, $\varepsilon \mapsto \tau_\varepsilon$ is a smooth function of ε and it satisfies : $\rho(\overline{\varphi}_{\tau_\varepsilon}(z)) = \varepsilon = \tau_\varepsilon + \mathcal{O}(\tau_\varepsilon^2)$. Thus $\partial_\varepsilon \tau_\varepsilon|_{\varepsilon=0} = 1$ and :

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon'/\varepsilon &= d\rho_{\gamma(x)} \left(d\gamma_x \left(d\pi_z \left(\frac{d\tau_\varepsilon}{d\varepsilon} \frac{\partial \overline{\varphi}_{\tau_\varepsilon}}{\partial \tau_\varepsilon}(z) \Big|_{\varepsilon=0} \right) \right) \right) \\ &= d\rho_{\gamma(x)} \left(d\gamma_x (d\pi_z(\overline{X}(z))) \right) \\ &= d\rho_{\gamma(x)} \left(d\gamma_x (\partial_\rho(x)) \right) \end{aligned}$$

Remark that at ∂M, $d\gamma_x(\partial_\rho(x)) = \lambda(x)\partial_\rho(\gamma(x))$ for some real number λ depending on x, since γ sends geodesics on geodesics. If $\eta_1, ..., \eta_n \in T_x(\partial M)$ is an orthonormal frame for the metric h, one can prove that $h(d\gamma_x(\eta_i), d\gamma_x(\eta_j)) = \lambda^2(x)\delta_{ij}$ by using the fact that $\gamma^* g = g$. As a consequence, the Jacobian of $\gamma|_{\partial M}$ at x with respect to the metric h is $\lambda^n(x)$. Thus :

$$\lim_{\varepsilon \to 0} \varepsilon'/\varepsilon = |d\gamma_x|^{\frac{1}{n}}$$

Replacing this in (8.2.2), and adding the other part of the geodesic, we find the sought result. □

8.2.3 Liouville current

We denote by $\widetilde{M}$ the universal cover of M : it is a topological disk on which we fix an orientation. All the objects $(g, \rho, X, ...)$ lift to $\widetilde{M}$ and their corresponding object in the universal cover is invariant by the action of the fundamental group $\pi_1(M)$. Since we will only work on $\widetilde{M}$ in the following, for the reader's convenience, we will often drop the notation $\widetilde{\cdot}$ when the context is clear, except for the universal cover itself $\widetilde{M}$. We define

$$\mathcal{G} := (\partial\widetilde{M} \times \partial\widetilde{M}) \setminus \text{diag},$$

which can be naturally identified with the set of untrapped geodesics (neither in the future nor the past) on $\widetilde{M}$. If $\mathcal{M}$ is the set of Borel measures on $\mathcal{G}$ which are invariant by the flip, then it is a classical fact from [Ota90] that the Liouville measure induces a measure $\eta \in \mathcal{M}$ called the *Liouville current* (see also [GM18] for a proof).

Expression in coordinates. Given $x, x' \in \widetilde{M}$, we can parametrize α, the unique geodesic joining x to x', in the following way : if $z = (x, \xi) \in \partial_- S^*\widetilde{M}$ denotes the point generating α, then we parametrize the geodesic by $\alpha(t) = \varphi_t(m(z))$, where $m(z) = \overline{\varphi}_{\tau_+(z)/2}(z)$ is the middle point (this is a smooth map according to Section §8.2.1). We set $\gamma(t) := \pi(\alpha(t))$. We define

$$V := \{(\tau, \theta) \in \mathbb{R} \times (0, \pi), (\gamma(\tau), R_\theta \dot{\gamma}(\tau)) \notin \Gamma_- \cup \Gamma_+\}, \tag{8.2.16}$$

where R_θ is the rotation by a positive angle θ in the fibers of $S^*\widetilde{M}$. For $x, x' \in \widetilde{M}$, we denote by $\mathcal{F}(x, x') \subset \mathcal{G}$ the open subsets of points $(y, y') \in \mathcal{G}$ such that the geodesic joining y to y' has a transverse and positive (with respect to the orientation) intersection with the geodesic α in $\widetilde{M}$. If we further assume that $x, x' \in \partial\widetilde{M}$, we can consider the diffeomorphism $\phi : V \mapsto \mathcal{F}(x, x')$ defined by $\phi(\tau, \theta) = (y, y')$, the two points in $\partial\widetilde{M}$ such that the geodesic connecting them passes through the point $(\gamma(\tau), R_\theta\dot{\gamma}(\tau)) \in S^*\widetilde{M}$. The following lemma is a well-known fact (see [GM18, Lemma 3.1] for instance) and we do not provide its proof.

Lemma 8.2.5. $\phi^*\eta = \sin(\theta)d\theta d\tau$

Remark 8.2.3. In negative curvature, the tails $\Gamma_- \cup \Gamma_+$ have zero Liouville measure. This implies that the set ${}^cV \subset \mathbb{R} \times (0, \pi)$ has zero measure in $\mathbb{R} \times (0, \pi)$ (for the measure $\sin(\theta)d\theta d\tau$). In particular, we will ignore trapped geodesics in the computations of the integrals of Section §8.3.4.

From the previous expression in coordinates, we recover the classical formula for $x, x' \in \widetilde{M}$ (see [Ota90]) :

$$\eta\left(\mathcal{F}(x, x')\right) = \int_0^\pi \int_0^{d(x,x')} \sin(\theta)d\theta d\tau = 2d(x, x'), \tag{8.2.17}$$

where $d(\cdot, \cdot)$ denotes the Riemannian distance between the two points. For $x, x' \in \partial\widetilde{M}$ and $\varepsilon > 0$ small enough, we denote by x_ε and x'_ε the two intersections of α (the geodesic joining x to x') with $\{\rho = \ \varepsilon\}$ in a respective neighborhood of x and x'. We have :

$$\begin{aligned}\eta(\mathcal{F}(x_\varepsilon, x'_\varepsilon)) + 4\log\varepsilon &= 2\left(d(x_\varepsilon, x'_\varepsilon) + 2\log\varepsilon\right) \\ &= 2\left(\ell(\alpha \cap \{\rho > \varepsilon\}) + 2\log\varepsilon\right) \\ &\to_{\varepsilon\to 0} 2L(\alpha)\end{aligned}$$

Liouville current and boundary distance. Let g_1 and g_2 be two negatively-curved metrics such that their renormalized lengths agree. We denote by η_1 and η_2 their respective Liouville currents.

Lemma 8.2.6. $\eta_1 = \eta_2$

Proof. We recall that $\partial\widetilde{M}$ is a countable union of real lines embedded in the circle $\mathbb{S}^1$. The topology on $\partial\widetilde{M}$ is that naturally induced by the topology on $\mathbb{S}^1$. It is sufficient to prove that the two measures coincide on *rectangles*, namely on subsets $(x_1, x_2) \times (x_3, x_4)$, such that $(x_1, x_2), (x_3, x_4) \subset \partial\widetilde{M}$ are two intervals with disjoint closure, since they generate the Borel σ-algebra. We actually prove the

Lemma 8.2.7. $\eta((x_1, x_2) \times (x_3, x_4)) = |L(x_1, x_3) + L(x_2, x_4) - L(x_2, x_3) - L(x_1, x_4)|$

Note that that $\eta((x_1, x_2) \times (x_3, x_4)) = |[x_1, x_2, x_3, x_4]|$, the cross-ratio of the four points (see [Led95]). In particular, this proves that the right-hand side of Lemma 8.2.7 is a cross-ratio in the sense of [Led95], which may not be obvious at first sight. Actually, the properties of symmetry are immediate and the invariance by the diagonal action of the fundamental group follows from Lemma 8.2.4.

Given some $\varepsilon > 0$, we introduce the four horospheres $H_i(\varepsilon), i \in \{1, ..., 4\}$ such that $H_i(\varepsilon)$ intercepts x_i and the point defined as the intersection of the geodesic $\alpha(x_i, x_{i+2})$ ($i + 2$ is taken modulo 4) with $\{\rho = \varepsilon\}$ in a neighborhood of x_i.

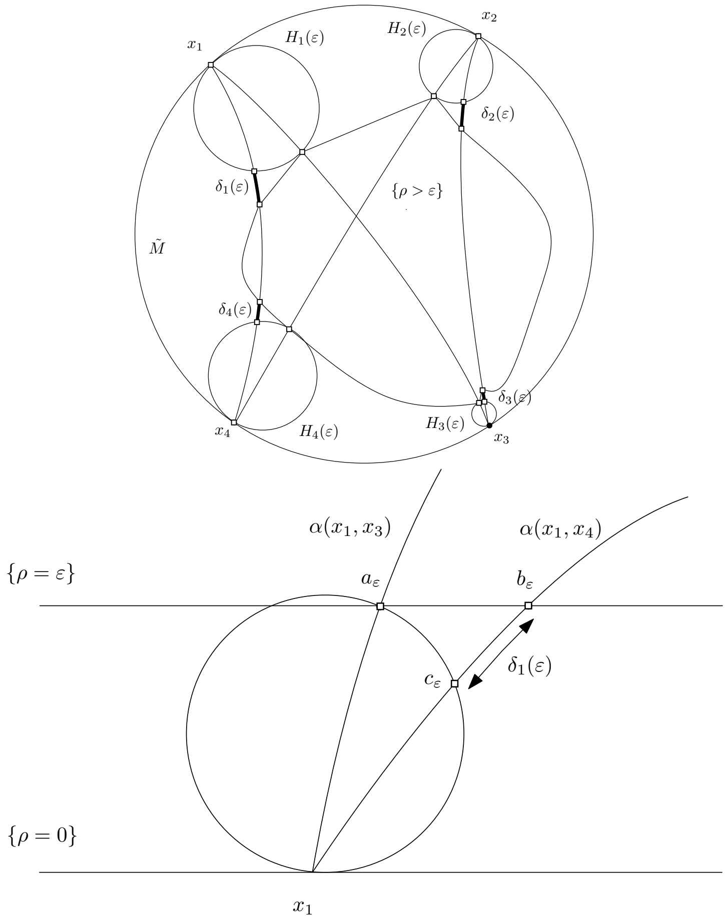

FIGURE 8.2 – Left : The four horospheres and the lengths $\delta_i(\varepsilon)$. Right : The horosphere $H_1(\varepsilon)$

We have :

$$\begin{aligned}
&L(x_1, x_3) + L(x_2, x_4) - L(x_2, x_3) - L(x_1, x_4) \\
&= \lim_{\varepsilon \to 0} \ell(\alpha(x_1, x_3) \cap \{\rho > \varepsilon\}) + 2 \log \varepsilon + \ell(\alpha(x_2, x_4) \cap \{\rho > \varepsilon\}) + 2 \log \varepsilon \\
&\qquad - \ell(\alpha(x_2, x_3) \cap \{\rho > \varepsilon\}) - 2 \log \varepsilon - \ell(\alpha(x_1, x_4) \cap \{\rho > \varepsilon\}) - 2 \log \varepsilon \\
&= \lim_{\varepsilon \to 0} \ell(\alpha(x_1, x_3) \cap \{\rho > \varepsilon\}) + \ell(\alpha(x_2, x_4) \cap \{\rho > \varepsilon\}) - \ell(\alpha(x_2, x_3) \cap \{\rho > \varepsilon\}) \\
&\qquad - \ell(\alpha(x_1, x_4) \cap \{\rho > \varepsilon\}) \\
&= \lim_{\varepsilon \to 0} \ell(\alpha(x_1, x_3) \cap H_{\mathrm{ext}}(\varepsilon)) + \ell(\alpha(x_2, x_4) \cap H_{\mathrm{ext}}(\varepsilon)) - \ell(\alpha(x_2, x_3) \cap H_{\mathrm{ext}}(\varepsilon)) \\
&\qquad - \ell(\alpha(x_1, x_4) \cap H_{\mathrm{ext}}(\varepsilon)) - \delta_1(\varepsilon) - \delta_2(\varepsilon) - \delta_3(\varepsilon) - \delta_4(\varepsilon),
\end{aligned}$$

where $\delta_i(\varepsilon)$ is the algebraic distance on the geodesic between its intersection with $H_i(\varepsilon)$ and $\{\rho = \varepsilon\}$, positively counted from x_i, and $H_{\mathrm{ext}}(\varepsilon) := \widetilde{M} \setminus \cup_{i=1}^4 H_i(\varepsilon)$. Now, we know that the quantity

$$\begin{aligned}
|\ell(\alpha(x_1, x_3) \cap H_{\mathrm{ext}}(\varepsilon)) + \ell(\alpha(x_2, x_4) \cap H_{\mathrm{ext}}(\varepsilon)) \\
- \ell(\alpha(x_2, x_3) \cap H_{\mathrm{ext}}(\varepsilon)) - \ell(\alpha(x_1, x_4) \cap H_{\mathrm{ext}}(\varepsilon))|
\end{aligned}$$

is actually independent of ε and equals $\eta([x_1, x_2] \times [x_3, x_4])$ (see [Wil14] for instance). It is thus sufficient to prove that $\delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let us consider $\delta_1(\varepsilon)$ and ε small enough so that we can work in the coordinates where the metric g can be written in the form $g = \rho^{-2}(d\rho^2 + h^2(\rho, y)dy^2)$ for some smooth positive function h^2 (down to the boundary).

We have :

$$\delta_1(\varepsilon) = d(c_\varepsilon, b_\varepsilon) \leq d(c_\varepsilon, a_\varepsilon) + d(a_\varepsilon, b_\varepsilon) \leq d(c_\varepsilon, a_\varepsilon) + l([a_\varepsilon, b_\varepsilon]),$$

where the points $a_\varepsilon, b_\varepsilon, c_\varepsilon, d_\varepsilon$ are introduced in Figure 8.2, $[a_\varepsilon, b_\varepsilon]$ denotes the Euclidean segment joining a_ε to b_ε. Note that by construction $d(c_\varepsilon, a_\varepsilon) \to 0$ as $\varepsilon \to 0$ (the points are on the same family of shrinking horospheres).

The two geodesics $\alpha(x_1, x_3)$ and $\alpha(x_1, x_4)$ with endpoint x_1, seen as curves in $\widetilde{M}$, can be locally parametrized by the respective smooth functions $(\rho, y_3(\rho))$ and $(\rho, y_4(\rho))$, according to the implicit function theorem since the geodesics intersect transversally the boundary (see Lemma 8.2.1). One has by derivating at $\rho = 0$ that $\lambda_i \partial_\rho = \partial_\rho + y_i'(0)\partial_y$ for some constant λ_i, that is $y_i'(0) = 0$ and $\lambda_i = 1$. In other words, we can parametrize locally both geodesics by $(\rho, y_0 + \mathcal{O}(\rho^2))$, where y_0 is some constant depending on the choice of coordinates. Thus $|y(a_\varepsilon) - y(b_\varepsilon)| = \mathcal{O}(\varepsilon^2)$. If we choose a parametrization $\gamma(t) = (\varepsilon, y(a_\varepsilon) + t(y(b_\varepsilon) - y(a_\varepsilon)))$, for $t \in [0,1]$, of the euclidean segment $[a_\varepsilon, b_\varepsilon]$, then one has :

$$\ell([a_\varepsilon, b_\varepsilon]) = \int_0^1 g(\dot\gamma(t), \dot\gamma(t))^{1/2} dt = \varepsilon^{-1} |y(b_\varepsilon) - y(a_\varepsilon)| \int_0^1 h(\gamma(t)) dt,$$

where the integral is uniformly bounded with respect to ε. Thus, by the previous remarks, $l([a_\varepsilon, b_\varepsilon]) = \mathcal{O}(\varepsilon)$, which concludes the proof.

□

8.3 Construction of the deviation κ

In this section, for the sake of simplicity, we will sometimes write $A = \mathcal{O}(\varepsilon^\infty)$ in order to denote the fact that for all $n \in \mathbb{N} \setminus \{0\}$, there exists $C_n > 0, \varepsilon_n > 0$ such that : $\forall \varepsilon \leq \varepsilon_n, |A| \leq C_n \varepsilon^n$.

8.3.1 Reducing the problem

Suppose g_1 and g_2 are two asymptotically hyperbolic metrics like in the setting of Theorem 8.1.1 that is, they are both negatively-curved and their renormalized distances coincide for some choices of conformal representatives in the conformal infinities. In local coordinates (ρ, y), for $i \in \{1, 2\}$, one can write $g_i = \rho^{-2}(d\rho^2 + h_{\rho,i})$, for some smooth metrics $h_{\rho,i}$ on ∂M (note that this is the same boundary defining function for both metrics, see [GGSU17, Section 4.2]). By [GGSU17, Theorem 2], there exists a smooth diffeomorphism $\psi : \overline{M} \to \overline{M}$ fixing the boundary such that $\psi^* g_1 - g_2 = \mathcal{O}(\rho^\infty)$ at ∂M (that is $h_{\rho,1} - h_{\rho,2} = \mathcal{O}(\rho^\infty)$). In the following, we will argue with this new metric $\psi^* g_1$ but we will still denote it g_1 for the sake of simplicity.

Remark 8.3.1. In particular, this implies that the respective renormalized vector fields satisfy $\overline{X}_1 - \overline{X}_2 = \mathcal{O}(\rho^\infty)$ at ∂M, that is their C^∞-jet coincide on the boundary. By Duhamel's formula (see [SUV16, Lemma 2.2] for instance) this implies that on the boundary $\partial_- S^*M$, for any $k \geq 0$, one has $||\overline{\varphi}^1_\tau - \overline{\varphi}^2_\tau||_{C^k} = \mathcal{O}(\tau^\infty)$.

8.3.2 The diffeomorphism κ

We denote by $M_\varepsilon := M \cap \{\rho \geq \varepsilon\}$ and by $\widetilde{M_\varepsilon}$ its lift to the universal cover. Like before, all the objects are lifted on the universal cover. Unless it is mentioned, we will drop the notation $\widetilde{\cdot}$, except for the universal cover itself. $S^*\widetilde{M}_i$ will denote the unit cotangent bundle with respect to the metric g_i. $\mathcal{G}_1$ and $\mathcal{G}_2$ denote the set of geodesics connecting points on the ideal boundary $\partial\widetilde{M}$, with respect to the metrics g_1 and g_2. They will sometimes be identified with $\partial\widetilde{M} \times \partial\widetilde{M} \setminus \text{diag}$.

Given $(x,\xi) \in S^*\widetilde{M}_1 \setminus \Gamma^1_- \cup \Gamma^1_+$, we denote by $(z,z') \in \partial\widetilde{M} \times \partial\widetilde{M}$ (resp. $(y,y') \in \partial\widetilde{M} \times \partial\widetilde{M}$) the two points on the ideal boundary induced by the geodesic carrying the point (x,ξ) (resp. $(x,R_\theta\xi)$ if $\theta \in (0,\pi)$ and $(x,R_\theta\xi) \in S^*\widetilde{M}_1 \setminus \Gamma^1_- \cup \Gamma^1_+$). This defines a map :

$$\kappa_1 : \widetilde{W}_1 \to \mathcal{G}_1 \times \mathcal{G}_1 \setminus \text{diag}, \qquad \kappa_1(x,\xi,\theta) = (z,z',y,y'),$$

where

$$\widetilde{W}_1 := \left\{(x,\xi,\theta) \in S^*\widetilde{M}_1 \times (0,\pi) \mid (x,\xi),(x,R_\theta\xi) \notin (\Gamma^1_- \cup \Gamma^1_+)\right\}$$

The map κ_1 is clearly bijective. It is smooth because each of the coordinates (z,z',y,y') is smooth. Indeed, one has for instance

$$z(x,\xi,\theta) = \pi(\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi)),$$

and this is a smooth application according to Section §8.2.1.

The g_2-geodesics with endpoints (z,z') and (y,y') intersect at a single point denoted $(\widetilde{\mathrm{x}}(x,\xi,\theta),\widetilde{\Xi}(x,\xi,\theta))$ (where $\widetilde{\Xi}$ is the covector on the g_2-geodesic with endpoints (z,z')) and form an angle $\widetilde{f}(x,\xi,\theta)$, which we call the *angle of deviation*. This defines a map

$$\widetilde{\kappa} := \kappa_2^{-1} \circ \kappa_1 : \widetilde{W}_1 \to \widetilde{W}_2, \qquad \widetilde{\kappa}(x,\xi,\theta) = (\widetilde{\mathrm{x}}(x,\xi,\theta),\widetilde{\Xi}(x,\xi,\theta),\widetilde{f}(x,\xi,\theta)), \tag{8.3.1}$$

where $\widetilde{W}_2$ is defined in the same fashion as $\widetilde{W}_1$. By the implicit function theorem, one can prove that κ_2^{-1} is smooth and thus $\widetilde{\kappa}$ too. It is a bijective map whose inverse $\widetilde{\kappa}^{-1} = \kappa_1^{-1} \circ \kappa_2$ is smooth by the same arguments. As a consequence, $\widetilde{\kappa}$ is a smooth diffeomorphism. Moreover, it is invariant by the action of the fundamental group and thus descends to the base as an application $\kappa : (x,\xi,\theta) \mapsto (\mathrm{x},\Xi,f)$.

8.3.3 Scattering on the universal cover

On the universal cover $\widetilde{M}$, the renormalized distance can actually be extended outside the boundary, namely we can set for $p,q \in \widetilde{M}$:

$$D_i(p,q) := d_i(p,q) + \log(\rho(p)) + \log(\rho(q)),$$

where d_i, $i \in \{1,2\}$ stands for the Riemannian distance induced by the metric g_i. D_i is clearly smooth on $\widetilde{M} \times \widetilde{M} \setminus \text{diag}$ and using the fact that there exists a unique geodesic connecting two points, one can prove like in [GGSU17, Proposition 5.15], that the extension of D_i to $\overline{\widetilde{M}} \times \overline{\widetilde{M}} \setminus \text{diag}$ is smooth. Now, as established in [GGSU17, Proposition 5.16] the renormalized distance on the boundary actually determines the scattering map σ_i (defined in (8.2.10)), that is :

Proposition 8.3.1. *If $L_1 = L_2$, then $\sigma_1 = \sigma_2$.*

The proof also applies here, in the universal cover. It is a standard computation since we know that D_i is differentiable, which relies on the fact that the gradient of $q \mapsto L_i(\alpha(p,q))$ (for $p,q \in \partial\widetilde{M}$) is the projection on the tangent space $T_q\partial\widetilde{M}$ of the gradient of $q \mapsto D_i(p,q)$ and the latter corresponds to the direction of the geodesic joining p to q when it exits $\widetilde{M}$.

We fix $\varepsilon > 0$ and define $S^*\widetilde{M}^i_\varepsilon := S^*\widetilde{M}_i \cap \{\rho \geq \varepsilon\}$. For $i \in \{1,2\}$, given $(x,\xi) \in \partial_- S^*\widetilde{M}^i_\varepsilon$ we can represent the vector $\xi = \xi(\omega)$ by the angle $\omega \in [0,\pi]$ such that $\sin\omega = |g_i(\nu_i(x),\xi)|$, where ν_i stands for the unit covector conormal to $\{\rho = \varepsilon\}$ (with respect to the metric g_i).

Lemma 8.3.1. *There exists an angle ω_ε (only depending on ε), such that for all $(x,\xi(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon \setminus \Gamma^1_-$, given by an angle $\omega \in [\omega_\varepsilon, \pi - \omega_\varepsilon]$, if $\alpha_1(p,q)$ denotes the g_1-geodesic generated by (x,ξ), with endpoints $(p,q) \in \partial\widetilde{M} \times \partial\widetilde{M}$, then the g_2-geodesic $\alpha_2(p,q)$ with endpoints p and q intercepts the set $\{\rho > \varepsilon\}$ for ϵ small enough. Moreover, for any $N \in \mathbb{N} \setminus \{0\}$, we can take $\omega_\varepsilon = \varepsilon^N$.*

Proof. Let $(x,\xi) \in \partial_- S^*\widetilde{M}^1_\varepsilon$. We set ourselves in the coordinates (ρ, y) induced by the conformal representative h. The trajectory

$$t \mapsto (\rho(t), y(t), \xi_0(t), \eta(t)) \in S^*\widetilde{M}$$

of the point (x,ξ) under the flow X is given by Hamilton's equation (see [GGSU17, Equation (2.8)]). Flowing backwards in time with φ_t, we know that (x,ξ) converges exponentially fast towards a point $(p,\zeta) \in \partial_- S^*\widetilde{M}$ (see [GGSU17, Equation (2.11)]) in the sense that there exists a constant C (uniform in the choice of points) such that :

$$\forall t \leq 0, \quad \rho(t) \leq C\rho(0)e^{-|t|} = \varepsilon C e^{-|t|}$$

In particular, the time $\tau_-(x,\xi)$ taken by the point (x,ξ) to reach (p,ζ) with the flow $\overline{\varphi}^1_\tau$ is (see (8.2.4)) :

$$\tau_-(x,\xi) = \int_{-\infty}^0 \rho(t)dt \leq C\varepsilon$$

We also know, according to Hamilton's equations (see [GGSU17, Equation (2.8)]) that

$$\dot\rho(0) = \rho^2(0)\xi_0(0) = \ \varepsilon\sin(\omega),$$

where ω satisfies $\overline{\xi}_0(0) = \rho\xi_0(0) = \sin(\omega) = |g_1(\xi, \nu_1(x))|$. Let us fix an integer $N > 0$ and assume that $\varepsilon^N \leq \omega \leq \pi - \varepsilon^N$. Then $\dot\rho(0) \geq 2/\pi \cdot \varepsilon^{N+1}$ so there exists an interval $[0,\delta]$ such that for $t \in [0,\delta]$:

$$\varepsilon + t/\pi \cdot \varepsilon^{N+1} \leq \varepsilon + t/2 \cdot \dot\rho(0) \leq \rho(t) \leq 2\varepsilon$$

In particular, $\rho(\delta) \geq \varepsilon + \delta/\pi \cdot \varepsilon^{N+1}$.

We go back to the flow $\overline{\varphi}^1_\tau$. By our previous remark, we know that there exists a time

$$\tau_0 \leq C\varepsilon + \int_0^\delta \rho(t)dt \leq C'\varepsilon,$$

such that $\rho(\overline{\varphi}^1_{\tau_0}(p,\zeta)) \geq \varepsilon + \delta/\pi \cdot \varepsilon^{N+2}$. But since $g_1 = g_2 + \mathcal{O}(\rho^\infty)$, we know that $X_1 = X_2 + \mathcal{O}(\rho^\infty)$ and $\overline{X}_1 = \overline{X}_2 + \mathcal{O}(\rho^\infty)$. Moreover, since the scattering maps agree according to Proposition 8.3.1, we know that the two geodesics $\alpha_1(p,q)$ and $\alpha_2(p,q)$ are both generated by (p,ζ). As a consequence, one has : $\rho(\overline{\varphi}^1_\tau(p,\zeta)) = \rho(\overline{\varphi}^2_\tau(p,\zeta)) + \mathcal{O}(\tau^\infty)$ (the remainder being independent of (p,ζ)). In particular, since $\tau_0 \leq C'\varepsilon$, there exists a constant $C'' > 0$ such that

$$|\rho(\overline{\varphi}^1_{\tau_0}(p,\zeta)) - \rho(\overline{\varphi}^2_{\tau_0}(p,\zeta))| \leq C''\varepsilon^{N+2}$$

Thus :

$$\rho(\overline{\varphi}^2_{\tau_0}(p,\zeta)) \geq \varepsilon + \frac{\delta}{\pi}\varepsilon^{N+1} - C''\varepsilon^{N+2} > \varepsilon,$$

if ε is small enough.

□

In the following, we assume that such an integer N is fixed (and taken large enough) and we apply the previous lemma with $N+1$, that is $\omega_\varepsilon = \varepsilon^{N+1}$.

This allows us to define a map $\widetilde{\psi}$ on $\mathcal{U} := \left\{(x,\xi(\omega)) \in S^*\widetilde{M}^1, \overline{\xi}_0 \geq 0, \omega \in [\rho(x)^{N+1}, \pi - \rho(x)^{N+1}]\right\}$, in the following way : to a point $(x,\xi) \in \mathcal{U}$, which we see as a boundary point $(x,\xi(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon$ for $\varepsilon = \rho(x)$, we associate the boundary point $(x',\xi') = \widetilde{\psi}(x,\xi)$ such that $\widetilde{\psi}(x,\xi) \in \partial_- S^*\widetilde{M}^2_\varepsilon$ is the point on the g_2-geodesic connecting p to q. A formal way to define $\widetilde{\psi}$ is to introduce another diffeomorphism $\widetilde{\psi}_1 : \mathcal{U} \to \partial_- S^*\widetilde{M} \times [0,\infty)$ such that $\widetilde{\psi}_1(x,\xi) = \left(\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi), \rho(x)\right)$ and to set

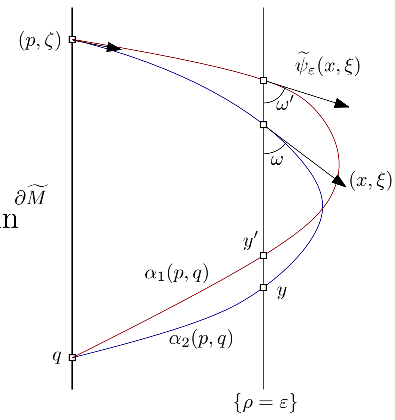

FIGURE 8.3 – The diffeomorphism $\widetilde{\psi}_\varepsilon$

$$\widetilde{\psi}(x,\xi) = \widetilde{\psi}_2^{-1} \circ \widetilde{\psi}_1(x,\xi) = \overline{\varphi}^2_{\tau_\rho}\left(\overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi)\right), \tag{8.3.2}$$

where $\widetilde{\psi}_2$ is defined in the same fashion and τ_ρ is the time taken to reach the hypersurface $\{\rho = \rho(x)\}$. Note that $\widetilde{\psi}(x,\xi)$ exists according to the previous lemma and this point is well-defined (it is unique) according to Lemma 8.2.2. Moreover, it is smooth on $\mathcal{U}$ thanks to the results of Section §8.2.1 (this mainly follows from the implicit function theorem). Eventually, it is invariant by the action of the fundamental group and descends on the base as a map ψ. We write $\mathcal{U}_\varepsilon := \mathcal{U} \cap \{\rho = \varepsilon\}$. What we need, is to prove that $\widetilde{\psi}$ is the identity plus a small remainder.

Lemma 8.3.2. $||\widetilde{\psi}_\varepsilon - Id\,||_{C^1} = \mathcal{O}(\varepsilon^\infty)$.

Proof. Since the two trajectories are $\mathcal{O}(\varepsilon^\infty)$ close, so will be the times τ_ρ and $-\tau_-(x,\xi)$ by which the g_1- and g_2-geodesics generated by (p,ζ) hit $\{\rho = \varepsilon\}$ (this can be proved by contradiction for instance, like in the proof of Lemma 8.3.1), which implies that $\widetilde{\psi}_\varepsilon(x,\xi) = (x,\xi) + \mathcal{O}(\varepsilon^\infty)$, where the remainder is uniform in (x,ξ). To obtain a bound on the derivatives, we see from the expression (8.3.2) and the fact that the two flows are $\mathcal{O}(\varepsilon^\infty)$ close in the C^1-topology (Remark 8.3.1), that it is sufficient to show that the times satisfy $\tau_\rho(x,\xi) = -\tau_-(x,\xi) + \mathcal{O}(\varepsilon^\infty)$ in the C^1-topology with a uniform remainder. Let $(p,\zeta) = \overline{\varphi}^1_{\tau_-(x,\xi)}(x,\xi)$. We have

$$\rho(\overline{\varphi}^1_{-\tau_-(x,\xi)}(p,\zeta)) = \varepsilon = \rho(\overline{\varphi}^2_{\tau_\rho}(p,\zeta))$$

We are interested in the variations of x along $\{\rho = \varepsilon\}$ and of the angle $\xi(\omega)$. If we denote by z any of these two parameters, we get by derivating the previous equality :

$$-\frac{\partial\tau_-}{\partial z}d\rho(\overline{X}_1) + d\rho(d\overline{\varphi}^1_{-\tau_-}(d_z(p,\zeta))) = \frac{\partial\tau_\rho}{\partial z}d\rho(\overline{X}_2) + d\rho(d\overline{\varphi}^2_{\tau_\rho}(d_z(p,\zeta)))$$

The two terms containing the differential of the flow coincide to order $\mathcal{O}(\varepsilon^\infty)$ and we also have $d\rho(\overline{X}_2) = d\rho(\overline{X}_1) + \mathcal{O}(\varepsilon^\infty)$ by Remark 8.3.1. Thus :

$$\left(-\frac{\partial \tau_-}{\partial z} - \frac{\partial \tau_\rho}{\partial z}\right) d\rho(\overline{X}_1) = \mathcal{O}(\varepsilon^\infty)$$

But $d\rho(\overline{X}_1)$ is precisely the sine of the angle with which the geodesic generated by (p,ζ) enters the set $\{\rho \geq \varepsilon\}$ and this angle is contained in $[\varepsilon^N, \pi - \varepsilon^N]$ by construction of the set $\mathcal{U}$, so $d\rho(\overline{X}_1) \geq \varepsilon^N$. By dividing by $d\rho(\overline{X}_1)$, this term is swallowed in the $\mathcal{O}(\varepsilon^\infty)$, which provides the sought result. □

Given $(x,\xi) \in \partial_- S^*\widetilde{M}^i_\varepsilon$, we denote by $\ell^i_{\varepsilon,+}(x,\xi)$ the length of the geodesic generated by this point in $\widetilde{M}_\varepsilon$. Note that by strict convexity of the sets $\{\rho \geq \varepsilon\}$ the intersections of the geodesics (for both metrics) with $\widetilde{M}_\varepsilon$ have a single connected component, so this length is well-defined.

Lemma 8.3.3. *$||\ell^1_{\varepsilon,+} - \ell^2_{\varepsilon,+} \circ \widetilde{\psi}_\varepsilon||_{C^0} = \mathcal{O}(\varepsilon^\infty)$, where the sup is computed over $\partial_- S^*\widetilde{M}^1_\varepsilon \setminus \Gamma^1_-$.*

Proof. Recall that $(p,\zeta) \in \partial_- S^*\widetilde{M}$ is the point obtained by flowing backwards (x,ξ) down to the boundary. If D_i denotes the renormalized distance for both metrics, then we have :

$$D_1(p,x) = D_2(p, x'(x,\omega)) + \mathcal{O}(\varepsilon^\infty),$$

where the remainder is independent of (x,ξ). Indeed, considering $0 < \varepsilon' < \varepsilon$, and denoting by $\alpha_1(p,x)$ the g_1-geodesic joining p to x, one has :

$$\begin{aligned}
\ell_1(\alpha_1(p,x) \cap \{\rho > \varepsilon'\}) + \log\varepsilon' &= \int_{\tau^1_{\varepsilon'}}^{\tau^1_\varepsilon} \frac{ds}{\rho(\overline{\varphi}^1_s(z))} + \log\varepsilon' \\
&= \int_{\varepsilon'}^{\varepsilon} \frac{(\psi_1^{-1})'(u)du}{u} + \log\varepsilon',
\end{aligned}$$

where τ^1_ε and $\tau^1_{\varepsilon'}$ are defined such that $\rho(\overline{\varphi}^1_{\tau^1_\varepsilon}(z)) = \varepsilon, \rho(\overline{\varphi}^1_{\tau^1_{\varepsilon'}}(z)) = \varepsilon'$, and $\psi_1 : s \mapsto \rho(\overline{\varphi}^1_s(z))$ is a diffeomorphism. Note that $\psi_1(0) = 0, \psi_1'(0) = 1$. By assumption, the two metrics are close, thus $\psi_1(s) = \psi_2(s) + \mathcal{O}(s^\infty)$ and one can check (by induction) that this implies that $(\psi_1^{-1})^{(k)}(0) = (\psi_2^{-1})^{(k)}(0)$ for all $k \in \mathbb{N}$, that is $\psi_1^{-1}(u) = \psi_2^{-1}(u) + \mathcal{O}(u^\infty)$. Inserting this into the previous integral expression, we get the claimed result.

The same occurs for the other bits of the geodesics : namely, if y and y' denote the exit points of $\alpha_1(p,q)$ and $\alpha_2(p,q)$ in $\widetilde{M}_\varepsilon$, then $D_1(q,y) = D_2(q,y') + \mathcal{O}(\varepsilon^\infty)$. Now, using the fact that the renormalized lengths agree on the boundary, we obtain :

$$\begin{aligned}
D_1(p,q) &= D_1(p,x) + d_1(x,y) + D_1(y,q) \\
&= D_1(p,x) + \ell^1_{\varepsilon,+}(x,\xi) + D_1(y,q) \\
&= D_2(p,q) \\
&= D_2(p,x') + \ell^2_{\varepsilon,+}(\widetilde{\psi}_\varepsilon(x,\xi)) + D_2(y',q)
\end{aligned}$$

Thus : $\ell^1_{\varepsilon,+}(x,\xi) = \ell^2_{\varepsilon,+}(\widetilde{\psi}_\varepsilon(x,\xi)) + \mathcal{O}(\varepsilon^\infty)$. □

8.3.4 The average angle deviation

The angle of deviation $\widetilde{f}$ satisfies two elementary properties :

Lemma 8.3.4. *1. It is π-symmetric, that is, for almost all $(x,\xi) \in S^*\widetilde{M}_1, \theta \in [0,\pi]$,*

$$\widetilde{f}(x,\xi,\theta) = \pi - \widetilde{f}(x, R_\theta \xi, \pi - \theta) \tag{8.3.3}$$

2. It is superadditive in the sense that, for almost all $(x,\xi) \in S^*\widetilde{M}_1, \theta_1, \theta_2 \in [0,\pi]$ such that $\theta_1 + \theta_2 \in [0,\pi]$,

$$\widetilde{f}(x,\xi,\theta_1) + \widetilde{f}(x, R_{\theta_1}\xi, \theta_2) \leq \widetilde{f}(x,\xi,\theta_1+\theta_2) \tag{8.3.4}$$

We will denote by $\mathcal{H} : \mathcal{G}_1 \to \mathcal{G}_2$ the map that associates to a g_1-geodesic with endpoints $z, z' \in \widetilde{M}$ the g_2-geodesic with same endpoints. Note that when $\mathcal{G}_1$ and $\mathcal{G}_2$ are identified with $\partial\widetilde{M} \times \partial\widetilde{M}$, $\mathcal{H}$ is simply the identity, but we will rather see $\mathcal{G}_i$ as the set of geodesics connecting two boundary points.

Proof. The π-symmetry is obtained from the very definition of $\widetilde{f}$. As to the superadditivity, it follows from Gauss-Bonnet formula in negative curvature. Indeed, consider the three geodesics $\alpha_1, \beta_1, \gamma_1$ of $\widetilde{M}_1$, carried by the points $(x,\xi), (x, R_{\theta_1}\xi), (x, R_{\theta_1+\theta_2}\xi)$. Their image by $\mathcal{H}$ (that is the corresponding g_2-geodesics with same endpoints) are three geodesics $\alpha_2 = \mathcal{H}(\alpha_1), \beta_2 = \mathcal{H}(\beta_2), \gamma_2 = \mathcal{H}(\gamma_2)$, forming a geodesic triangle which we denote by T, with angles

$$\widetilde{f}(x,\xi,\theta_1), \widetilde{f}(x, R_{\theta_1}\xi, \theta_2), \widetilde{f}(x, R_{\theta_1+\theta_2}\xi, \pi - \theta_1 - \theta_2)$$

Now, we have by Gauss-Bonnet formula :

$$0 \geq \int_T \kappa \ \ d\mathrm{vol}_g = \widetilde{f}(x,\xi,\theta_1) + \widetilde{f}(x, R_{\theta_1}\xi, \theta_2) + \widetilde{f}(x, R_{\theta_1+\theta_2}\xi, \pi - \theta_1 - \theta_2) - \pi \tag{8.3.5}$$

Using π-symmetry, we obtain inequality (8.3.3). □

Note that the inequality (8.3.4) is saturated if and only if the geodesic triangle is degenerate, that is it is reduced to a single point, since the curvature is negative. As mentioned previously, $\widetilde{f}$ descends on the base as a function f which also satisfies the properties of Lemma 8.3.4.

One of the ideas of Otal was to introduce the *average angle of deviation*. Since we work in a non-compact setting, we are forced to consider partial averages depending on ε. We define for fixed $\varepsilon > 0$:

$$\Theta_\varepsilon(\theta) := \frac{1}{\mathrm{vol}_{g_1}(S^*M^1_\varepsilon)} \int_{S^*M^1_\varepsilon} f(x,\xi,\theta) d\mu_1(x,\xi) \tag{8.3.6}$$

It also satisfies

$$\Theta_\varepsilon(0) = 0, \Theta_\varepsilon(\pi) = \pi \tag{8.3.7}$$

Since the rotations R_θ preserve the Liouville measure, by integrating over $S^*M^1_\varepsilon$ the relations (8.3.3) and (8.3.4) given in Lemma 8.3.4, we see that Θ_ε also satisfies the π-symmetry :

$$\forall \theta \in [0,\pi], \quad \Theta_\varepsilon(\theta) = \pi - \Theta_\varepsilon(\pi - \theta), \tag{8.3.8}$$

and the superadditivity :

$$\forall \theta_1, \theta_2 \in [0,\pi], \text{ s.t. } \theta_1 + \theta_2 \in \ [0,\pi], \quad \Theta_\varepsilon(\theta_1) + \Theta_\varepsilon(\theta_2) \leq \Theta_\varepsilon(\theta_1 + \theta_2) \tag{8.3.9}$$

We now show that Θ_ε satisfies the following

Lemma 8.3.5. *Let $J : [0, \pi] \to \mathbb{R}$ be a convex continuous function. Then :*

$$\int_0^{\pi} J(\Theta_\varepsilon(\theta)) \sin(\theta) d\theta \leq \int_0^{\pi} J(\theta) \sin(\theta) d\theta + ||J||_{L^\infty} \mathcal{O}(\varepsilon^N), \tag{8.3.10}$$

where the remainder only depends on ε, N is fixed by Lemma 8.3.1.

The proof of this lemma relies on the use of Santaló's formula, together with the fact that the Liouville currents coincide. But let us make a preliminary remark. Consider $(x, \xi(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon$ with $\omega \in [\omega_\varepsilon, \pi - \omega_\varepsilon]$. It generates the g_1-geodesic $\alpha_1(p, q)$ with endpoints $p, q \in \partial\widetilde{M}$ which enters (resp. exits) $\widetilde{M}_\varepsilon$ at x (resp. y). We denote by α_2 the g_2-geodesic joining p and q which enters (resp. exits) $\widetilde{M}_\varepsilon$ at $x' = x'(\widetilde{\psi}_\varepsilon(x, \xi))$ (resp. y'). Let us denote by $\mathcal{F}_1(x, y) \subset \mathcal{G}$ the g_1-geodesics which have a positive transverse intersection with the geodesic segment $\alpha^1_\varepsilon := \alpha_1 \cap \widetilde{M}_\varepsilon$. $\mathcal{F}_2(x', y')$ denotes its analogue for the second metric, that is the g_2-geodesics having a positive transverse intersection with $\alpha^2_\varepsilon := \alpha_2 \cap \widetilde{M}_\varepsilon$.

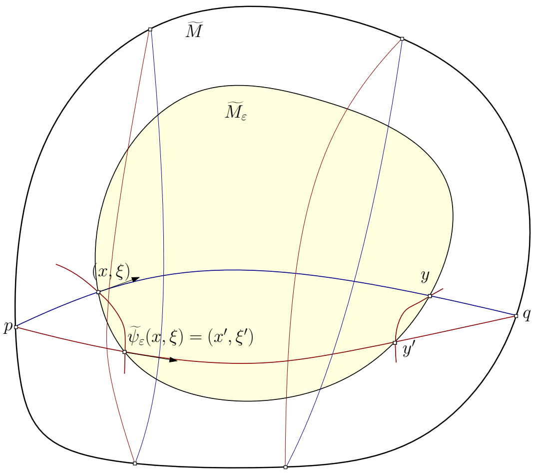

FIGURE 8.4 – A picture of the situation : in red, the g_2-geodesics, in blue, the g_1-geodesics

Since $\mathcal{H}$ preserves the Liouville measure (that is $\mathcal{H}_*\eta_1 = \eta_2$), we have :

$$\eta_1(\mathcal{F}_1(x, y)) = \eta_2(\mathcal{H}(\mathcal{F}_1(x, y)))$$

We could hope that $\mathcal{H}(\mathcal{F}_1(x, y)) = \mathcal{F}_2(x', y')$ but this is not the case (see Figure 8.4), insofar as there is a slight defect due to the fact that we are not looking at points on the boundary, and this is where the arguments of Otal fail to apply immediately. However, we have :

Lemma 8.3.6.

$$\eta_1(\mathcal{F}_1(x, y)) = \eta_2(\mathcal{F}_2(x', y')) + \mathcal{O}(\varepsilon^\infty),$$

where the remainder is independent of (x, ξ).

Proof. It follows from Lemma 8.3.3, combined with equation (8.2.17). □

We can now establish the lemma on convexity. We will denote with a tilde $\tilde{\cdot}$ the objects on the universal cover.

Proof. $d\mu_1/\mathrm{vol}_{g_1}(S^*M^1_\varepsilon)$ is a probability measure on $S^*M^1_\varepsilon$ and by Jensen inequality, we have, for all $\theta \in [0, \pi]$:

$$J(\Theta_\varepsilon(\theta)) \leq \frac{1}{\mathrm{vol}_{g_1}(S^*M^1_\varepsilon)} \int_{S^*M^1_\varepsilon} J(f(x, \xi, \theta)) d\mu_1(x, \xi)$$

Multiplying by $\sin(\theta)$, integrating over $[0, \pi]$ and applying Fubini's Theorem, we obtain :

$$\int_0^{\pi} J(\Theta_\varepsilon(\theta)) \sin(\theta) d\theta \leq \frac{1}{\mathrm{vol}_{g_1}(S^*M^1_\varepsilon)} \int_{S^*M^1_\varepsilon} \int_0^{\pi} J(f(x, \xi, \theta)) \sin(\theta) d\theta d\mu_1(x, \xi)$$

Using Santaló's formula, we obtain for the last integral :

$$\begin{aligned}\int_{S^*M^1_\varepsilon}\int_0^\pi & J(f(x,\xi,\theta))\sin(\theta)d\theta d\mu_1(x,\xi) \\ &= \int_{\partial_- S^*M^1_\varepsilon}\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi^1_\tau(x,\xi),\theta))\sin(\theta)d\theta d\tau d\mu_{1,\nu}(x,\xi)\end{aligned},$$

where $d\mu_{1,\nu}(x,\xi) = |g_1(\xi,\nu_1)| i^*_{\partial S^*M^1_\varepsilon}(d\mu_1)$, ν_1 is unit covector conormal to the boundary, $i^*_{\partial S^*M^1_\varepsilon}(d\mu_1)$ is the restriction of the Liouville measure to the boundary (the measure induced by the Sasaki metric restricted to $\partial S^*M^1_\varepsilon$), and $\ell^1_{\varepsilon,+}(x,\xi)$ is the length of the geodesic starting from (x,ξ) in M_ε. Note that we would formally need to remove the set of trapped geodesics when applying Santaló's formula. However, as mentionned in Remark 8.2.3, they have zero measure and do not influence the computation, so we forget them in order not to complicate the notations. By parametrizing each fiber $\partial_- S^*_x M^1_\varepsilon$ with an angle $\omega \in [0,\pi]$, we can still disintegrate the measure $d\mu_{1,\nu} = \sin(\omega)d\omega dx$, where dx is the measure induced by the metric g_1 on ∂M_ε and $d\omega$ is the measure in the fiber $\partial_- S^*M^1_\varepsilon$, so that :

$$\begin{aligned}&\int_{S^*M^1_\varepsilon}\int_0^\pi J(f(x,\xi,\theta))\sin(\theta)d\theta d\mu_1(x,\xi) \\ &= \int_{\partial M_\varepsilon}\int_0^\pi\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi^1_\tau(x,\xi),\theta))\sin(\theta)d\theta d\tau\sin(\omega)d\omega dx \\ &= \int_{\partial M_\varepsilon}\int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon}\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi^1_\tau(x,\xi),\theta))\sin(\theta)d\theta d\tau\sin(\omega)d\omega dx + ||J||_{L^\infty}\mathcal{O}(\varepsilon^N),\end{aligned}$$

Recall that we applied Lemma 8.3.1 with $\omega_\varepsilon = \mathcal{O}(\varepsilon^{N+1})$. The loss of 1 in the exponent is due to the fact that we have to swallow uniformly the lengths $\ell^1_{\varepsilon,+}(x,\xi) = \mathcal{O}(-\log\varepsilon)$ in the integral.

Let us fix $(x,\xi(\omega)) \in \partial_- S^*M^1_\varepsilon \setminus \Gamma_-$ and consider one of its lift on the universal cover $(\widetilde{x},\widetilde{\xi}(\omega)) \in \partial_- S^*\widetilde{M}^1_\varepsilon \setminus \widetilde{\Gamma}^1_-$. It generates a geodesic with endpoints $(p,q) \in \partial\widetilde{M} \times \partial\widetilde{M}$. We can rewrite the integral

$$\int_0^{\ell^1_{\varepsilon,+}(x,\xi)}\int_0^\pi J(f(\varphi^1_\tau(x,\xi),\theta))\sin(\theta)d\theta d\tau = \int_0^{\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})}\int_0^\pi J(\widetilde{f}(\widetilde{\varphi}^1_\tau(\widetilde{x},\widetilde{\xi}),\theta))\sin(\theta)d\theta d\tau.$$

We will now use the diffeomorphisms $\phi_i : V_i \to \mathcal{F}(p,q)$ (for $i = 1,2$) introduced in Section §8.2.3 (see equation (8.2.16)). The $\widetilde{g}_1$-geodesic joining p to q is denoted by $\alpha_1(p,q)$: we choose a parametrization $\gamma : \mathbb{R} \to \alpha_1(p,q)$ by arc-length using the middle point (see Section §8.2.3). Remark that the composition $\phi_2^{-1} \circ \phi_1 : V_1 \to V_2$ has the form $(\tau,\theta) \mapsto (\,\cdot\,, \widetilde{f}(\gamma(\tau),\dot{\gamma}(\tau),\theta))$ (the first coordinate is of no interest to us). Moreover,

$$\left(\phi_2^{-1}\circ\phi_1\right)^*\sin(\theta)d\theta d\tau = \phi_1^*\eta_2 = \phi_1^*\eta_1 = \sin(\theta)d\theta d\tau,$$

since the two Liouville currents agree according to Lemma 8.2.6. We have :

$$\begin{aligned}
&\int_0^{\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})} \int_0^\pi J(\widetilde{f}(\widetilde{\varphi}^1_\tau(\widetilde{x},\widetilde{\xi}),\theta)) \sin(\theta) d\theta d\tau \\
&\qquad = \phi_1^* \eta_1 (J \circ \phi_2^{-1} \circ \phi_1 \cdot \mathbf{1}_{[T, T+\widetilde{\ell}^1_{\varepsilon,+}(\widetilde{x},\widetilde{\xi})]\times[0,\pi]}) \\
&\qquad = \eta_1 (J \circ \phi_2^{-1} \cdot \mathbf{1}_{\mathcal{F}_1(\widetilde{x},\widetilde{y})}) \\
&\qquad = \eta_2 (J \circ \phi_2^{-1} \cdot \mathbf{1}_{\mathcal{H}(\mathcal{F}_1(\widetilde{x},\widetilde{y}))}) \\
&\qquad = \eta_2 (J \circ \phi_2^{-1} \cdot \mathbf{1}_{\mathcal{F}_2(\widetilde{x}',\widetilde{y}')}) + ||J||_{L^\infty} \mathcal{O}(\varepsilon^\infty) \\
&\qquad = \int_0^{\widetilde{\ell}^2_{\varepsilon,+}(\widetilde{x}',\widetilde{\xi}')} \int_0^\pi J(\theta) \sin(\theta) d\theta d\tau + ||J||_{L^\infty} \mathcal{O}(\varepsilon^\infty) \\
&\qquad = \widetilde{\ell}^2_{\varepsilon,+}(\widetilde{x}',\widetilde{\xi}') \int_0^\pi J(\theta)\sin(\theta) d\theta \ + ||J||_{L^\infty} \mathcal{O}(\varepsilon^\infty),
\end{aligned}$$

where the fourth equality follows from Lemma 8.3.6. The constant T on the second line is unknown and appears in the choice of parametrization of the geodesic segment $\alpha_1(\widetilde{x},\widetilde{y})$ but does not influence the computation. The point $(\widetilde{x}',\widetilde{\xi}') = \widetilde{\psi}_\varepsilon(\widetilde{x},\widetilde{\xi})$ is the image of $(\widetilde{x},\widetilde{\xi})$ by the diffeomorphism $\widetilde{\psi}_\varepsilon$ defined in Section §8.3.3. We recall that this diffeomorphism is invariant by the fundamental group and descends on the base as ψ_ε.

Inserting this into the previous integrals, we obtain :

$$\begin{aligned}
&\int_{S^*M^1_\varepsilon} \int_0^\pi J(f(x,\xi,\theta)) \sin(\theta) d\theta d\mu_1(x,\xi) \\
&= \int_0^\pi J(\theta)\sin(\theta) d\theta \int_{\partial M_\varepsilon} \int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon} \ell^2_{\varepsilon,+}(\psi_\varepsilon(x,\xi(\omega))) \sin(\omega) d\omega dx \ + ||J||_{L^\infty} \mathcal{O}(\varepsilon^N)
\end{aligned}$$

According to Lemma 8.3.2, we know that $\psi_\varepsilon = \mathrm{Id} + \mathcal{O}(\varepsilon^\infty)$ in the C^1 topology. In particular, the Jacobian of ψ_ε is $1 + \mathcal{O}(\varepsilon^\infty)$ and by a change of variable :

$$\begin{aligned}
\int_{\partial M_\varepsilon} \int_{\omega_\varepsilon}^{\pi-\omega_\varepsilon} \ell^2_{\varepsilon,+}(\psi_\varepsilon(x,\xi(\omega))) \sin(\omega) d\omega dx &= \int_{\partial M_\varepsilon} \int_0^\pi l^2_\varepsilon(x',\xi') \sin(\omega') d\omega' dx' + \mathcal{O}(\varepsilon^N) \\
&= \mathrm{vol}_{g_2}(S^*M^2_\varepsilon) + \mathcal{O}(\varepsilon^N) \\
&= \mathrm{vol}_{g_1}(S^*M^1_\varepsilon) + \mathcal{O}(\varepsilon^N),
\end{aligned}$$

where the two volumes agree to order $\mathcal{O}(\varepsilon^N)$ according to the same computation with $J \equiv 1$. Inserting this into the previous integrals, we obtain the sought result.

□

Remark that we can actually consider in Lemma 8.3.5 a family of functions J_ε, instead of a single function. We can assume that $||J_\varepsilon||_{L^\infty} = \mathcal{O}(1/\varepsilon^\alpha)$, for some $\alpha > 0$ which we may take as large as we want. Then, we can always apply the lemma with $N' := N + \lfloor \alpha \rfloor + 1$, so that in the end, the sup norm $||J_\varepsilon||_{L^\infty}$ is swallowed in the term $\mathcal{O}(\varepsilon^N)$. We actually obtain for free a better version :

Lemma 8.3.7. *Let $N \in \mathbb{N} \setminus \{0\}$ be an integer and $\alpha > 0$. Let $J_\varepsilon : [0,\pi] \to \mathbb{R}$ be a family of convex continuous function such that $||J_\varepsilon||_{L^\infty} = \mathcal{O}(\varepsilon^{-\alpha})$. Then :*

$$\int_0^\pi J_\varepsilon(\Theta_\varepsilon(\theta)) \sin(\theta) d\theta \leq \int_0^\pi J_\varepsilon(\theta) \sin(\theta) d\theta + \mathcal{O}(\varepsilon^N), \tag{8.3.11}$$

where the remainder only depends on ε.

8.4 Estimating the average angle of deviation

As mentioned previously, we are unable to prove a priori that the Θ_ε are uniformly Lipschitz. Nevertheless, we can show that they decompose as a sum $\Theta_\varepsilon^{(a)} + \Theta_\varepsilon^{(b)}$ where the $\Theta_\varepsilon^{(a)}$ are Lipschitz (and their Lipschitz constant is controlled) and the $\Theta_\varepsilon^{(b)}$ have a "small" C^0 norm. This will be sufficient to apply our version of Otal's estimate (see Proposition 8.4.1).

8.4.1 Derivative of the angle of deviation

The purpose of this paragraph is to estimate the derivative (with respect to θ) of the angle of deviation f. We recall that

$$W_1 = \left\{(x,\xi,\theta) \in S^*M_1 \times (0,\pi) \mid (x,\xi),(x,R_\theta\xi) \notin (\Gamma_-^1 \cup \Gamma_+^1)\right\}$$

Lemma 8.4.1. *There exist constants $C, k > 0$ (independent of ε) such that for all $(x,\xi,\theta) \in S^*M_\varepsilon^1 \cap W_1$:*

$$\left|\frac{\partial f}{\partial \theta}(x,\xi,\theta)\right| \leq C \exp\left(k\left(\ell_{\varepsilon,+}^1(x,R_\theta\xi) + |\ell_{\varepsilon,-}^1(x,R_\theta\xi)|\right)\right)$$

Proof. We can write the derivative of f as :

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial y'}\left(\frac{\partial y'}{\partial \theta}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial \theta}\right), \tag{8.4.1}$$

where y and y' are defined in Section §8.3.2 and study the different terms separately.

The idea is to study the behaviour (and more precisely the growth) of Jacobi vector fields in a neighborhood of the boundary. Given a geodesic which enters the set $\{\rho \geq \varepsilon\}$, we will use the bounds (8.2.13) to estimate the Jacobi vector fields on the segment contained in $\{\rho \geq \varepsilon\}$. Then, by convexity, the geodesic exits $\{\rho \geq \varepsilon\}$ with a coordinate $\overline{\xi}_0 \leq 0$. On the set $\mathcal{C} = \{\rho < \delta\} \cap \left\{\overline{\xi}_0 \leq 0\right\}$ (for some $\delta > 0$ small enough), we can study the behaviour of the geodesics more explicitly. Namely, given any point $(x,\xi) \in S^*M$ in $\mathcal{C}$, we know that it converges uniformly exponentially fast to the boundary in the sense that there exists $C > 0$ (uniform in (x,ξ)) such that if $\rho(t) := \rho(\varphi_t(x,\xi))$, then one has $\rho(0)e^{-t} \leq \rho(t) \leq C\rho(0)e^{-t}$ for $t \geq 0$ (see [GGSU17, Lemma 2.3]). From the expression of the metric (8.1.1) in local coordinates, one can check that the curvature is given by $\kappa = -1 + \rho \cdot \mathcal{O}(1)$. As a consequence, if $\kappa(t) = \kappa(\pi(\varphi_t(x,\xi)))$ and $\delta > 0$ is chosen small enough at the beginning, one has that $-1 - \frac{1}{10}e^{-t} \leq \kappa(t) \leq -1 + \frac{1}{10}e^{-t}$, for any such (x,ξ). If $t \mapsto \gamma(t)$ denotes the geodesic generated by this point and J is a normal Jacobi vector field along γ, we write $J(t) = j(t)R_{\pi/2}\dot{\gamma}(t)$, where j satisfies the Jacobi equation $\ddot{j}(t) + \kappa(t)j(t) = 0$. Assume $j(0) = 0, \dot{j}(0) = 1$, then $j(t) > 0$ (there are no conjugate points) and thus $\ddot{j}(t) \leq (1 + \frac{1}{10}e^{-t})j(t)$. By a comparison argument, $j(t) \leq z(t)$ where z is the solution to $\ddot{z}(t) - (1 + \frac{1}{10}e^{-t})z(t) = 0$ with $z(0) = j(0), \dot{z}(0) = \dot{j}(0)$.

But making the change of variable $u = 2\sqrt{10}e^{-t/2}$, $\widetilde{z}(u) = z(t)$, one can prove that $\widetilde{z}$ solves the modified Bessel equation of parameter 2 that is

$$u^2\frac{d^2\widetilde{z}}{du^2} + u\frac{d\widetilde{z}}{du} - (u^2 + 2^2)\widetilde{z} = 0$$

and thus $\widetilde{z}(u) = A \cdot I_2(u) + B \cdot K_2(u)$ for some parameters $A, B \in \mathbb{R}$ depending on $\widetilde{z}(0)$, $\dot{\widetilde{z}}(0)$, I_2 and K_2 being the modified Bessel functions of first and second kind.

Thus : $z(t) = A \cdot I_2(2\sqrt{10}e^{-t/2}) + B \cdot K_2(2\sqrt{10}e^{-t/2})$ where $I_2(2\sqrt{10}e^{-t/2}) \sim_{t\to+\infty} Ce^{-t}$, $K_2(2\sqrt{10}e^{-t/2}) \sim_{t\to+\infty} Ce^{t}$ (see [AS64, 9.6.7-9.6.9]) For instance, if $j(0) = 0, \dot{j}(0) = 1$, which corresponds to a vertical variation of geodesics, then we obtain $|d\pi \circ d\varphi_t(V)| = |J(t)| \leq Ce^t$ for some constant $C > 0$ independent of the point. Using this technique of comparison and decomposing any vector by its vertical and horizontal components, one obtains that $||d\varphi_t(x,\xi)|| \leq Ce^t$ for $(x,\xi) \in \mathcal{C}$, where the constant $C > 0$ is uniform in (x,ξ).

We fix (x_0,ξ_0,θ_0) and look at the variation $\theta \mapsto (x_0, R_{\theta_0+\theta}\xi_0)$. For each θ, we thus have a g_1-geodesic $t \mapsto \gamma_\theta(t)$ generated by this point and it hits the boundary in the future at $y'(\theta)$. We set $\gamma := \gamma_0$. We denote by $J(t) := \partial_\theta\gamma_\theta(t)$ the Jacobi vector field along γ. Writing in short $\ell^1_{+,\varepsilon} = \ell^1_{+,\varepsilon}(x_0, R_{\theta_0}\xi_0), V = V(x_0, R_{\theta_0}\xi_0)$, we have for $t = s + l_\varepsilon, s \geq 0$:

$$|J(t)|_{g_1} = \left|d\pi \circ d\varphi_{s+\ell^1_{+,\varepsilon}}(V)\right| \leq Ce^s|d\pi \circ d\varphi_{l_\varepsilon}(V)| \leq Ce^se^{k\ell^1_{+,\varepsilon}}$$

The first inequality follows from our previous remarks whereas the second one is a consequence of (8.2.13). Now, we know that $\rho(\ell^1_{+,\varepsilon})e^{-s} = \varepsilon e^{-s} \leq \rho(t) \leq C\varepsilon e^{-s} = C\rho(\ell^1_{+,\varepsilon})e^{-s}$. As a consequence, for t large enough, we have : $|J(t)|_{\overline{g}_1} = \rho(t)|J(t)|_{g_1} \leq C \cdot \varepsilon e^{k\ell^1_{+,\varepsilon}}$. By making $t \to +\infty$, we obtain that $\left|\frac{\partial y'}{\partial\theta}\right|_h \leq C \cdot \varepsilon e^{k\ell^1_{+,\varepsilon}}$.

Conversely, we consider a family of points $y'(u)$ in a neighborhood of y'_0 on the boundary (such that $\left|\frac{\partial y'}{\partial u}\right|_h = 1$) and we look at the g_2-geodesics joining y to $y'(u)$. They intersect the g_2-geodesic joining z to z' (the endpoints of the geodesic genreated by (x,ξ)) at some point $\mathrm{x}(u)$, and we obtain $(\mathrm{x}(u),\Xi(u))$ and an angle $f(u)$. From another perspective, we have a family of points $(\mathrm{x}(u), R_{f(u)}\Xi(u))$ which generate geodesics joining $y'(u)$ (in the future) to y (in the past). Like before, we denote by γ the geodesic obtained for $u = 0$ and by J the Jacobi vector field along γ. Since the point y joined in the past by the geodesic is fixed (it does not depend on u), J (more precisely, its lift in TS^*M) lies in the unstable bundle. We write

$$\partial_u(\mathrm{x}(u), R_{f(u)}\Xi(u)) = d\pi^{-1}(J(0)) + \mathcal{K}^{-1}(\nabla_t J(0)) = \lambda \cdot \xi_u,$$

where ξ_u is one of the two unit vectors (with respect to the g_2-Sasaki metric) generating E_u. Note that the vertical component of this vector is precisely $\frac{\partial f}{\partial u}V$ and thus $|\lambda| \geq \left|\frac{\partial f}{\partial u}\right|$. We write $\ell^2_{+,\varepsilon} = \ell^2_{+,\varepsilon}(\mathrm{x}, R_f\Xi)$. For $t = s + \ell^2_{+,\varepsilon}, s \geq 0$:

$$\begin{aligned} |J(t)|_{g_2} = |d\pi \circ d\varphi_t(\lambda\xi_u)| &= |\lambda| \cdot |d\pi \circ d\varphi_s\left(d\varphi_{\ell^2_{+,\varepsilon}}(\xi_u)\right)| \\ &\geq |\lambda| \cdot e^s|d\varphi_{\ell^2_{+,\varepsilon}}(\xi_u)| \\ &\geq C|\lambda|e^se^{k\ell^2_{+,\varepsilon}} \geq C\left|\frac{\partial f}{\partial u}\right|e^se^{k\ell^2_{+,\varepsilon}} \end{aligned}$$

The term in $e^{k\ell^2_{+,\varepsilon}}$ follows from (8.2.13) whereas the term e^s is a consequence on the bounds of the curvature. More precisely, for fixed bounds, that is $-k_0^2 \leq \kappa \leq -k_1^2$, such a lower bound is obtained in [Kli95, Theorem 3.2.17], and the same proof applies here, except that we have bounds $-1 - \frac{1}{10}e^{-t} \leq \kappa(t) \leq -1 + \frac{1}{10}e^{-t}$. But the argument of Klingenberg is based on Gronwall lemma and $t \mapsto e^{-t}$ is integrable, so we get the same result in the end. Multiplying by $\rho(t)$ and taking the limit as $t \to +\infty$, we eventually obtain that $\left|\frac{\partial y'}{\partial u}\right|_h = 1 \geq C\varepsilon e^{k\ell^2_{+,\varepsilon}} \left|\frac{\partial f}{\partial u}\right|$.

Putting the previous bounds together, and using (8.4.1), we obtain the sought result. □

8.4.2 Derivative of the exit time

We set $T_\varepsilon = -N_0 \log \varepsilon$ for some integer N_0, like in the proof of Lemma 8.2.3.

Lemma 8.4.2. *There exist constants $C, k > 0$ (independent of ε) such that for all $(x, \xi, \theta) \in S^*M^1_\varepsilon \cap W_1$ such that*

$$T_\varepsilon \leq \ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|,$$

one has :

$$\partial_\theta \left(\ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|\right) \leq C \exp\left(k\left(\ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|\right)\right)$$

Proof. Let us deal with the case of the exit time in the future, the other case being similar. The exit time is defined by the implicit equation :

$$\rho\left(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi)\right) = \varepsilon$$

Differentiating with respect to θ, we obtain :

$$\partial_\theta\left(\ell^1_{\varepsilon,+}(x, R_\theta\xi)\right) d\rho\left(X_1(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi))\right) + d\rho\left(d\left(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\right)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\right) = 0,$$

where $V(x, \xi) \in \mathbb{V}$ is the vertical vector in (x, ξ) (it is unitary with respect to the Sasaki metric G_1). But :

$$\left|d\rho\left(X_1(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}(x, R_\theta\xi))\right)\right| = \varepsilon|d\rho(\overline{X}_1)|,$$

and $d\rho(\overline{X}_1)$ is the sine of the angle with which the geodesic exits the region $\{\rho \geq \varepsilon\}$. If this angle is less than $\frac{1}{10}$ (any small constant works as long as the geodesics concerned stay in a region where the metric still has the usual expression (8.1.1)), then the geodesic will spend at most a bounded (independently of ε) amount of time in the region $\{\rho \geq \varepsilon\}$, thus contradicting the condition :

$$T_\varepsilon = -N_0 \log(\varepsilon) \leq \ell^1_{\varepsilon,+}(x, R_\theta\xi) + |\ell^1_{\varepsilon,-}(x, R_\theta\xi)|$$

This can be proved using the Hamilton's equations, similarly to the proof of Lemma 8.3.1 for instance. Thus $|d\rho(\overline{X}_1)| \geq \frac{1}{10}$.

As to the second term, using the fact that $d\rho/\rho$ is unitary (with respect to the dual metric of g_1 on the cotangent space), we obtain that :

$$\begin{aligned}\left|\rho\frac{d\rho}{\rho}\left(d\left(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\right)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\right)\right| &\leq \varepsilon\left|d\left(\varphi^1_{\ell^1_{\varepsilon,+}(x,R_\theta\xi)}\right)_{(x,R_\theta\xi)} V(x, R_\theta\xi)\right|_{G_1} \\ &\leq \varepsilon e^{k\ell^1_{\varepsilon,+}(x,R_\theta\xi)},\end{aligned}$$

for some constant k, following (8.2.13). This provides the sought result. □

8.4.3 An inequality on the average angle of deviation

We know that f is almost everywhere continuous and bounded, so Θ_ε is continuous by Lebesgue theorem. We now prove that the homeomorphism Θ_ε satisfies the following estimate :

Lemma 8.4.3. *For any $\delta \in (Q, 0)$ (defined in Lemma 8.2.3), for all $\beta > 0$ small enough, there exists $\beta' > 0$ (depending on β and converging towards 0 as $\beta \to 0$) such that :*

$$\forall \theta_1, \theta_2 \in [0, \pi], \quad |\Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2)| \lesssim \varepsilon^{-\beta'} |\theta_1 - \theta_2|^\beta + \varepsilon^\delta$$

Proof. First, remark that it is sufficient to prove the lemma for $\theta_1, \theta_2 \in [0, \pi/2]$, since the result will follow from the π-symmetry of the homeomorphism Θ_ε. We fix $\varepsilon > 0$. We introduce the smooth cutoff function χ_T (for some $T > 0$ which will be chosen to depend on ε later) such that $\chi_T(s) \equiv 1$ on $[0, T]$ and $\chi_T(s) \equiv 0$ on $[2T, +\infty)$. Note that we can always construct such a χ_T so that $||\partial_s \chi_T||_{L^\infty} \leq 1$ (as long as $T > 1$, which we can assume since it will be chosen growing to infinity as $\varepsilon \to 0$). We write $\Theta_\varepsilon = \Theta_\varepsilon^{(a),T} + \Theta_\varepsilon^{(b),T}$, where :

$$\begin{aligned}
\Theta_\varepsilon^{(a),T}(\theta) &:= \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \chi_T\left(\ell_{\varepsilon,+}^1(x, R_\theta \xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\right) f(x, \xi, \theta) d\mu_1(x, \xi) \\
&= \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \psi_T(x, \xi, \theta)
\end{aligned}$$

where ψ_T is defined to be the integrand and

$$\Theta_\varepsilon^{(b),T}(\theta) := \Theta_\varepsilon - \Theta_\varepsilon^{(a),T}$$

Morally, the cutoff function mean that we integrate over the compact region

$$\left\{\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)| \leq T\right\}$$

By Lebesgue theorem, $\Theta_\varepsilon^{(a),T}$ is C^1 on $[0, \pi/2]$. For $\beta > 0$, $\theta_1, \theta_2 \in [0, \pi/2]$, one has :

$$|\Theta_\varepsilon^{(a),T}(\theta_1) - \Theta_\varepsilon^{(a),T}(\theta_2)| \lesssim \sup_{\theta \in [0,\pi/2]} \left|\partial_\theta \Theta_\varepsilon^{(a),T}(\theta)\right|^\beta |\theta_1 - \theta_2|^\beta$$

Let us estimate the former derivative. We have :

$$\partial_\theta \Theta_\varepsilon^{(a),T}(\theta) = \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \int_{S^*M_\varepsilon^1} \partial_\theta \psi_T(x, \xi, \theta) d\mu_1(x, \xi),$$

and the derivative under the integral is composed of a sum of two terms which we now estimate separately.

1. By Lemma 8.4.1, the first term is bounded by :

$$\begin{aligned}
&\left|\chi_T\left(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\right) \partial_\theta f(x, \xi, \theta)\right| \\
&\lesssim \exp\left(k(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|)\right) \lesssim e^{2kT}
\end{aligned}$$

2. And the second term is bounded by Lemma 8.4.2 :

$$\left|\partial_\theta\left(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)\right) \partial_s \chi_T\left(\ell_{\varepsilon,+}^1(x, R_\theta\xi) + |\ell_{\varepsilon,-}^1(x, R_\theta\xi)|\right) f(x, \xi, \theta)\right| \lesssim e^{2kT}$$

Note that the constant $k > 0$ may be different from one line to another. Gathering everything, we obtain that for all $\theta \in [0, \pi/2]$, $|\partial_\theta \Theta_\varepsilon^{(a),T}(\theta)| \lesssim e^{2kT}$ and thus :

$$|\Theta_\varepsilon^{(a),T}(\theta_1) - \Theta_\varepsilon^{(a),T}(\theta_2)| \lesssim e^{2k\beta T}|\theta_1 - \theta_2|^\beta$$

As to $\Theta_\varepsilon^{(b),T}$, we can write :

$$\Theta_\varepsilon^{(b),T}(\theta) \leq \frac{1}{\mathrm{vol}_{g_1}(S^*M_\varepsilon^1)} \left(\int_{S^*M_\varepsilon^1 \cap \left\{\ell_\varepsilon^{1,+}(x,R_\theta\xi) > T\right\}} f d\mu_1 + \int_{S^*M_\varepsilon^1 \cap \left\{|\ell_\varepsilon^{1,-}(x,R_\theta\xi)| > T\right\}} f d\mu_1 \right)$$

If $T \geq -N_0 \log(\varepsilon)$ (N_0 is a large integer defined in Lemma 8.2.3, independent of ε), then the two integrals can be estimated by Lemma 8.2.3 (note that we here divide by the volume which is bounded by $\mathcal{O}(\varepsilon)$). We obtain :

$$|\Theta_\varepsilon^{(b),T}(\theta)| \lesssim e^{-\delta T}\varepsilon^{-4\delta}$$

We choose $T := T_\varepsilon = -N_0 \log(\varepsilon)$ and set $\Theta_\varepsilon^{(a)} := \Theta_\varepsilon^{(a),T_\varepsilon}, \Theta_\varepsilon^{(b)} := \Theta_\varepsilon^{(b),T_\varepsilon}$. Since N_0 is taken large enough (greater than 5 at least to swallow the $\varepsilon^{-4\delta}$), we obtain $||\Theta_\varepsilon^{(b)}||_{L^\infty} \lesssim \varepsilon^\delta$. And :

$$|\Theta_\varepsilon^{(a),T}(\theta_1) - \Theta_\varepsilon^{(a),T}(\theta_2)| \leq \varepsilon^{-2\beta k N_0}|\theta_1 - \theta_2|^\beta,$$

which provides the sought result by going back to Θ_ε. □

8.4.4 Otal's lemma revisited

In the spirit of Otal's lemma (see [Ota90, Lemma 8]), we prove :

Proposition 8.4.1. *Assume $\Theta_\varepsilon : [0, \pi] \to [0, \pi]$ is a family of increasing homeomorphisms for $\varepsilon \in (0, \delta)$ such that :*

1. $\Theta_\varepsilon(0) = 0, \Theta_\varepsilon(\pi) = \pi$,
2. *For all $\theta \in [0, \pi], \Theta_\varepsilon(\pi - \theta) = \pi - \Theta_\varepsilon(\theta)$,*
3. *For all $\theta_1, \theta_2 \in [0, \pi]$ such that $\theta_1 + \theta_2 \in [0, \pi]$,*
$$\Theta_\varepsilon(\theta_1) + \Theta_\varepsilon(\theta_2) \leq \Theta_\varepsilon(\theta_1 + \theta_2)$$
4. *There exists constants $C, \beta, \beta' > 0$ and $\delta > 0$ (independent of ε), such that for all $\theta_1, \theta_2 \in [0, \pi]$:*
$$|\Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2)| \leq C \left(\varepsilon^\delta + \varepsilon^{-\beta'}|\theta_1 - \theta_2|^\beta\right)$$
5. *There exists $\alpha > 2\beta'/\beta - 1$ such that for all family of continuous convex functions $J_\varepsilon : [0, \pi] \to \mathbb{R}$ such that $||J_\varepsilon||_{L^\infty} = \mathcal{O}(1/\varepsilon^\alpha)$,*
$$\int_0^\pi J_\varepsilon(\Theta_\varepsilon(\theta)) \sin(\theta) d\theta \leq \int_0^\pi J_\varepsilon(\theta) \sin(\theta) d\theta + \mathcal{O}(\varepsilon)$$

Then $\Theta_\varepsilon = Id + \mathcal{O}(\varepsilon^\gamma)$, where we can take any γ up to the critical exponent

$$\hat{\gamma} := \frac{1 + \alpha - 2\beta'/\beta}{1 + 2/\beta},$$

as long as $\gamma < \delta$.

Proof. We argue by contradiction. Assume there there exists a sequence $\varepsilon_n \to 0$ such that $||\Theta_n - \mathrm{Id}||_{L^\infty} > n\varepsilon_n^\gamma$ (where $\Theta_n := \Theta_{\varepsilon_n}$). By π-symmetry, there exists an interval $[a_n, A_n]$ such that for all $\theta \in (a_n, A_n)$, $\Theta_n(\theta) < \theta - n\varepsilon_n^\gamma$ and we can choose $\Theta_n(a_n) = a_n - n\varepsilon_n^\gamma, \Theta_n(A_n) = A_n - n\varepsilon_n^\gamma$.

We also construct the largest interval $[b_n, B_n] \supset [a_n, A_n]$ such that for all $\theta \in (b_n, B_n)$, $\Theta_n(\theta) < \theta - \varepsilon_n^\gamma$ and $\Theta_n(b_n) = b_n - \varepsilon_n^\gamma, \Theta_n(B_n) = B_n - \varepsilon_n^\gamma$. Eventually, we define the largest interval $[c_n, C_n] \supset [b_n, B_n]$ such that for all $\theta \in (c_n, C_n)$, $\Theta_n(\theta) < \theta$ and $\Theta_n(c_n) = c_n, \Theta_n(C_n) = C_n$. The π-symmetry implies that $\Theta(\pi/2) = \pi/2$ and since $\Theta(0) = 0, \Theta(\pi) = \pi$, we know that the points $c_n < b_n < a_n < A_n < B_n < C_n$ all lie either in $[0, \pi/2]$ or in $[\pi/2, \pi]$.

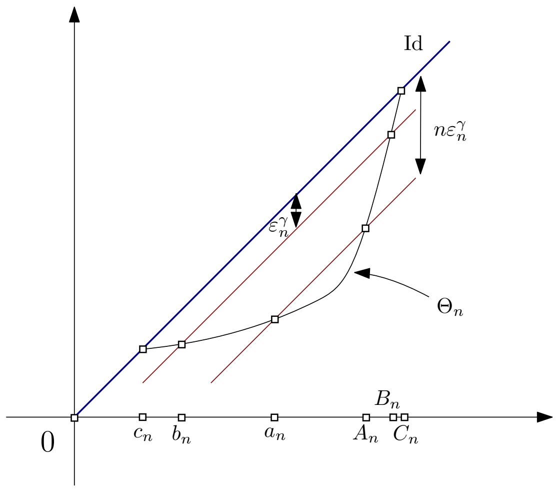

FIGURE 8.5 – The points $c_n < b_n < a_n < A_n < B_n < C_n$

Remark that $\Theta_n - \mathrm{Id}$ also satisfies the fifth item, namely :

$$\begin{aligned}|(\Theta_n - \mathrm{Id})(\theta_1) - (\Theta_n - \mathrm{Id})(\theta_2)| &\lesssim |\Theta_n(\theta_1) - \Theta_n(\theta_2)| + |\theta_1 - \theta_2| \\ &\lesssim \left(\varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}|\theta_1 - \theta_2|^\beta\right) + (2\pi)^{1-\beta}|\theta_1 - \theta_2|^\beta \\ &\lesssim \varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}|\theta_1 - \theta_2|^\beta\end{aligned}$$

This implies that :

$$|(\Theta_n - \mathrm{Id})(a_n) - (\Theta_n - \mathrm{Id})(b_n)| = (n-1)\varepsilon_n^\gamma \lesssim \varepsilon_n^\delta + \frac{1}{\varepsilon_n^{\beta'}}(a_n - b_n)^\beta$$

Thus :

$$(a_n - b_n)^\beta \gtrsim (n-1)\varepsilon_n^{\gamma+\beta'} - \varepsilon_n^{\delta+\beta'} \gtrsim (n-1)\varepsilon_n^{\gamma+\beta'},$$

for n large enough since $\delta > \gamma$. The same inequalities hold for the other points and we get, for n large enough :

$$a_n - b_n \gtrsim (n-1)^{1/\beta}\varepsilon_n^{(\gamma+\beta')/\beta}, \qquad B_n - A_n \gtrsim (n-1)^{1/\beta}\varepsilon_n^{(\gamma+\beta')/\beta}$$

$$b_n - c_n \gtrsim \varepsilon_n^{(\gamma+\beta')/\beta}, \qquad C_n - B_n \gtrsim \varepsilon_n^{(\gamma+\beta')/\beta}$$

Now, for $h \in (0, C_n - c_n)$, by superadditivity :

$$c_n + h > \Theta_n(c_n + h) \geq \Theta_n(c_n) + \Theta_n(h) = c_n + \Theta_n(h),$$

that is $\Theta_n(h) < h$. In the same fashion, we have for $h \in (b_n - c_n, B_n - c_n)$, $\Theta_n(h) < h - \varepsilon_n^\gamma$.

Let us now consider the continuous convex functions $J_n(x) := \varepsilon_n^{-\alpha} \sup(C_n - c_n - x, 0) = \varepsilon_n^{-\alpha} \widetilde{J}_n(x)$ on $[0, \pi]$. Using :

$$\int_0^\pi \widetilde{J}_n(\Theta_n(\theta)) \sin(\theta) d\theta \leq \int_0^\pi \widetilde{J}_n(\theta) \sin(\theta) d\theta + C\varepsilon_n^{1+\alpha},$$

where $C > 0$ is a constant independent of n, we obtain :

$$\begin{aligned} 0 &\leq \int_0^{C_n - c_n} (\Theta_n(\theta) - \theta) \sin(\theta) d\theta + C\varepsilon_n^{1+\alpha} \\ &= \int_0^{b_n - c_n} (\Theta(\theta) - \theta) \sin(\theta) d\theta + \int_{b_n - c_n}^{B_n - c_n} \text{"} + \int_{B_n - c_n}^{C_n - c_n} \text{"} + C\varepsilon_n^{1+\alpha} \\ &< C\varepsilon_n^{1+\alpha} - \varepsilon_n^\gamma \int_{b_n - c_n}^{B_n - c_n} \sin(\theta) d\theta, \end{aligned}$$

where we used the bounds stated above and the fact that both $b_n - c_n$ and $B_n - c_n$ are in $[0, \pi/2]$. But remark that :

$$\begin{aligned} \int_{b_n - c_n}^{B_n - c_n} \sin(\theta) d\theta &\geq ((B_n - c_n) - (b_n - c_n)) \sin(b_n - c_n) \\ &\geq C'(n-1)^{1/\beta} \varepsilon_n^{2(\gamma + \beta')/\beta}, \end{aligned}$$

for some constant $C' > 0$, by inserting the previous bounds and using the inequality $\sin(x) \geq 2x/\pi$ on $[0, \pi/2]$. Thus, we obtain :

$$0 < \varepsilon_n^{1+\alpha} \left(C - C'(n-1)^{1/\beta} \varepsilon_n^{(2/\beta + 1)\gamma + 2\beta'/\beta - 1 - \alpha} \right),$$

and $(2/\beta + 1)\gamma + 2\beta'/\beta - 1 - \alpha \leq 0$ by the definition of γ, so the right-hand side is negative as n goes to infinity. ☐

Remark 8.4.1. Let us mention that the result is still valid in the limit $\delta = +\infty, \beta = 1, \beta' = 0$ (the Θ_ε are uniformly Lipschitz) and $\alpha = 0$. It provides an exponent $\gamma = 1/3$. Had we been able to prove a priori that the family Θ_ε was uniformly Lipschitz, this would have been enough to conclude.

8.5 End of the proof

We can now conclude the proof.

Proof. Combining Lemmas 8.3.7, 8.4.3 and Proposition 8.4.1, we conclude that $\Theta_\varepsilon = \mathrm{Id} + \mathcal{O}(\varepsilon^N)$, for some N which we can choose large enough. Thus for $\theta_1, \theta_2 \in [0, \pi]$ such that $\theta_1 + \theta_2 \in [0, \pi]$:

$$\begin{aligned} 0 \leq \frac{1}{\mathrm{vol}(S^*\widetilde{M}_\varepsilon^1)} \int_{S^*\widetilde{M}_\varepsilon^1} f(x, \xi, \theta_1 + \theta_2) - f(x, \xi, \theta_1) - f(x, R_{\theta_1}\xi, \theta_2) \;\; d\mu_1(x, \xi) \\ = \Theta_\varepsilon(\theta_1 + \theta_2) - \Theta_\varepsilon(\theta_1) - \Theta_\varepsilon(\theta_2) \\ = \mathcal{O}(\varepsilon^N) \end{aligned}$$

Since the integrand is positive and the inverse of the volume can be estimated by $\mathcal{O}(\varepsilon)$, this implies by taking $\varepsilon \to 0$ that

$$f(x,\xi,\theta_1+\theta_2) - f(x,\xi,\theta_1) - f(x,R_{\theta_1}\xi,\theta_2) = 0$$

so the inequality is saturated in Gauss-Bonnet formula. As a consequence, three intersecting g_1-geodesics correspond to three intersecting g_2-geodesics with same endpoints.

We can now construct the isometry Φ between (M,g_1) and (M,g_2). We will use in this paragraph the notation $\widetilde{\cdot}$ to refer to objects considered on the universal cover $\widetilde{M}$. Given $p \in \widetilde{M}$, we choose three g_1-geodesics α, β and γ passing through p with respective endpoints (x,x'), (y,y') and (z,z') in $\partial\widetilde{M} \times \partial\widetilde{M}$. By the previous section, we know that the g_2-geodesics with same endpoints meet in a single point which we define to be $\widetilde{\Phi}(p)$. Now, $\widetilde{\Phi}(p)$ is well-defined (it does not depend on the choice of the geodesics) and remark that for $(x,\xi) \notin \widetilde{\overline{\Gamma}}_- \cup \widetilde{\overline{\Gamma}}_+$ (such a covector always exists) and θ such that $(x,R_\theta\xi) \notin \widetilde{\overline{\Gamma}}_- \cup \widetilde{\overline{\Gamma}}_+$, we have $\widetilde{\Phi}(p) = \mathrm{x}(x,\xi,\theta)$, where x is defined in (8.3.1) (in other words, κ maps fibers to fibers). Thus $\widetilde{\Phi}$ is C^∞ in the interior (see Section §8.3.2) and extends continuously down to the boundary as $\widetilde{\Phi}|_{\partial\widetilde{M}} = \mathrm{Id}$.

Moreover, $\widetilde{\Phi}^*(\widetilde{g}_2) = \widetilde{g}_1$. Indeed, it is sufficient to prove that $\widetilde{\Phi}$ preserves the distance. Given $p,q \in \widetilde{M}$, we have $\widetilde{\mathcal{F}}_1(p,q) = \widetilde{\mathcal{F}}_2(\widetilde{\Phi}(p),\widetilde{\Phi}(q))$ and thus :

$$d_{\widetilde{g}_1}(p,q) = \frac{1}{2}\eta_{\widetilde{g}_1}\left(\widetilde{\mathcal{F}}_1(p,q)\right) = \frac{1}{2}\eta_{\widetilde{g}_2}\left(\widetilde{\mathcal{F}}_2(\widetilde{\Phi}(p),\widetilde{\Phi}(q))\right) = d_{\widetilde{g}_2}(\widetilde{\Phi}(p),\widetilde{\Phi}(q))$$

Now, observe that $\widetilde{\Phi}$ is invariant by the action of the fundamental group : it thus descends to a smooth diffeomorphism $\Phi : M \to M$ which extends continuously down to the boundary and satisfies $\Phi^* g_2 = g_1$.

We now conclude the argument by proving that Φ is actually smooth on $\overline{M}$. In the compact setting, it is a classical fact that an isometry which is at least differentiable is actually smooth and our argument somehow follows the idea of proof of this statement. More precisely, we show that a smooth isometry on an asymptotically hyperbolic manifold actually extends as a smooth application on the compactification $\overline{M}$. The proof does not rely on the dimension two. Note that another proof could be given in this case using the fact that Φ is a conformal map.

Consider a fixed point $p \in M$ in a neighborhood of the boundary. For any point $q \in \overline{M}$ in a neighborhood of p, we denote by $\xi(q)$ the unique covector such that $w(q) := (p,\xi(q))$ generates the geodesic joining p to q. The map $q \mapsto \xi(q)$ is smooth down to the boundary by [GGSU17, Proposition 5.13]. Let us denote by $\tau_1(q)$ the time such that $q = \pi\left(\overline{\varphi}^1_{\tau_1(q)}(w(q))\right)$. It is smooth down to the boundary too. Since Φ conjugates the two geodesic flows, we can write :

$$\Phi(q) = \pi\left(\overline{\varphi}^2_{\tau_2(q)}(z(q))\right),$$

where $z(q) := (\Phi(p), d\Phi_p(\xi(q)))$, for some time $\tau_2(q)$. All is left to prove, is thus that τ_2 is smooth down to the boundary. If $t(q)$ denotes the g_1-geodesic distance between p and q (which is also that between $\Phi(p)$ and $\Phi(q)$ for g_2), one has :

$$t(q) = \int_0^{\tau_1(q)} \frac{ds}{\rho(\overline{\varphi}^1_s(p,\xi(q)))} = -\log\left(1 - \frac{\tau_1(q)}{\tau^1_+(w(q))}\right) + G(\tau_1(q), w(q)),$$

for some smooth function $(\tau,z) \mapsto G(\tau,z)$ down to the boundary (this is a computation similar to the one carried out in Section §8.2.2, see also [GGSU17, Lemma 2.7]). And :

$$\tau_2(q) = \tau^2_+(z(q)) - e^{-t(q)}\tau^2_+(z(q))H(e^{-t}, z(q)),$$

for some smooth positive function H on $[0,1) \times \overline{S^*M} \setminus (\partial_- S^*M \cup \Gamma_-)$ (this stems from the previous equality, or see also [GGSU17, Lemma 2.7]). As a consequence :

$$\tau_2(q) = \tau_+^2(z(q)) - \left(1 - \frac{\tau_1(q)}{\tau_+^1(w(q))}\right) I(q),$$

for some smooth function I down to the boundary, which can be expressed in terms of H and G. This concludes the proof.

□

Chapitre 9

Conclusion

As a conclusion, we indicate some open questions which are of interest :

1. **Local rigidity of the length spectrum :** The main result of [GL19d] adresses the question of the local rigidity of the marked length spectrum for negatively-curved manifolds. It is conjectured that the length spectrum (without any homotopy consideration) should also locally determines the geometry. This is however globally false insofar as [Vig80] constructs examples of pairs of non-isometric isospectral Riemann surfaces (which have thus the same length spectrum, using the trace formula). Nevertheless, one can hope that the rigidity of the length spectrum still holds locally. As an application, this result would imply the proof of a conjecture due to Sarnak [Sar90], asserting that there only exists a finite number of isospectral isometry classes in dimension 2. Indeed, isopectral family of metrics are compact by [OPS88]. Thus, arguing by contradiction, one would obtain an infinite number of isospectral non-isometric metrics for which one could extract a converging subsequence. But all the metrics in this converging subsequence would be non-isometric and share the same length spectrum which would contradict the local rigidity of the length spectrum.
2. **Injectivity of the X-ray transform on Anosov manifolds :** As detailed in the Appendix B and following [GK80a, CS98, PSU14b, Gui17a], the s-injectivity of the X-ray transform of symmetric m-tensors on Anosov manifolds is known to hold for $\dim(M) = 2, m \in \mathbb{N}$ and $\dim(M) > 2, m = 0, 1$, and $m \in \mathbb{N}$ under the additional assumption that the curvature is non-positive. However, it is conjectured that this assumption is unnecessary. According to the discussion in Appendix B, there are two main obstacles one needs to overcome : first of all, one needs to prove that there are no non-trivial Conformal Killing Tensors (CKTs) for $m \geq 0$; then, one needs to obtain an effective estimate for the constant $C_m > 0$ such that $\|X_- u\|_{L^2} \leq C_m \|X_+ u\|_{L^2}$, where $X_\pm : \Omega_m \to \Omega_{m\pm1}$. This may be done by considering $m \in \mathbb{N}$ as a semiclassical parameter.
3. **Rigidity problems for non-uniformly hyperbolic geodesic flows :** As explained in the introductory chapter of this manuscript, the existence of an embedded flat cylinder in a surface clearly prevents the X-ray transform to be s-injectivity. Nevertheless, one could push this assumption to the extremal case where there exists only a finite number of closed geodesics on the surface on which the curvature vanishes, the rest of the surface being negatively-curved. This non-uniformly hyperbolic surface is one of the easiest examples of manifolds for which the hyperbolic behavior of the geodesic degenerates. It is very likely that some features of the hyperbolic case still persist in this context. For instance : is the

X-ray transform still s-injective on tensors ? What can be said about the resolvent of the geodesic vector field ? Can one obtain regularity properties for the solutions to the cohomological equation ?

4. **Thurston's distance in variable curvature :** In Chapter 3, we have proved that Thurston's distance, initially defined on Teichmüller space, also extends as a genuine distance in a neighbourhood of the diagonal in the space of negatively-curved metrics in any dimension. Infinitesimally, this asymmetric distance is induced by an asymmetric Finsler norm and Thurston proved that on Teichmüller space, the distance induced by the Finsler norm and his distance coincide. We conjecture that this is still true in variable curvature. This would solve the marked length spectrum rigidity conjecture.
5. **Geometry in the space of metrics :** The *pressure metric* introduced in Chapter 3 induces a metric on the space of isometry classes of negatively-curved metrics on a fixed closed manifold M. It could be interesting to understand the geodesics in this space of metrics, computed with respect to the pressure metric. This would be a generalization of the geodesic flow induced by the Weil-Petersson, initially defined for constant hyperbolic metrics on surfaces. This may provide new ideas as to the resolution of the marked length spectrum rigidity conjecture

Annexe A

Pseudodifferential operators and the wavefront set of distributions

The aim of this section is to briefly introduce pseudodifferential operators and recall elementary properties on the wavefront set of distributions. We refer to [Abe12, Hö03, Mel03, Shu01] for a detailed treatment. Like in the previous chapter, (M, g) is a smooth closed Riemannian manifold. Most of the notions here do not actually need such a strong structure to be defined but their definition is actually easier in the context of a Riemannian manifold.

A.1 Pseudodifferential operators

A.1.1 Pseudodifferential operators in Euclidean space

Although the identification $T^*\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is immediate, we will keep the notation $T^*\mathbb{R}^{n+1}$ in order to stay coherent with the rest of this chapter. We first recall the definition of pseudodifferential operators in the Euclidean space $\mathbb{R}^{n+1}$. We start with the usual classes of symbols.

Definition A.1.1. Let $m \in \mathbb{R}, \rho \in (1/2, 1]$. We define $S^m_\rho(\mathbb{R}^{n+1})$ to be the set of smooth functions $p \in C^\infty(T^*\mathbb{R}^{n+1})$ such that for all $\alpha, \beta \in \mathbb{N}$:

$$\|p\|_{\alpha,\beta} := \sup_{|\alpha'| \leq \alpha, |\beta'| \leq \beta} \sup_{(x,\xi) \in T^*\mathbb{R}^{n+1}} \langle \xi \rangle^{-(m-\rho|\alpha'|+(1-\rho)|\beta'|)} |\partial_\xi^{\alpha'} \partial_x^{\beta'} p(x, \xi)| < \infty, \qquad \text{(A.1.1)}$$

where $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$. For $\rho = 1$, we will simply write $S^m(\mathbb{R}^{n+1})$. We also introduce the class $S^m_{(0)}(\mathbb{R}^{n+1})$ of smooth functions $p \in C^\infty(T^*\mathbb{R}^{n+1})$ such that for all $\alpha, \beta \in \mathbb{N}$:

$$\|p\|^{(0)}_{\alpha,\beta} := \sup_{|\alpha'| \leq \alpha, |\beta'| \leq \beta} \sup_{(x,z) \in T^*(\mathbb{R}^{n+1} \setminus \{0\}} |z|^{-(m-|\alpha'|)} |\partial_z^{\alpha'} \partial_x^{\beta'} p(x, \xi)| < \infty.$$

These classes are invariant by the action by pullback of properly supported diffeomorphisms. As a consequence, they are intrinsically defined on smooth closed manifolds. Namely, if M is a smooth closed manifold, then $p \in S^m(M)$ if and only if, in any local trivialization, $p \in S^m(\mathbb{R}^{n+1})$. These classes of symbols form a graded algebra of Fréchet spaces (for each $m \in \mathbb{R}$) with semi-norms given by (A.1.1).

Remark A.1.1. The order $m \in \mathbb{R}$ is fixed in the previous definition but it can actually be chosen to vary. Namely, if $m \in S^0(\mathbb{R}^{n+1})$, then we define $S^m_\rho(\mathbb{R}^{n+1})$ to be the set of smooth functions $p \in C^\infty(T^*\mathbb{R}^{n+1})$ such that for all indices α, β, there exists a constant $C_{\alpha\beta} > 0$ such that :

$$\forall (x,\xi) \in T^*\mathbb{R}^{n+1}, \qquad |\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m(x,\xi) - \rho|\alpha| + (1-\rho)|\beta|}.$$

We refer to [FRS08] for further details. This class of symbols will appear in the proofs of the meromorphic extension of the generator of Anosov flows. It enjoys the usual features of more classical classes of symbols like the parametrix construction for instance.

We say that P is a pseudodifferential operator of order $m \in \mathbb{R}$ on $\mathbb{R}^{n+1}$ if there exists $p \in S^m(\mathbb{R}^{n+1})$ such that for any function $f \in C_c^\infty(\mathbb{R}^{n+1})$:

$$Pf(x) = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} e^{i\xi\cdot(x-y)} p(x,\xi) f(y) dy d\xi \tag{A.1.2}$$

This integral does not converge absolutely and has to be understood as an *oscillatory integral* : for further details, we refer to [Abe12, Shu01]. In this case, we write $P = \mathrm{Op}(p)$ and we say that the operator P is the *quantization* of p. We denote by $\Psi^m(\mathbb{R}^{n+1})$ the set of pseudodifferential operators of order m and we set $\Psi^{-\infty}(\mathbb{R}^{n+1}) := \cap_{m\in\mathbb{R}} \Psi^m(\mathbb{R}^{n+1})$. These are operators with smooth Schwartz kernel (and fast decay at infinity off the diagonal $\{x = y\}$ in $\mathbb{R}^{(n+1)} \times \mathbb{R}^{n+1}$). Eventually, we denote by $\sigma_P : \Psi^m(\mathbb{R}^{n+1}) \to S^m(\mathbb{R}^{n+1})/S^{m-1}(\mathbb{R}^{n+1})$ the principal symbol of P, defined by

$$\sigma_P(x,\xi) := \lim_{h\to 0} h^m e^{-iS/h} P(e^{iS/h})(x),$$

for $(x,\xi) \in T^*\mathbb{R}^{n+1}$, if $S : C^\infty(\mathbb{R}^{n+1})$ is such that $dS(x) = \xi$.

The space $\Psi^m(\mathbb{R}^{n+1})$ is in one-to-one correspondance with $S^m(\mathbb{R}^{n+1})$ (see [Mel03, Theorem 2.1]) via the quantization formula (A.1.2). This allows to transfer the Fréchet topology of $S^m(\mathbb{R}^{n+1})$ to the space $\Psi^m(\mathbb{R}^{n+1})$. As a consequence, $\Psi^m(\mathbb{R}^{n+1})$ is a Fréchet space endowed with the topology given by the semi-norms of its full symbol (A.1.1).

A symbol $p \in S^m(\mathbb{R}^{n+1})$ is said to be *globally elliptic* if there exists constants $C, R > 0$ such that :

$$\forall |\xi| \geq R, \forall x \in \mathbb{R}^{n+1}, \qquad |p(x,\xi)| \geq C\langle \xi \rangle^m.$$

It is said to be *locally elliptic* at (x_0, ξ_0) if there exists a conic neighborhood V of (x_0, ξ_0) [1] such that :

$$\forall (x,\xi) \in V, |\xi| \geq R, \qquad |p(x,\xi)| \geq C\langle \xi \rangle^m.$$

Given $P \in \Psi^m(\mathbb{R}^{n+1})$, we say that it is locally elliptic at (x_0, ξ_0) if its principal symbol σ_P is. We denote by $\mathrm{ell}(P)$ the set of points $(x_0, \xi_0) \in T^*M$ at which P is locally elliptic. Note that this is by construction an open conic subset of $T^*M \setminus \{0\}$.

A.1.2 Pseudodifferential operators on compact manifolds

We now move to the case of pseudodifferential operators on a smooth closed manifold M. There is no *intrinsic way* of defining pseudodifferential operators on compact manifolds (although some constructions may look more natural than others, there is

1. V is an open conic neighborhood of (x_0, ξ_0) of $T^*\mathbb{R}^{n+1} \setminus \{0\}$ if it is open in $T^*\mathbb{R}^{n+1} \setminus \{0\}$ and contains for some $\varepsilon > 0$ small enough the set of points $(x,\xi) \in T^*\mathbb{R}^{n+1} \setminus \{0\}$ such that $|x - x_0| < \varepsilon$ and $|\xi/|\xi| - \xi_0/|\xi_0|| < \varepsilon$.

always a part of choice in the definitions) but what is important is that the resulting class of operators $\Psi^m(M)$ obtained in the end *is independent* of all the choices made. Moreover, all the important features of the calculus (principal symbol, ellipticity) are independent of the choices made in the constructions.

We consider a cover of M by a finite number of open sets $M = \cup_i U_i$ such that there exists a smooth diffeomorphism $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{R}^{n+1}$. By assumption, since M is smooth, the transition maps $\phi_i \circ \phi_j^{-1}$ are smooth whenever they are defined. We consider a smooth partition of unity $\sum_i \Phi_i = \mathbf{1}$ subordinated to this cover of M and smooth functions Ψ_i supported in each patch U_i, defined such that $\Psi_i \equiv 1$ on the support of Φ_i. We call these elements $(U_i, \Phi_i, \Psi_i)_i$ a family of cutoff charts.

Definition A.1.2. A linear operator $P : C^\infty(M) \to C^\infty(M)$ is a pseudodifferential of order m on M if and only if there exists a family of cutoff charts $(U_i, \Phi_i, \Psi_i)_i$ such that, in the decomposition

$$P = \sum_i \Psi_i P \Phi_i + (1 - \Psi_i) P \Phi_i, \tag{A.1.3}$$

the operators $\Psi_i P \Phi_i$ can be written in coordinates

$$\Psi_i P \Phi_i f(\phi_i^{-1}(x)) = \psi_i \operatorname{Op}(p_i) \varphi_i f_i(x), \tag{A.1.4}$$

for some symbols $p_i \in S^m(\mathbb{R}^{n+1})$ (Op being the quantization (A.1.2) in Euclidean space), where $x \in \phi_i(U_i)$, $f_i := f \circ \phi_i^{-1}$ and $f \in C^\infty(M)$ is arbitrary, $\psi_i := \Psi_i \circ \phi_i^{-1}$, $\varphi_i := \Phi_i \circ \phi_i^{-1}$ and the operators $(1 - \Psi_i) P \Phi_i$ have smooth Schwartz kernel. We denote by $\Psi^m(M)$ the class of such operators.

Another formulation is the following : if one chooses a family of cutoff charts, given a symbol $p \in S^m(M)$, (A.1.4) provides a formula of quantization $\operatorname{Op}(p)$ (which depends on the choice of cutoff charts). Then the equality

$$\Psi^m(M) = \left\{ \operatorname{Op}(p) + R \mid p \in S^m(M), R \in \Psi^{-\infty}(M) \right\}$$

holds (here R is a smoothing operator, that is an operator with smooth Schwartz kernel), that is any other choice of cutoff charts produces the same class of operators. Note that *once a family of cutoff charts is chosen*, the decomposition (A.1.3) of P is unique and one can endow the Fréchet space $\Psi^m(M)$ with the semi-norms in local coordinates

$$\|P\|_{\alpha,\beta,\gamma} = \sum_i \|p_i\|_{\alpha,\beta} + \|(1 - \Psi_i) P \Phi_i\|_\gamma, \tag{A.1.5}$$

where $\|p_i\|_{\alpha,\beta}$ is given by (A.1.1) and, confusing $(1 - \Psi_i) P \Phi_i$ with its smooth Schwartz kernel, we define for $K \in C^\infty(M \times M)$ the semi-norms :

$$\|K\|_\gamma := \sup_{|j|+|k| \leq \gamma} \sup_{x,y \in M} |\partial_x^j \partial_y^k K(x,y)|$$

The principal symbol map $\sigma_m : \Psi^m(M) \to S^m(M)/S^{m-1}(M)$ is a well-defined map, independent of the quantization chosen. Let us recall some elementary properties of pseudodifferential operators :

Proposition A.1.1. • *If $P \in \Psi^m(M)$, then $P : H^s(M) \to H^{s-m}(M)$ is bounded for all $s \in \mathbb{R}$,*

- *If* $P_1 \in \Psi^{m_1}(M), P_2 \in \Psi^{m_2}(M)$*, then* $P_1 \circ P_2 \in \Psi^{m_1+m_2}(M)$ *and* $\sigma_{P_1 \circ P_2} = \sigma_{P_1}\sigma_{P_2}$*,*
- *If* $P \in \Psi^m(M)$ *is globally elliptic, there exists* $Q \in \Psi^{-m}(M)$ *such that* $PQ = \mathbb{1} + R_1, QP = \mathbb{1} + R_2$*, where* $R_1, R_2 \in \Psi^{-\infty}(M)$*.*

An operator $R \in \Psi^{-\infty}(M)$ is bounded and compact as a map $H^r(M) \to H^s(M)$, for all $s, r \in \mathbb{R}$. We will denote by $C^{-\infty}(M) := \cup_{s\in\mathbb{R}} H^s(M)$. The following lemma on elliptic estimates is crucial :

Lemma A.1.1. *Let* $P \in \Psi^m(M)$ *be an elliptic pseudodifferential operator. For all* $s, r \in \mathbb{R}$*, there exists a constant* $C := C(r, s)$ *such that for all* $f \in C^{-\infty}(M)$ *such that* $Pf \in H^{s-m}(M)$ *:*

$$\|f\|_{H^s} \leq C\left(\|Pf\|_{H^{s-m}} + \|f\|_{H^r}\right)$$

Moreover, if $P : H^s(M) \to H^{s-m}(M)$ *is injective for some (and thus any)* $s \in \mathbb{R}$*, then :*

$$\|f\|_{H^s} \leq C\|Pf\|_{H^{s-m}}$$

Proof. Let $Q \in \Psi^{-m}(M)$ be a parametrix for P, i.e. such that $QP = \mathbb{1} + R$, where $R \in \Psi^{-\infty}(M)$. Then :

$$\|f\|_{H^s} \lesssim \|QPf\|_{H^s} + \|Rf\|_{H^s} \lesssim \|Pf\|_{H^{s-m}} + \|f\|_{H^r},$$

since $R : H^r(M) \to H^s(M)$ is bounded and $Q : H^{s-m}(M) \to H^s(M)$ is bounded.

We now assume that P is invertible and we take $r = s$. Assume that the bound $\|f\|_{H^s} \lesssim \|Pf\|_{H^{s-m}}$ does not hold, so we can find a family of elements $f_n \in H^s(M)$ such that $\|f_n\|_{H^s} = 1$ and $\|f_n\|_{H^s} = 1 \geq n\|Pf_n\|_{H^{s-m}}$. So $Pf_n \to 0$ in $H^{s-m}(M)$. But $R : H^s(M) \to H^s(M)$ is compact and $(f_n)_{n\in\mathbb{N}}$ is bounded in $H^s(M)$ so we can assume (up to extraction) that $Rf_n \to v \in H^s(M)$. By the elliptic estimate

$$\|f_n\|_{H^s} \lesssim \|Pf_n\|_{H^{s-m}} + \|Rf_n\|_{H^s},$$

we obtain that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^s(M)$ which thus converges to $w \in H^s(M)$. But by continuity of P, $Pf_n \to 0 = Pw$ so $w \equiv 0$ since P is injective. This is contradicted by the fact that $\|w\|_{H^s} = 1$. □

As usual, one can define pseudodifferential operators $P : C^\infty(M, E) \to C^\infty(M, F)$ acting on vector bundles $E, F \to M$ by taking local coordinates and matrix-valued pseudodifferential operators in these coordinates. All the results previously stated still hold in this general context. The principal symbol is then a map $\sigma_P : T^*M \to \mathrm{Hom}(E, F)$. When the vector bundles E and F have different ranks, ellipticity is replaced by injectivity of the principal symbol with a coercive estimate, that is

$$\|\sigma_P(x, \xi)\|_{E_x \to F_x} \geq C\langle\xi\rangle^m,$$

for $|\xi| > R, C > 0$.

A.2 Wavefront set : definition and elementary operations

A.2.1 Definition

Definition A.2.1 (Wavefront set of a distribution)**.** Let $u \in C^{-\infty}(M)$. A point $(x_0, \xi_0) \in T^*M \setminus \{0\}$ *is not* in the wavefront set $\mathrm{WF}(u)$ of u, if there exists a conic neighborhood U of (x_0, ξ_0) such that for any smooth functions $\chi \in C_c^\infty(\pi_0(U))$

($\pi_0 : T^*M \to M$ being the projection), in any set of local coordinates, one has :

$$\forall N \in \mathbb{N}, \qquad \sup_{\xi \in U} |\widehat{\chi u}(\xi)||\xi|^N < \infty.$$

This is well-defined i.e. independent of the choice of coordinates. An equivalent definition is that $(x_0, \xi_0) \notin \mathrm{WF}(u)$ if and only if there exists a pseudodifferential operator A of order 0 microlocally supported in the conic neighborhood U, elliptic at (x_0, ξ_0) such that $Au \in C^\infty(M)$. By construction, the wavefront set of a distribution is *a conic set* in $T^*M \setminus \{0\}$. We will say that $u \in C^{-\infty}(M)$ is *smooth at* (x_0, ξ_0) if $(x_0, \xi_0) \notin \mathrm{WF}(u)$.

If $\imath : Y \to M$ is a smooth submanifold of M, then the *conormal* to Y is the set

$$N^*Y := \{(x, \xi) \in T^*M \mid \forall x \in Y, \forall Z \in T_xY, \langle \xi, Z \rangle = 0\} \subset T^*M$$

It is a smooth vector bundle over Y. We will say that a distribution $u \in C^{-\infty}(M)$ is *conormal to* Y if $\mathrm{WF}(u) \subset N^*Y$.

Example A.2.1 (Surface density)**.** Let $\imath : Y \to M$ be a submanifold. If σ is a smooth density on Y, then σ can be seen as a distribution $\overline{\sigma} \in C^{-\infty}(M)$ on M by setting $\langle \overline{\sigma}, f \rangle := \langle \sigma, f|_Y \rangle$, for $f \in C^\infty(M)$. Then $\mathrm{WF}(\overline{\sigma}) = N^*Y$, i.e. $\overline{\sigma}$ is conormal to Y.

Indeed, by taking local coordinates, the computation actually boils down to considering the case $\sigma = \phi(x)\delta(x' = 0)$, with $x' \in \mathbb{R}^{n-k}, x \in \mathbb{R}^k$, where $M \simeq \mathbb{R}^n$ and $N \simeq \{x' = 0\}$, $\phi \in C^\infty(\mathbb{R}^k)$. But then, for $\chi \in C^\infty(\mathbb{R}^n)$ localized in a neighborhood of $(x, 0)$, and denoting $\eta = (\xi, \xi'), e_\eta : (x, x') \mapsto e^{i\eta \cdot (x, x')}$, one has :

$$\widehat{\chi\overline{\sigma}}(\xi, \xi') = \langle \overline{\sigma}, \chi e_\eta \rangle = \int_{\mathbb{R}^k} \phi(x)\chi(x, 0)e^{ix\cdot\xi}dx = \mathcal{O}(|\eta|^{-\infty})$$ [2]

by the non-stationary phase lemma, unless $\xi = 0$. Thus

$$\mathrm{WF}(\overline{\sigma}) = \big\{(0, \xi'), \xi' \in \mathbb{R}^{n-k} \setminus \{0\}\big\} = N^*\mathbb{R}^k$$

We can refine the definition of the wavefront set in order to evaluate the frequency behavior of the distribution at infinity :

Definition A.2.2 (H^s-wavefront set)**.** Let $u \in C^{-\infty}(M)$. A point $(x, \xi) \notin \mathrm{WF}_s(u)$ if there exists a conic neighborhood of (x, ξ) and a pseudodifferential operator A of order 0 microlocally supported in this conic neighborhood, elliptic at (x, ξ) such that $Au \in H^s(M)$. We will say that $u \in C^{-\infty}(M)$ is *microlocally* H^s *at* (x_0, ξ_0) if $(x_0, \xi_0) \notin \mathrm{WF}_s(u)$.

Example A.2.2. Let δ_0 be the Dirac mass at 0 in $\mathbb{R}^n$. Then

$$\mathrm{WF}_{-n/2}(\delta_0) = \{(0, \xi), \xi \in \mathbb{R}^n \setminus \{0\}\},$$

but $\mathrm{WF}_s(\delta_0) = \emptyset$ for all $s < -n/2$.

A.2.2 Elementary operations on distributions

This paragraph mainly follows the lecture notes by Melrose [Mel03].

2. By this, we mean that for all $N \in \mathbb{N}$, there exists a constant $C_N > 0$ such that the right-hand side is bounded by $C_N|\eta|^{-N}$

Multiplication of distributions. We will denote by $\mathrm{d}\,\mathrm{vol}$ the smooth Riemannian density on M. Given $u_1, u_2 \in C^\infty(M)$, the (complex) pairing

$$\langle u_1, u_2 \rangle_{\mathbb{C}} := \int_M u_1(x)\overline{u_2(x)}\mathrm{d}\,\mathrm{vol}(x)$$

is always well-defined (note that M is compact). We want to understand to what extent this can be generalized to distributions $u_1, u_2 \in C^{-\infty}(M)$.

Lemma A.2.1. *Given $u_1, u_2 \in C^{-\infty}(M)$ such that $\mathrm{WF}(u_1) \cap \mathrm{WF}(u_2) = \emptyset$, there exists $A \in \Psi^0(M)$ such that*

$$\mathrm{WF}(u_1) \cap \mathrm{WF}(A)\,^{3} = \emptyset, \qquad \mathrm{WF}(u_2) \cap \mathrm{WF}(\mathbb{1} - A^*) = \emptyset.$$

Then :

$$\langle u_1, u_2 \rangle_{\mathbb{C}} := \overline{\langle u_2, Au_1 \rangle_{\mathbb{C}}} + \langle u_1, (\mathbb{1} - A^*)u_2 \rangle_{\mathbb{C}}$$

is well-defined and independent of the choice of A, where the right-hand side is understood as the pairing of a distribution with a smooth function.

To construct A, one can take $A = \mathrm{Op}(a)$ for some $a \in S^0(M)$ supported in a conic neighborhood of $\mathrm{WF}(u_1)$ (in particular, $a \equiv 0$ on $\mathrm{WF}(u_2)$ since $\mathrm{WF}(u_1) \cap \mathrm{WF}(u_2) = \emptyset$) and such that $a \equiv 1$ on $\mathrm{WF}(u_1)$. We do not detail the proof which can be found in [Mel03, Proposition 4.9]. Then the real pairing is just $\langle u_1, u_2 \rangle := \langle u_1, \overline{u_2} \rangle_{\mathbb{C}}$. Since

$$\mathrm{WF}(\overline{u}) = \{(x, -\xi) \mid (x, \xi) \in \mathrm{WF}(u)\},$$

it is defined as long as $\mathrm{WF}(u_1) \cap i(\mathrm{WF}(u_2)) = \emptyset$, where $i : T^*M \to T^*M$ stands for the involution $i(x, \xi) = (x, -\xi)$. This provides the

Lemma A.2.2. *Given $u_1, u_2 \in C^{-\infty}(M)$ such that $\mathrm{WF}(u_1) \cap i(\mathrm{WF}(u_2)) = \emptyset$, the multiplication $u_1 \times u_2 \in C^{-\infty}(M)$ is well-defined by*

$$\forall f \in C^\infty(M), \qquad \langle u_1 u_2, f \rangle := \langle u_1, f u_2 \rangle = \langle f u_1, u_2 \rangle$$

and coincides with the usual multiplication for $u_1, u_2 \in C^\infty(M)$. Moreover :

$$\begin{aligned}\mathrm{WF}(u_1 u_2) \subset & \{(x, \xi) \mid x \in \mathrm{supp}(u_1), (x, \xi) \in \mathrm{WF}(u_2)\} \\ & \cup \{(x, \xi) \mid x \in \mathrm{supp}(u_2), (x, \xi) \in \mathrm{WF}(u_1)\} \\ & \cup \{(x, \xi) \mid \xi = \eta_1 + \eta_2, (x, \eta_i) \in \mathrm{WF}(u_i), i \in \{1, 2\}\}\end{aligned}$$

The proof of the first part of the lemma simply follows from the previous computation. As to the wavefront set computation, it can be done directly in coordinates by using the definition.

Pushforward. The pushforward is one of the easiest operations one can define. Let $\pi : M \times N \to M$ be the left-projection, where N is a smooth closed manifold [4]. We denote by (x, y) the coordinates on $M \times N$, dx and dy are smooth measures on M and

3. See Example A.2.5 below for a definition of $\mathrm{WF}(A)$.

4. Once again, this can be generalized to the non-compact case, but then one has to consider distributions with compact support in the product.

N. The *pushforward* $\pi_* u$ of a distribution $u \in C^{-\infty}(M)$ is defined by duality according to the formula :

$$\forall f \in C^\infty(M), \qquad \langle \pi_* u, f \rangle := \langle u, \pi^* f \rangle,$$

where $\pi^* f := f \circ \pi$ is the pullback of f. In particular, if $u \in C^\infty(M \times N)$, this definition coincides with

$$\pi_* u(x) = \int_N u(x, y) dy$$

The wavefront set of the pushforward is characterized by the following lemma :

Lemma A.2.3.

$$\mathrm{WF}(\pi_* u) \subset \{(x, \xi) \in T^* M \mid \exists y \in N, (x, \xi, y, 0) \in T^*(M \times N)\}$$

We omit the proof, which can be done directly by using the characterization of the wavefront set with the Fourier transform. Morally, integration kills all the singularities except the ones which are *really conormal* to N i.e. the manifolds along which we integrate.

Restriction. Let $\imath : Y \to M$ be the embedding of the smooth submanifold Y into M. Given $u \in C^{-\infty}(M)$, the pullback $\imath^* u$, that is the restriction of u to Y, is not always well-defined. We denote by δ_Y the smooth Riemannian density obtained by restricting the metric g to Y and then taking the Riemannian volume form induced. Morally, given $f \in C^\infty(Y)$, we want to define $\langle \imath^* u, f \rangle = \langle u \times \delta_Y, \tilde{f} \rangle$, where $\tilde{f}$ is any smooth extension in a neighborhood of Y (under the condition that the multiplication $u \times \delta_Y$ is defined). Note that by Example A.2.1, $\mathrm{WF}(\delta_Y) \subset N^* Y$.

Lemma A.2.4. *Assume $u \in C^{-\infty}(M)$ satisfies $\mathrm{WF}(u) \cap N^* Y = \emptyset$ (so u is not conormal* at all*). Then $u \times \delta_Y$ makes sense by Lemma A.2.2 and*

$$\forall f \in C^\infty(Y), \qquad \langle \imath^* u, f \rangle := \langle u \times \delta_Y, \tilde{f} \rangle,$$

is well-defined, independently of the extension $\tilde{f}$. Moreover,

$$\mathrm{WF}(\imath^* u) \subset \{(x, \xi) \in T^* Y \mid \exists \eta \in N_x^* Y, (x, (\xi, \eta)) \in \mathrm{WF}(u)\},$$

where (ξ, η) is seen as an element of $T_x^* M$.

It is actually not obvious that this definition is independent of the extension $\tilde{f}$ of f : the proof can be done by an approximation argument (see [Hö3, Theorem 8.2.3]).

Pullback. Let $f : M \to N$ be a smooth map between the two smooth compact manifolds M and N [5]. The *normals of the map* (or the conormal to $f(M)$) is the set

$$N_f := N^* f(M) = \left\{(f(x), \xi) \in T^* N \mid x \in M, df^\top \xi = 0\right\}$$

The pullback $f^* u$ of a distribution $u \in C^{-\infty}(N)$ is not always defined, whereas that of a smooth function is. If f is a diffeomorphism, then it is an elementary result that $f^* u$

5. If M and N are not compact, then one has to assume f is *proper*, i.e. the preimage of a compact subset is a compact subset.

makes sense in a unique way : this amounts to saying that distributions are intrinsically defined i.e. are invariant by a change of coordinates. Moreover, the wavefront set of a distribution $u \in C^{-\infty}(N)$ is simply moved to

$$\mathrm{WF}(f^*u) \subset f^* \mathrm{WF}(u) = \left\{(x,\xi) \in T^*M \mid (f(x), df_x^{-\top}\xi) \in T^*N\right\},$$

where $df^{-\top}$ stands for the inverse transpose. But if f is no longer a diffeomorphism, if it maps spaces of different dimensions for instance, then the result may not be obvious.

We consider the graph

$$\mathrm{Graph}(f) := \{(x,y) \in M \times N \mid y = f(x)\} \xrightarrow{\imath} M \times N$$

which is an embedded submanifold of $M \times N$ (even if f is not a diffeomorphism !). We denote by $\pi_2 : M \times N \to N$ the right-projection and by $g : M \to \mathrm{Graph}(f)$ the diffeomorphism $g : x \mapsto (x, f(x))$. Then $f = \pi_2 \circ i \circ g$. For $u \in C^{-\infty}(N)$, we thus want to define f^*u by $g^* \circ \imath^* \circ \pi_2^* u$. So we have to study separately these three maps and understand under which conditions we can compose them. First, $\pi_2^* u = \mathbf{1} \otimes u$ is always defined and

$$\mathrm{WF}(\pi_2^* u) \subset \{(x, 0, y, \eta) \mid (y, \eta) \in \ \mathrm{WF}(u)\}$$

In the same fashion, the pullback of a distribution by g^* is always so one has to understand when the restriction $\imath^*$ is defined. But according to Lemma A.2.4, it is the case if $\mathrm{WF}(\pi_2^* u) \cap N^* \mathrm{Graph}(f) = \emptyset$. Note that

$$T \,\mathrm{Graph}(f) = \{(x, Z, f(x), df(Z)) \mid (x, Z) \in TM\} \subset T(M \times N).$$

Thus $N^* \mathrm{Graph}(f) = \{(x, 0, f(x), \eta) \mid (f(x), \eta) \in N_f\}$ so $\imath^* \circ \pi_2^* u$ is well-defined if $\mathrm{WF}(u) \cap N_f = \emptyset$.

Lemma A.2.5. *Let $u \in C^{-\infty}(N)$. If $\mathrm{WF}(u) \cap N_f = \emptyset$, then $f^*u := g^* \circ \imath^* \circ \pi_2^* u$ is well-defined and coincides for $u \in C^{\infty}(N)$ with $f^*u = u \circ f$. Moreover,*

$$\mathrm{WF}(f^*u) \subset f^* \mathrm{WF}(u) = \left\{(x, df^{\top}\xi) \mid (f(x), \xi) \in \mathrm{WF}(u)\right\}.$$

Example A.2.3. Let $\imath : M \to M \times M$ be the embedding $\imath : x \mapsto (x, x)$ of the diagonal $\imath(M) =: \Delta(M) \subset M \times M$. Note that $N^*\Delta(M) = \{(x, \xi, x, -\xi) \mid (x, \xi) \in T^*M\}$. Let $A : C^{\infty}(M) \to C^{-\infty}(M)$ be a linear operator with kernel K_A. Assume

$$\mathrm{WF}(K_A) \cap N^*\Delta(M) = \emptyset$$

Then $\imath^*(K_A) \in C^{-\infty}(M)$ is a well-defined distribution. We define the *flat trace* of A by

$$\mathrm{Tr}^{\flat}(A) := \langle \imath^*(K_A), \mathbf{1}\rangle.$$

One can prove that the flat trace is independent of the density chosen on M to define the Schwartz kernel. If $A \in \Psi^{-\infty}$, then A is a compact operator with smooth Schwartz kernel — in particular, it is trace class and its trace coincides with its flat trace.

A.2.3 The canonical relation

Linear operators. If $A : C^{\infty}(M) \to C^{-\infty}(M)$ is a linear operator, we denote by $K_A \in C^{-\infty}(M \times M)$ its Schwartz kernel. We define the *canonical relation* $\mathrm{WF}'(A)$ of A (also denoted by C_A) by

$$\mathrm{WF}'(A) := \{(x, \xi, y, \eta) \mid (x, \xi, y, -\eta) \in \mathrm{WF}(K_A)\}$$

Given $f \in C^\infty(M)$, using the Schwartz kernel theorem, we know that

$$Au(x) = \langle K_A(x,\cdot), u\rangle = \int_M K_A(x,y)u(y)dy,$$

where this equality has to be understood in a formal sense. By the previous operations introduced, we can rewrite this as $\pi_{2*}(K_A \times \pi_2^* u)$, where $\pi_2 : M \times M \to M$ is the projection on the second coordinate. If we want to extend A to $C^{-\infty}(M)$, then we have to understand this decomposition of A in light of the elementary operations seen so far. Recall that $\pi_2^* f = \mathbf{1} \otimes f$ has wavefront set included in $\{(x,0,y,\eta) \mid (y,\eta) \in \mathrm{WF}(u)\}$. As a consequence, $K_A \times \pi_2^* u$ makes sense as a distribution if

$$\mathrm{WF}(K_A) \cap \{(x,0,y,-\eta) \mid (y,\eta) \in \mathrm{WF}(u)\} = \emptyset,$$

and by Lemma A.2.2 :

$$\begin{aligned}\mathrm{WF}(K_A \times \pi_2^* u) \subset \{(x,\xi,y,\eta) \mid y \in \mathrm{supp}(u), (x,\xi,y,\eta) \in \mathrm{WF}(K_A)\} \\ \cup \{(x,0,y,\eta) \mid (x,y) \in \mathrm{supp}(K_A), (y,\eta) \in \mathrm{WF}(u)\} \\ \{(x,\xi,y,\eta) \mid y \in \mathrm{supp}(u), (x,\xi,y,\eta) \in \mathrm{WF}(K_A)\}\end{aligned} \tag{A.2.1}$$

By Lemma A.2.3, we know that :

$$\mathrm{WF}(\pi_{2*}(K_A \times \pi_2^* u) \subset \{(x,\xi) \mid \exists y \in M, (x,\xi,y,0) \in \mathrm{WF}(K_A \times \pi_2^* u)\}$$

As a consequence, in (A.2.1), the first set in the union of the right-hand side is immediately ruled-out. We obtain :

$$\begin{aligned}\mathrm{WF}(\pi_{2*}(K_A \times \pi_2^* u) \subset \{(x,\xi) \mid \exists y \in \mathrm{supp}(u), (x,\xi,y,0) \in \mathrm{WF}(K_A)\} \\ \cup \{(x,\xi) \mid \exists (y,\eta) \in T^*M, (x,\xi,y,-\eta) \in \mathrm{WF}(K_A), (y,\eta) \in \mathrm{WF}(u)\}\end{aligned}$$

We introduce the compact notation

$$\mathrm{WF}'(A) \circ \mathrm{WF}(u) := \{(x,\xi) \mid \exists (y,\eta) \in \mathrm{WF}(u), (x,\xi,y,\eta) \in \mathrm{WF}'(A)\}$$

Note that this is precisely the last set on the right-hand side of the previous formula. We write

$$\mathrm{WF}(K_A, u)_1 := \{(x,\xi) \mid \exists y \in \mathrm{supp}(u), (x,\xi,y,0) \in \mathrm{WF}(K_A)\}.$$

These points are the singularities created by A, no matter the regularity of u. In other words, if $u \in C^\infty(M)$, then $\mathrm{WF}(Au) \subset \mathrm{WF}(K_A, u)_1$. We sum up this discussion in the

Lemma A.2.6. *Let $A : C^\infty(M) \to C^{-\infty}(M)$ be a linear operator. Then A extends by continuity to a linear map*

$$A : \left\{u \in C^{-\infty}(M) \mid \mathrm{WF}(K_A) \cap \{(x,0,y,-\eta) \mid (y,\eta) \in \mathrm{WF}(u)\} = \emptyset\right\} \to C^{-\infty}(M)$$

and $\mathrm{WF}(Au) \subset \mathrm{WF}(K_A, u)_1 \cup \mathrm{WF}'(A) \circ \mathrm{WF}(u)$.

As we will see, given a general operator A, there is no practical way to characterize its Schwartz kernel by testing it against well-chosen distributions (unless we are given other informations on A). To do this, one has to resort to semiclassical analysis which we do not want to introduce here.

Example A.2.4. Let
$$\Lambda \subset T^*(M \times M) \setminus \{0\} \tag{A.2.2}$$
be a conic Lagrangian submanifold (i.e. such that the canonical symplectic form $\omega \oplus -\omega$ vanishes on Λ). We say that $K \in C^{-\infty}(M \times M)$ is *Lagrangian* with respect to Λ if $\mathrm{WF}(K) \subset \Lambda$. The Fourier Integral Operators (FIOs) are the operators having Lagrangian distribution kernels with Lagrangian included in $T^*M \setminus \{0\} \times T^*M \setminus \{0\}$ [6] (and an order condition on the symbol of their quantification, see [Hö3, Chapter XXV]). In particular, if Λ is the Lagrangian of a FIO A, then
$$\mathrm{WF}(K_A)_1 := \{(x,\xi) \mid \exists y \in M, (x,\xi,y,0) \in \mathrm{WF}(K_A)\} = \emptyset$$
As a consequence, the wavefront set relation in Lemma A.2.6 is simply : $\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u)$. Here $\mathrm{WF}'(A) = \{(x,\xi,y,-\eta) \mid (x,\xi,y,\eta) \in \Lambda\}$ is the canonical relation. In other words, a FIO does not create singularities from scratch. It may only kill or duplicate (and propagate) already existing singularities.

Example A.2.5. If P is a pseudodifferential operator on M, then K_P is a distribution which is conormal to the diagonal $\Delta(M) \subset M \times M$, i.e. $\mathrm{WF}(K_P) \subset N^*\Delta(M)$. In other words, its canonical relation $\mathrm{WF}'(P)$ satisfies
$$\mathrm{WF}'(P) \subset \Delta(T^*M \setminus \{0\})$$
We can define the *wavefront set of* P by
$$\mathrm{WF}(P) := \{(x,\xi) \in T^*M \setminus \{0\} \mid (x,\xi,x,\xi) \in \mathrm{WF}'(P)\}$$
This has to be understood in the following way : the operator P is smoothing outside its wavefront set $\mathrm{WF}(P)$. The wavefront set $\mathrm{WF}(P)$ is also called the *essential support* of P or the *microlocal support*. If $P = \mathrm{Op}(p)$ is a quantization of $p \in C^\infty(T^*M)$, then $\mathrm{WF}(P)$ coincides with the *cone support* of p, namely the complementary of the set of directions in T^*M for which p, as well as all its derivatives (both in the x and ξ variables), decays like $\mathcal{O}(|\xi|^{-\infty})$.

Composition of linear operators. If $A, B : C^\infty(M) \to C^{-\infty}(M)$ are linear operators with smooth Schwartz kernel, then
$$K_{A\circ B}(x,y) = \int_M K_A(x,z)K_B(z,y)dz$$
Using the previous operations, this can be written as $K_{A\circ B} = \pi_{2*}(\pi_{1,2}^* K_A \times \pi_{2,3}^* K_B)$, where $\pi_{1,2}(x,z,y) = (x,z), \pi_{2,3}(x,z,y) = (z,y)$. This formula allows to generalize the composition to operators with non-smooth Schwartz kernel. Repeating the arguments of Lemma A.2.6, one can prove the

Lemma A.2.7. *Assume A and B satisfy the condition*
$$\begin{aligned}&\{(z,\theta) \mid \exists x \in M, (x,0,z,-\theta) \in \mathrm{WF}(K_A)\} \\ &\cap \{(z,\theta) \mid \exists y \in M, (z,\theta,y,0) \in \mathrm{WF}(K_B)\} = \emptyset.\end{aligned}$$
Then, $A \circ B$ extends continuously as a linear operator on distributions satisfying Lemma A.2.6 and
$$\begin{aligned}\mathrm{WF}'(A\circ B) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(B) &\\ \cup \{(x,\xi,z,0) \mid z \in M, \exists z' \in M, (x,\xi,z',0) \in \mathrm{WF}(K_A)\}&\\ \cup \{(z,0,y,\eta) \mid z \in M, \exists z' \in M, (z',0,y,\eta) \in \mathrm{WF}(K_B)\}&\end{aligned}$$

6. Note that this is stronger than (A.2.2).

A.3 The propagator of a pseudodifferential operator

Let P be a pseudodifferential operator of order 1 with principal symbol p. We will denote by $(\Phi_t)_{t\in\mathbb{R}}$ the Hamiltonian flow on T^*M generated by p. We assume that P is formally self-adjoint on $C^\infty(M)$ but not necessarily elliptic. This implies that P is closed and self-adjoint on its domain $\mathcal{D}(P) := \{u \in L^2(M) \mid Pu \in L^2(M)\}$ [7] (see [FS11, Lemma 29]).

By Stone's theorem, $U : t \mapsto e^{-itP}$ is a unitary group on $L^2(M)$ which can be obtained as the unique solution of $(\partial_t + iP)U(t) = 0, U(0) = \mathbb{1}$. We will rather see U as map $C_c^\infty(\mathbb{R}\times M) \to C^\infty(M)$. Note that if P is elliptic and its spectrum is discrete [8], then it consists of isolated eigenvalues $\{\lambda_j\}_{j=0}^{+\infty}$ (with finite multiplicity) on $\mathbb{R}$ with corresponding normalized eigenvectors $\{e_j\}_{j=0}^{+\infty}$ forming a Hilbertian basis of L^2. Then if $f = \sum_{j=0}^{+\infty} f_j e_j \in L^2(M)$, one has the explicit expression $U(t)f = \sum_{j=0}^{+\infty} e^{-i\lambda_j t} f_j e_j$.

Theorem A.3.1. *[DG75, Theorem 1.1] The operator $U : C_c^\infty(\mathbb{R}\times M) \to C^\infty(M)$ is a Fourier Integral Operator with canonical relation*

$$\mathrm{WF}'(U) \subset \{(\Phi_t(x,\xi),(x,\xi),(t,\lambda)) \mid t\in\mathbb{R},(x,\xi)\in T^*M, \lambda = -p(x,\xi)\}$$

In particular, for all $t\in\mathbb{R}$, $U(t) : C^\infty(M) \to C^\infty(M)$ is a Fourier Integral Operator with canonical relation

$$\mathrm{WF}'(U) \subset \{(\Phi_t(x,\xi),(x,\xi)) \mid (x,\xi)\in T^*M\}$$

Example A.3.1. We will mostly be interested in the case $P = -iX$, where X is a vector field preserving a smooth measure (and P is thus selfadjoint). Its symbol is $\sigma_{-iX} : (x,\xi) \mapsto \langle \xi, X(x)\rangle$ and has a non-trivial characteristic set $\Sigma := \{\langle \xi, X(x)\rangle = 0\}$. Then $U(t) = e^{-tX}$. Using Lemma A.2.3 and the previous theorem, we obtain that for all $\chi \in C_c^\infty(\mathbb{R})$, if $A := \int_{-\infty}^{+\infty} \chi(t)e^{-tX}dt$, then :

$$\mathrm{WF}'(A) \subset \{(\Phi_t(x,\xi),(x,\xi)) \mid (x,\xi)\in\Sigma, t\in \mathrm{supp}(\chi)\}$$

In other words, the operator A is smoothing in the flow-direction (since it is obtained by integration in this direction) and propagates forward singularities (by the Hamiltonian flow $(\Phi_t)_{t\in\mathbb{R}}$) in the orthogonal directions to the flow. The operator Π in Chapter 2 is morally the operator A with $\chi \equiv 1$ on $\mathbb{R}$. This is no longer a FIO : indeed Π not only propagates forward the singularities in the orthogonal directions to the flow, but it also creates (from scratch) singularities in the stable and unstable bundles $E_s^* \cup E_u^*$.

A.4 Propagation of singularities

In this paragraph, we state results concerning the propagation of singularities for pseudodifferential operators. We omit the proofs, but all of them can be found in [DZ, Appendix E].

7. This is specific to pseudodifferential operators of order 1.
8. Which is the case for instance if $p(T^*M) \subset T^*m$ avoids a conic neighborhood $\Lambda \subset T^*M$ (see [Shu01, Chapter 9]), the most common case being that of a positive symbol p.

We consider P, a pseudodifferential operator of order 1 [9], with *real* principal symbol σ_P [10]. Like in the previous paragraph, we denote by $(\Phi_t)_{t\in\mathbb{R}}$ the Hamiltonian flow induced by the Hamiltonian σ_P.

Theorem A.4.1. *[DZ, Theorem E.49] Let $A, B, B_1 \in \Psi^0(M)$. Assume we have the following control condition : for all $(x,\xi) \in \mathrm{WF}(A)$, there exists $T \geq 0$ such that $\Phi_{-T}(x,\xi) \in \mathrm{ell}(B)$ and $\Phi_{-t}(x,\xi) \in \mathrm{ell}(B_1)$ for all $t \in [0,T]$. Then, for all $s \in \mathbb{R}, N \geq 0$, if $u \in H^{-N}(M), Bu \in H^s(M), Pu \in H^s(M)$:*

$$\|Au\|_{H^s} \leq C(\|Bu\|_{H^s} + \|B_1Pu\|_{H^s} + \|u\|_{H^{-N}}),$$

for some constant $C > 0$ independent of u.

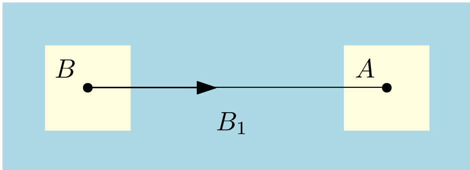

Figure A.1 – A picture of the situation (Figure extracted from [DZ16]).

In particular, the previous Theorem implies the usual result of propagation of singularities for operators with real principal symbol : if $\gamma : [0,T] \to \Sigma$ is a flowline in the characteristic set $\Sigma := \sigma_P^{-1}(\{0\})$ and $u \in C^{-\infty}(M)$ is such that $\gamma(0) \notin \mathrm{WF}(u), \gamma(t) \notin \mathrm{WF}(Pu)$ for all $t \in [0,T]$, then $\gamma(T) \notin \mathrm{WF}(u)$.

We denote by S^*M the unit cosphere bundle over M (induced by the metric g) and by $\kappa : T^*M \setminus \{0\} \to S^*M$ the canonical projection.

Definition A.4.1. Let $L \subset T^*M$ be a closed conic subset, invariant by the flow $(\Phi_t)_{t\in\mathbb{R}}$. We say that L is a *radial source* if there exists an open conic neighborhood U of L in $T^*M \setminus \{0\}$ and constants $C, \theta > 0$, such that :

$$\lim_{t\to+\infty} d(\kappa(\Phi_{-t}(U)), \kappa(L)) = 0, \tag{A.4.1}$$

$$\forall (x,\xi) \in U, \forall t \geq 0, \qquad Ce^{\theta t}|\xi| \leq |\Phi_{-t}(x,\xi)| \tag{A.4.2}$$

Reversing the time direction, we obtain the definition of a *radial sink*.

Example A.4.1. If $P = -iX$ is a hyperbolic flow on M, then $L = E_s^*$ is a radial source (E_u^* is a radial sink).

We have a high regularity estimate in a neighborhood of radial sources :

Theorem A.4.2. *[DZ, Theorem E.54] Assume L is a radial source for the flow $(\Phi_t)_{t\in\mathbb{R}}$ induced by the Hamiltonian σ_P. Then, there exists a threshold $s_0 > 0$ such that for any $s \geq s_0, N \geq s$ and $B_1 \in \Psi^0(M)$ elliptic near L, there exists $A \in \Psi^0(M)$ elliptic near L such that for any distribution $u \in H^{-N}(M)$, if $B_1Pu \in H^s(M)$ and $Au \in H^{s_0}(M)$, then one has :*

$$\|Au\|_{H^s} \leq C(\|B_1Pu\|_{H^s} + \|u\|_{H^{-N}}),$$

for some constant $C > 0$ independent of u.

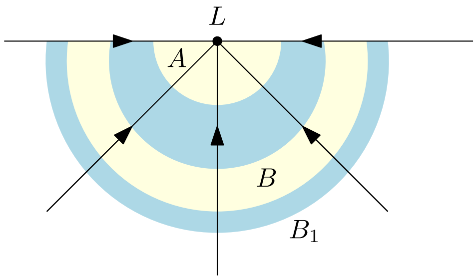

FIGURE A.2 – A picture of the situation for Theorem A.4.3 (Figure extracted from [DZ16]).

Note that one has to assume an *a priori* regularity on u in a conic neighborhood of L, i.e. $Au \in H^{s_0}(M)$. In the case where P is selfadjoint or anti-selfadjoint, the threshold s_0 can be taken arbitrarily small. In particular, this result asserts that if $Au \in H^{s_0}(M)$ and Pu is smooth near L, then so is u. In other words, we retrieve an *essential feature* of ellipticity near L although the operator P *may not be elliptic* near L. We also have a low regularity estimate in a neighborhood of radial sinks — it will not be used in these notes but we state it for the sake of completeness :

Theorem A.4.3. *[DZ, Theorem E.56] Assume L is a radial sink for the flow $(\Phi_t)_{t\in\mathbb{R}}$. Then there exists a threshold $s_0 > 0$ such that for any $s > s_0, N \geq s$ and $B_1 \in \Psi^0(M)$ elliptic near L, there exists $A \in \Psi^0(M)$ elliptic near L, $B \in \Psi^0(M)$ with $\mathrm{WF}(B) \cap L = \emptyset$, microsupported in the region of ellipticity of B_1 such that for any distribution $u \in H^{-N}(M)$, if $B_1 Pu \in H^{-s}(M), Bu \in H^{-s}(M)$:*

$$\|Au\|_{H^{-s}} \leq C(\|Bu\|_{H^{-s}} + \|B_1 Pu\|_{H^{-s}} + \|u\|_{H^{-N}}),$$

for some constant $C > 0$ independent of u.

9. This is actually not a constraint and the order k of P does not need to be equal to 1. Since we will apply the results in this case, we chose to state the theorems with $k = 1$. We refer to [DZ] for further details.

10. Most of the arguments can be extended to the case where σ_P is complex-valued under a suitable sign condition on $\Im(\sigma_P)$. In Theorem A.4.1, this condition is $\Im(\sigma_P) \leq 0$ on $\mathrm{WF}(B_1)$. The Hamiltonian flow one has to consider is then the one induced by the Hamiltonian $\Re(\sigma_P)$.

Annexe B

On symmetric tensors

The purpose of this chapter is to describe elementary properties of symmetric tensors on Riemannian manifolds which are used throughout this manuscript. The context chosen is that of a smooth compact Riemannian manifold without boundary (M, g). However, most of the results extend to the case of a compact manifold with boundary by adding a Dirichlet-type condition on the boundary.

B.1 Definitions and first properties

B.1.1 Symmetric tensors in euclidean space

Let E be a Euclidean $(n+1)$-dimensional vector space endowed with a metric g and let $(\mathbf{e}_1, ..., \mathbf{e}_{n+1})$ be an orthonormal basis. We say that a tensor $f \in \otimes^m E^*$ is symmetric if $f(v_1, ...v_m) = f(v_{\tau(1)}, ..., v_{\tau(m)})$, for all $v_1, ..., v_m \in E$ and $\tau \in \mathfrak{S}_m$, the group of permutations of order m. We denote by $\otimes_S^m T^*E$ the vector space of symmetric m-tensors on E. There is a natural projection $\sigma : \otimes^m E^* \to \otimes_S^m E^*$ given by

$$\sigma\left(v_1^* \otimes ... \otimes v_m^*\right) = \frac{1}{m!} \sum_{\tau \in \mathfrak{S}_m} v_{\tau(1)}^* \otimes ... \otimes v_{\tau(m)}^*,$$

for all $v_1^*, ..., v_m^* \in E^*$. The metric g induces a scalar product $\langle \cdot, \cdot \rangle$ on $\otimes^m E^*$ by declaring the basis $(\mathbf{e}_{i_1}^* \otimes ... \otimes \mathbf{e}_{i_m}^*)_{1 \leq i_1, ..., i_m \leq n+1}$ to be orthonormal which yields

$$\langle u_1^* \otimes ... \otimes u_m^*, v_1^* \otimes ... \otimes v_m^* \rangle = \prod_{i=1}^{m} g^{-1}(u_i^*, v_i^*),$$

where g^{-1} is the dual metric, that is the natural metric on E^* which makes the musical isomorphism $\flat : E \to E^*$ an isometry. Since σ is self-adjoint with respect to this metric, it is an orthogonal projection. Let $(g_{ij})_{1 \leq i,j \leq n+1}$ denote the metric g in the coordinates $(x_1, ..., x_{n+1})$. Then the metric can be expressed as

$$\langle f, h \rangle = f_{i_1 ... i_m} h^{i_1 ... i_m},$$

where $h^{i_1 ... i_m} = g^{i_1 j_1} ... g^{i_m j_m} h_{j_1 ... j_m}$. In particular, if $m = 2$, then

$$\langle f, h \rangle = \mathrm{Tr}_g(fh) = \mathrm{Tr}(g^{-1} fh)$$

More generally, we will define the trace $\mathrm{Tr}_g : \otimes_S^m E^* \to \otimes_S^{m-2} E^*$ of a symmetric tensor by

$$\mathrm{Tr}_g(f) = \sum_{i=1}^{n+1} f(\mathbf{e}_i, \mathbf{e}_i, \cdot, ..., \cdot).$$

In coordinates, $\mathrm{Tr}_g(f)(v_2, ..., v_m) = \mathrm{Tr}(g^{-1} f(\cdot, \cdot, v_2, ..., v_m))$. Its adjoint with respect to the scalar products is the map $I : \otimes_S^{m-2} E^* \to \otimes_S^m E^*$ given by $I(u) = \sigma(g \otimes u)$.

Symmetric tensors can also be seen as polynomials on the unit sphere of the euclidean space. We denote by $\mathbb{S}_E$ the n-dimensional unit sphere on (E, g) and by dS the Riemannian measure on the sphere induced by the metric $g|_{\mathbb{S}_E}$. We define $\pi_m : (x, v) \mapsto (x, \otimes^m v)$ for $v \in E$; it induces a canonical morphism $\pi_m^* : \otimes_S^m E^* \to C^\infty(\mathbb{S}_E)$ given by $\pi_m^* f(v) = f(v, ..., v)$. Its formal adjoint is $\langle \pi_m^* f, h \rangle_{L^2(\mathbb{S}_E, dS)} = \langle f, \pi_{m*} h \rangle_{\otimes^m T^* M}$, where $f \in \otimes_S^m T^* M, h \in C^\infty(\mathbb{S}_E)$. In coordinates,

$$(\pi_{m*} h)_{i_1 \dots i_m} := \pi_{m*} h(\partial_{i_1}, ..., \partial_{i_m}) = g_{i_1 j_1} ... g_{i_m j_m} \int_{\mathbb{S}_E} h(v) v^{j_1} ... v^{j_m} dS \qquad \text{(B.1.1)}$$

Any Euclidean space (E, g) is always isometric to $(\mathbb{R}^{n+1}, g_{\mathrm{euc}})$ so it may be cumbersome to bother with coordinates insofar as all the objects are coordinate-invariant. However, on a Riemannian manifold, this is no longer possible : given a point $p \in M$, one can always choose a trivialization (the *normal coordinates*) $\psi : U \to \psi(U) \subset \mathbb{R}^{n+1}$ (where U is a neighborhood of p) so that $\psi_* g|_{\psi(p)} = g_{\mathrm{euc}}$ but this cannot be true on a neighborhood of $\psi(p)$ (otherwise the metric would necessarily be flat !). Thus, bothering with coordinates *has* an interest as we will see on Riemannian manifolds. Also remark that (B.1.1) can be rewritten intrinsically as

$$\forall u_1, ..., u_m \in E, \qquad \pi_{m*} h(u_1, ..., u_m) = \int_{\mathbb{S}_E} h(v) g(v, u_1) ... g(v, u_m) dv \qquad \text{(B.1.2)}$$

The map $\pi_{m*} \pi_m^*$ is an isomorphism which we will study in the next paragraph. Also note that $\pi_m^*(\sigma f) = \pi_m^* f$ (since all the antisymmetric parts of the tensor f vanish by plugging m times the same vector v).

We denote by j_ξ the multiplication by ξ, that is $j_\xi : f \mapsto \xi \otimes f$, and by i_ξ the contraction, that is $i_\xi : f \mapsto u(\xi^\sharp, \cdot, ..., \cdot)$. The adjoint of i_ξ on symmetric tensors with respect to the L^2-scalar product is σj_ξ, that is

$$\forall f \in \otimes_S^{m-1} E^*, h \in \otimes_S^m E^*, \qquad \langle \sigma j_\xi f, h \rangle = \langle f, i_\xi h \rangle.$$

The space $\otimes_m^S E^*$ can thus be decomposed as the direct sum

$$\otimes_S^m E^* = \mathrm{ran} \left(\sigma j_\xi |_{\otimes_S^{m-1} E^*} \right) \oplus^\perp \ker \left(i_\xi |_{\otimes_S^m E^*} \right)$$

We denote by $\pi_{\ker i_\xi}$ the projection onto the right space, parallel to the left space. We will need the following

Lemma B.1.1. *For all $f, h \in \otimes_S^m E^*$,*

$$C_{n,m} \int_{\langle \xi, v \rangle = 0} \pi_m^* f(v) \pi_m^* h(v) dS_\xi(v) = \langle \pi_{\ker i_\xi} \pi_{m*} \pi_m^* \pi_{\ker i_\xi} f, h \rangle,$$

where

$$C_{n,m} = \int_0^\pi \sin^{n-1+2m}(\varphi) d\varphi = \sqrt{\pi} \frac{\Gamma((n+2m)/2)}{\Gamma((n+1+2m)/2)},$$

and dS_ξ is the canonical measure induced on the $n-1$ dimensional sphere $\mathbb{S}_{E,\xi} := \mathbb{S}_E \cap \{\langle \xi, v \rangle = 0\}$.

Proof. We can write $h = \sigma j_\xi h_1 + h_2$ where $h_1 \in \otimes_S^{m-1} E^*, h_2 \in \ker\left(i_\xi|_{\otimes_S^m T_x^* M}\right)$. Note that $\pi_m^*(\sigma j_\xi h_1)(v) = \pi_m^*(j_\xi h_1)(v) = \langle \xi, v\rangle \pi_{m-1}^* h_1(v)$ and this vanishes on $\{\langle \xi, v\rangle = 0\}$ (and the same holds for f). In other words, $\pi_m^* h = \pi_m^* \pi_{\ker i_\xi}$ on $\{\langle \xi, v\rangle = 0\}$. We are thus left to check that for $f, h \in \ker i_\xi$,

$$C_{n,m} \int_{\langle \xi, v\rangle = 0} \pi_m^* f(v) \pi_m^* h(v) dS_\xi(v) = \int_{\mathbb{S}_E} \pi_m^* f(v) \pi_m^* h(v) dS(v)$$

We will use the coordinates $v' = (v, \varphi) \in \mathbb{S}_{E,\xi} \times [0, \pi]$ on $\mathbb{S}_E$ which allow to decompose $v' = \sin(\varphi) v + \cos(\varphi)\xi^\sharp/|\xi|$. Then the measure on $\mathbb{S}_E$ disintegrates as $dS = \sin^{n-1}(\varphi) d\varphi dS_\xi(v)$. Also remark that $\pi_m^* f(v + \cos(\varphi)\xi^\sharp/|\xi|) = \pi_m^* f(v)$. Then, if $C_{n,m} := \int_0^\pi \sin^{n-1+2m}(\varphi) d\varphi$, we obtain :

$$\begin{aligned}
&\int_{\langle \xi, v\rangle = 0} \pi_m^* f(v) \pi_m^* h(v) dS_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_0^\pi \sin^{d-1+2m}(\varphi) d\varphi \int_{\langle \xi, v\rangle = 0} \pi_m^* f(v) \pi_m^* h(v) dS_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_0^\pi \int_{\langle \xi, v\rangle = 0} \pi_m^* f(\sin(\varphi) v + \cos(\varphi)\xi^\sharp/|\xi|) \\
&\qquad\qquad\qquad \times \pi_m^* h(\sin(\varphi) v + \cos(\varphi)\xi^\sharp/|\xi|) \sin^{n-1}(\varphi) d\varphi dS_\xi(v) \\
&\qquad = C_{n,m}^{-1} \int_{\mathbb{S}_E} \pi_m^* f(v') \pi_m^* h(v') dS(v')
\end{aligned}$$

□

B.1.2 Spherical harmonics

Let $\Delta|_{\mathbb{S}_E} := \operatorname{div}_{\mathbb{S}_E} \nabla_{\mathbb{S}_E}$ be the Laplacian on the unit sphere $\mathbb{S}_E$ induced by the metric $g|_{\mathbb{S}_E}$ and Δ be the usual Laplacian on E induced by g. Let

$$L^2(\mathbb{S}_E) = \bigoplus_{m=0}^{+\infty} \Omega_m$$

be the spectral break up in spherical harmonics, where $\Omega_m := \ker(\Delta|_{\mathbb{S}_E} + m(m+n-1))$ are the eigenspaces of the Laplacian. We denote by E_m the vector space of trace-free symmetric m-tensors, where the trace is, as before, taken over the first two coordinates.

Lemma B.1.2. *$\pi_m^* : E_m \to \Omega_m$ is an isomorphism.*

Proof. For $f \in \otimes_S^m E^*$, we can see $v \mapsto f(v, ..., v) = \bar{f}(v)$ as a homogeneous polynomial of order m in the variables $(v_1, ..., v_{n+1})$ (and its restriction to $\mathbb{S}_E$ is $\pi_m^* f$, that is for $v \in \mathbb{S}_e, r > 0, \bar{f}(rv) = r^m \pi_m^* f(v)$). For any smooth function u on E

$$\Delta(u)|_{\mathbb{S}_E} = \Delta_{\mathbb{S}_e}(u|_{\mathbb{S}_e}) + \left.\frac{\partial^2 u}{\partial r^2}\right|_{\mathbb{S}_E} + n \left.\frac{\partial u}{\partial r}\right|_{\mathbb{S}_E},$$

if r is the radial coordinate (see [GHL04, Proposition 4.48]). Then the homogeneity provides

$$\Delta(\bar{f})|_{\mathbb{S}_E} = \Delta_{\mathbb{S}_e}(\pi_m^* f) + m(m+n-1)\pi_m^* f.$$

But if f is trace-free, we claim that $\Delta \bar{f} = 0$ so $\pi_m^* f \in \Omega_m$. Since π_m^* is clearly injective, it is sufficient to prove equality of the dimensions. The dimension of trace-free symmetric

m-tensors in a $(n+1)$-dimensional space is $\binom{n+m}{m} - \binom{n+m-2}{m-2}$ (see [DS10, Lemma 2.3]) which turns out to be that of Ω_m.

We are thus left to prove the equality $\Delta \bar{f} = 0$. Let us write the symmetric tensor $f = \sum_{i_1...i_m} f_{i_1...i_m} e_{i_1}^* \otimes ... \otimes e_{i_m}^*$, then $\mathrm{Tr}(f) = \sum_{k,i_3,...,i_m} f_{kki_3...i_m} e_{i_3}^* \otimes ... \otimes e_{i_m}^*$. Thus $\mathrm{Tr}(f) = 0$ implies that $f_{kki_3...i_m} = 0$ for all indices $k, i_3, ..., i_m \in \{1, ..., n+1\}$. Since f is symmetric, this implies that $f_{i_1...i_m} = 0$ as long as there exists $k, l \in \{1, ..., m\}$ such that $i_k = i_l$. As a consequence $\bar{f} : v \mapsto \sum_{i_1 \neq ... \neq i_m} f_{i_1...i_m} v_{i_1}...v_{i_m}$ and for such a m-tuple $(i_1, ..., i_m)$, one has $\partial_k^2(v_{i_1}...v_{i_m}) = 0$ for any $k \in \{1, ..., n+1\}$ (since the index k appears at most once) so $\Delta(v_{i_1}...v_{i_m}) = 0$ and $\Delta \bar{f} = 0$. □

The group of linear (orientation-preserving) isometries $\mathrm{Isom}_0(E)$ acts on the right by pullback both on $C^\infty(\mathbb{S}_E)$ and $\otimes_S^m E^*$ and it is immediate that its action commutes with π_m^*. It also commutes with π_{m*}. Indeed, since $(E, g) \simeq (\mathbb{R}^{n+1}, g_{\mathrm{euc}})$, it is sufficient to compute in this case and given $S \in SO(n+1)$, one has for $u_1, ..., u_m \in \mathbb{R}^{n+1}$:

$$\begin{aligned} S^* \pi_{m*} h(u_1, ..., u_m) &= \pi_{m*} h(Su_1, ..., Su_m) \\ &= \int_{\mathbb{S}_E} h(v) \langle v, Su_1 \rangle ... \langle v, Su_m \rangle dv \\ &= \int_{\mathbb{S}_E} h(v) \langle S^\top v, u_1 \rangle ... \langle S^\top v, u_m \rangle dv \\ &= \int_{\mathbb{S}_E} h(v) \langle S^{-1} v, u_1 \rangle ... \langle S^{-1} v, u_m \rangle dv \\ &= \int_{\mathbb{S}_E} h(Sv) \langle v, u_1 \rangle ... \langle v, u_m \rangle dv = \pi_{m*}(S^* h)(u_1, ..., u_m), \end{aligned}$$

were $S^\top$ stands for the transpose of S and the penultimate equality follows from a change of variable (S preserves the Lebesgue measure dv).

Lemma B.1.3. $\pi_{m*} \pi_m^*|_{E_m} = \lambda_{m,n} \mathbb{1}_{E_m}$.

Proof. This is an immediate consequence of Schur's lemma. Indeed, $L^2(\mathbb{S}_E) = \bigoplus_{m=0}^{+\infty} \Omega_m$ and Ω_m is an irreducible $SO(n+1)$-module[1] (if $n+1 \geq 3$). The map $\pi_{m*}\pi_m^*$ can be conjugated via π_m^* to a map $\Omega_m \to \Omega_m$ which commutes with the $SO(n+1)$-action and is thus a multiple of the identity by Schur's lemma. The sought result follows. □

We will not bother with the computation of the constant $\lambda_{m,n}$: this can be done by evaluating the map on a particular element (see [DS10]). This also shows that, up to rescaling by the constant $\lambda_{m,n}$, $\pi_m^* : E_m \to \Omega_m$ is an isometry. One could be more accurate and actually show that the maps

$$\pi_m^* : \otimes_S^m E^* \to \oplus_{k=0}^{[m/2]} \Omega_{m-2k}, \qquad \pi_{m*} : \oplus_{k=0}^{[m/2]} \Omega_{m-2k} \to \otimes_S^m E^* \tag{B.1.3}$$

are isomorphisms, where $[m/2]$ stands for the integer part of $m/2$. This follows from the (unique) decomposition of a symmetric tensor into a trace-free part and a rest (which lies in the image of the adjoint of Tr). More precisely, by iterating this process, one can decompose u as $u = \sum_{k=0}^{[m/2]} I^k(u_k)$, where $I : \otimes_S^\bullet E^* \to \otimes_S^{\bullet+2} E^*$ is the adjoint of Tr with respect to the scalar products (one has $I = \sigma(g \otimes \cdot)$) and $u_k \in \otimes_S^{m-2k} E^*, \mathrm{Tr}(u_k) = 0$

1. If $n+1 = 2$, then $\Omega_m = H_m \oplus H_{-m}$, where $H_{\pm m}$ is the one-dimensional space spanned by $\theta \mapsto e^{\pm im\theta}$ and $H_{\pm m}$ is an irreducible $SO(2)$-module.

and $\pi_m^* I^k(u_k) \in \Omega_{m-2k}$. Then (B.1.3) is an immediate consequence of Lemma B.1.3. The map $\pi_{m*}\pi_m^*$ acts by scalar multiplication on each component $I^k(u_k)$ (but with a different constant though, so $\pi_{m*}\pi_m^*$ is not a multiple of the identity). Since we will only need the fact that $\pi_{m*}\pi_m^*$ is an isomorphism, we do not provide further details.

B.1.3 Symmetric tensors on a Riemannian manifold

Decomposition in solenoidal and potential tensors. We now consider the Riemannian manifold (M, g) and denote by $d\mu$ the Liouville measure on the unit tangent bundle SM. All the previous definitions naturally extend to the vector bundle $TM \to M$. For $f, h \in C^\infty(M, \otimes^m T^*M)$, we define the L^2-scalar product

$$\langle f, h \rangle = \int_M \langle f_x, h_x \rangle_x d\operatorname{vol}(x),$$

where $\langle \cdot, \cdot \rangle_x$ is the scalar product on T_xM introduced in the previous paragraph. The map $\pi_m^* : C^\infty(M, \otimes^m T^*M) \to C^\infty(SM)$ is the canonical morphism given by $\pi_m^* f(x, v) = f_x(v, ..., v)$, whose formal adjoint with respect to the two L^2-inner products (on $L^2(SM, d\mu)$ and $L^2(\otimes^m T^*M, d\operatorname{vol})$) is π_{m*}, i.e.

$$\langle \pi_m^* f, h \rangle_{L^2(SM, d\mu)} = \langle f, \pi_{m*} h \rangle_{L^2(\otimes^m T^*M, d\operatorname{vol})}.$$

If ∇ denotes the Levi-Civita connection, we set $D := \sigma \circ \nabla : C^\infty(M, \otimes^m T^*M) \to C^\infty(M, \otimes^{m+1} T^*M)$, the symmetrized covariant derivative. Its formal adjoint with respect to the L^2-scalar product is $D^* = -\operatorname{Tr}(\nabla)$ where the trace is taken with respect to the two first indices, like in B.1.1. One has the following relation between the geodesic vector field X on SM and the operator D :

Lemma B.1.4. $X \pi_m^* = \pi_{m+1}^* D$

Proof. Since $D = \sigma \nabla$ and $\pi_{m+1}^* \sigma = \pi_{m+1}^*$, it is sufficient to prove that $X \pi_m^* = \pi_{m+1}^* \nabla$. Remark that these two maps satisfy the Leibniz rule, namely for $f_1 \in C^\infty(M, \otimes_S^{m_1} T^*M)$, $f_2 \in C^\infty(M, \otimes_S^{m_2} T^*M)$ such that $m_1 + m_2 = m$:

$$X \pi_m^* (f_1 \otimes f_2) = X(\pi_{m_1}^* f_1 \pi_{m_2}^* f_2) = X \pi_{m_1}^* f_1 \pi_{m_2}^* f_2 + \pi_{m_1}^* f_1 X \pi_{m_2}^* f_2,$$

and for $v \in C^\infty(M, SM)$,

$$\begin{aligned} \pi_{m+1}^* \nabla (f_1 \otimes f_2)(v) &= \nabla (f_1 \otimes f_2)(v, ..., v) \\ &= (\nabla_v f_1 \otimes f_2)(v, ..., v) + (f_1 \otimes \nabla_v f_2)(v, ..., v) \\ &= \pi_{m_1+1}^* \nabla f_1(v) \pi_{m_2}^* f_2(v) + \pi_{m_1}^* f_1(v) \pi_{m_2+1}^* f_2(v), \end{aligned}$$

that is $\pi_{m+1}^* \nabla (f_1 \otimes f_2) = \pi_{m_1+1}^* \nabla f_1 \pi_{m_2}^* f_2 + \pi_{m_1}^* f_1 \pi_{m_2+1}^* f_2$. It is thus sufficient to prove the result for $m = 0, 1$, but then the result is immediate computing in local coordinates. □

The operator D can be seen as a differential operator of order 1. Its principal symbol is given by $\sigma(D)(x, \xi) f \mapsto \sigma(\xi \otimes f) = \sigma j_\xi f$ (see [Sha94, Theorem 3.3.2]).

Lemma B.1.5. *D is elliptic. It is injective on tensors of odd order, and its kernel is reduced to $\mathbb{R} g^{\otimes m/2}$ on even tensors.*

When m is even, we will denote by $K_m = c_m \sigma(g^{\otimes m/2})$, with $c_m > 0$, a unitary vector in the kernel of D.

Proof. We fix $(x,\xi) \in T^*M$. For a tensor $u \in \otimes_s^m T_x^* M$, using the fact that the anti-symmetric part of $\xi \otimes u$ vanishes in the integral :

$$\langle \sigma(D)u, \sigma(D)u \rangle = \int_{\mathbb{S}_x^n} \langle \xi, v\rangle^2 \pi_m^* u^2(v) dS_x(v) = |\xi|^2 \int_{\mathbb{S}_x^n} \langle \xi/|\xi|, v\rangle^2 \pi_m^* u^2(v) dS_x(v) > 0,$$

unless $u \equiv 0$. Since $\otimes_s^m T_x^* M$ is finite dimensional, the map

$$(u, \xi/|\xi|) \mapsto \langle \sigma(D)(x,\xi/|\xi|)u, \sigma(D)(x,\xi/|\xi|)u \rangle,$$

defined on the compact set $\{u \in \otimes_S^m T_x^* M, |u|^2 = 1\} \times \mathbb{S}^n$ is bounded and attains its lower bound $C^2 > 0$ (which is independent of x). Thus $\|\sigma(x,\xi)\| \geq C|\xi|$, so the operator is uniformly elliptic and can be inverted (on the left) modulo a compact remainder : there exists pseudodifferential operators Q, R of respective order $-1, -\infty$ such that $QD = 1 + R$.

As to the injectivity of D : if $Df = 0$ for some tensor $f \in C^{-\infty}(M, \otimes_S^m T^*M)$, then f is smooth and $\pi_{m+1}^* Df = X\pi_m^* f = 0$. By ergodicity of the geodesic flow, $\pi_m^* f = c \in \Omega_0$ is constant. If m is odd, then $\pi_m^* f(x,v) = -\pi_m^* f(x,-v)$ so $f \equiv 0$. If m is even, then, by §B.1.2, $f = I^{m/2}(u_{m/2})$ where $u_{m/2} \in \otimes_S^0 E^* \simeq \mathbb{R}$ so $f = c'\sigma(g^{\otimes m/2})$. ☐

By classical elliptic theory, the ellipticity and the injectivity of D imply that

$$H^s(M, \otimes_S^m T^*M) = D(H^{s+1}(M, \otimes_S^{m-1} T^*M)) \oplus \ker D^*|_{H^s(M, \otimes_S^m T^*M)}, \tag{B.1.4}$$

and the decomposition still holds in the smooth category and in the $C^{k,\alpha}$-topology for $k \in \mathbb{N}, \alpha \in (0,1)$. This is the content of the following theorem :

Theorem B.1.1 (Tensor decomposition)**.** *Let $s \in \mathbb{R}$ and $f \in H^s(M, \otimes_S^m T^*M)$. Then, there exists a unique pair of symmetric tensors*

$$(p,h) \in H^{s+1}(M, \otimes_S^{m-1} T^*M) \times H^s(M, \otimes_S^m T^*M),$$

such that $f = Dp + h$ and $D^*h = 0$. Moreover, if $m = 2l+1$ is odd, $\langle p, K_{2l}\rangle = 0$.

The proof will be an immediate consequence of the following dicussion. When m is even, we denote by $\Pi_{K_m} := \langle K_m, \cdot\rangle K_m$ the orthogonal projection on $\mathbb{R}K_m$. We define $\Delta_m := D^*D + \varepsilon(m)\Pi_{K_m}$, where $\varepsilon(m) = 1$ for m even, $\varepsilon(m) = 0$ for m odd. The operator Δ_m is an elliptic differential operator of order 2 which is invertible : as a consequence, its inverse is also pseudodifferential of order -2 (see [Shu01, Theorem 8.2]). We can thus define the operator

$$\pi_{\ker D^*} := \mathbb{1} - D\Delta_m^{-1} D^*. \tag{B.1.5}$$

One can check that this is exactly the L^2-orthogonal projection on solenoidal tensors, it is a pseudodifferential operator of order 0 (as a composition of pseudodifferential operators).

Since $\sigma(D)(x,\xi) = \sigma j_\xi$, we know by §B.1.1 that given $(x,\xi) \in T^*M$, the space $\otimes_S^m T_x^* M$ breaks up as the direct sum

$$\begin{aligned}\otimes_S^m T_x^* M &= \operatorname{ran}\left(\sigma(D)(x,\xi)|_{\otimes_S^{m-1} T_x^* M}\right) \oplus \ker\left(\sigma(D^*)(x,\xi)|_{\otimes_S^m T_x^* M}\right) \\ &= \operatorname{ran}\left(\sigma j_\xi|_{\otimes_S^{m-1} T_x^* M}\right) \oplus \ker\left(i_\xi|_{\otimes_S^m T_x^* M}\right)\end{aligned}$$

We recall that $\pi_{\ker i_\xi}$ is the projection on $\ker\left(i_\xi|_{\otimes_S^m T_x^* M}\right)$ parallel to $\operatorname{ran}\left(\sigma j_\xi|_{\otimes_S^{m-1} T_x^* M}\right)$.

Lemma B.1.6. *The principal symbol of* $\pi_{\ker D^*}$ *is* $\sigma_{\pi_{\ker D^*}} = \pi_{\ker i_\xi}$.

Proof. First, observe that :

$$\begin{aligned} D\Delta_m^{-1}D^*D\Delta_m^{-1}D^* &= D\Delta_m^{-1}(\Delta_m - \varepsilon(m)\Pi_{K_m})\Delta_m^{-1}D^* \\ &= D\Delta_m^{-1}D^* - \varepsilon(m)D\Delta_m^{-1}\Pi_{K_m}\Delta_m^{-1}D^* \end{aligned}$$

The second operator is smoothing so at the principal symbol level

$$\sigma_{(D\Delta_m^{-1}D^*)^2} = \sigma^2_{D\Delta_m^{-1}D^*} = \sigma_{D\Delta_m^{-1}D^*},$$

which implies that $\sigma_{D\Delta_m^{-1}D^*}$ is a projection. Moreover, $\sigma_{D\Delta_m^{-1}D^*} = \sigma_D\sigma_{\Delta_m^{-1}}\sigma_{D^*} = \sigma j_\xi \sigma_{\Delta_m^{-1}} i_\xi$, so it is the projection onto $\operatorname{ran}\sigma j_\xi$ with kernel $\ker i_\xi$. Since $\pi_{\ker D^*} = \mathbb{1} - D\Delta_m^{-1}D^*$, the result is immediate. □

Tensorial distributions. The spaces $H^s(M, \otimes_S^m T^*M)$ that have been mentioned so far are the L^2-based Sobolev spaces of order $s \in \mathbb{R}$. They can be defined in coordinates (each coordinate of the tensor has to be in $H^s_{\mathrm{loc}}(\mathbb{R})$) or more intrinsically by setting $H^s(M, \otimes_S^m T^*M) := (\mathbb{1} - D^*D)^{-s/2}L^2(M, \otimes_S^m T^*M)$. These two definitions are equivalent by [Shu01, Proposition 7.3], following the properties of the operator $-D^*D$ (it is elliptic, invertible, positive). In the same fashion, the spaces $L^p(M, \otimes_S^m T^*M)$, for $p \geq 1$ can be defined in coordinates. Note that the maps

$$\pi_m^* : H^s(M, \otimes_S^m T^*M) \to H^s(SM), \quad \pi_{m*} : H^s(SM) \to H^s(M, \otimes_S^m T^*M).$$

are bounded for all $s \in \mathbb{R}$ (and they are bounded on L^p-spaces for $p \geq 1$). The operator π_{m*} acts by duality on distributions, namely :

$$\pi_{m*} : C^{-\infty}(SM) \to C^{-\infty}(M, \otimes_S^m T^*M), \qquad \langle \pi_{m*}f_1, f_2\rangle := \langle f_1, \pi_m^* f_2\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the distributional pairing.

The solenoidal gauge. It is immediate that the metric g is solenoidal with respect to itself since $D^*g = \operatorname{Tr}(\underbrace{\nabla g}_{=0}) = 0$. The following lemma will be useful : it asserts that in a neighborhood of the metric g, we can always set ourselves in the solenoidal gauge. We recall that the metric g is assumed to be smooth.

Lemma B.1.7. *Let k be an integer ≥ 2 and $\alpha \in (0,1)$. There exists a neighborhood $\mathcal{U}$ of g in the $C^{k,\alpha}$-topology such that for any $g' \in \mathcal{U}$, there exists a unique $C^{k+1,\alpha}$-diffeomorphism ψ such that ψ^*g' is solenoidal with respect to g. Moreover, the map $C^{k,\alpha}(M, \otimes_S^2 T^*M) \ni g' \mapsto \psi \in \mathrm{Diff}^{k+1,\alpha}(M)$ is smooth.*

The idea is to apply the inverse function theorem in Banach spaces.

Proof. Consider the map $C^{k+1,\alpha}(M, TM) \ni V \mapsto e_V := x \mapsto \exp_x(V(x)) \in \mathrm{Diff}^{k+1,\alpha}(M)$; it is a well-defined smooth diffeomorphism for $V \in \mathcal{U}_0$ a small $C^{k+1,\alpha}$-neighborhood of the zero section onto a neighborhood of the identity in $\mathrm{Diff}^{k+1,\alpha}(M)$. We define

$$F_1 : \mathcal{U}_0 \times C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M), \qquad F_1(V, f) = D_g^*(e_V^*(g+f))$$

and we want to solve locally the equation $F_1(V(f), f) = 0$. Note that $e_V^*(g+f) \in C^{k,\alpha}(M, \otimes_S^2 T^*M)$ if $V \in C^{k+1,\alpha}(M, TM)$. However, there is a subtle problem here coming from the fact that F_1 *is not smooth* in a neighborhood of $(0,0)$ but only differentiable. This would not prevent us from applying the inverse function theorem, but the regularity of the map $g' \mapsto \psi$ would only be C^1. Indeed, if we take $f \neq 0$, then $g' := g + f \in C^{k,\alpha}(M, \otimes_S^2 T^*M)$ and in local coordinates

$$(e_V^* g')_{kl}(x) = g'_{ij}(e_V(x)) \frac{\partial e_V^i}{\partial x_k}(x) \frac{\partial e_V^j}{\partial x_l} \tag{B.1.6}$$

As a consequence, by the chain rule, differentiating with respect to V makes a term $Z \mapsto d_{e_V(x)} g'_{ij}(d_V e(Z)) \in C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$ appear and differentiating twice, we would obtain a term in $C^{k-2,\alpha}(M, \otimes_S^2 T^*M)$ (so we would leave the Banach space $C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$). However, remark that

$$e_{V*} \circ D_g^* \circ e_V^* = D_{e_{V*}g}^* \tag{B.1.7}$$

Thus, solving $D_g^* e_V^*(f+g) = 0$ is equivalent to solving $D_{e_{V*}g}^*(f+g) = 0$. Therefore, we rather consider

$$F_2 : \mathcal{U}_0 \times C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M), \qquad F_2(V,f) = D_{e_{V*}g}^*(f+g)$$

and we want to solve $F_2(V(f), f) = 0$ in a neighborhood of $(0,0)$. The map F_2 is *smooth*. Indeed, it is immediately smooth in f, since it is linear and by (B.1.6), since g is smooth, it is smooth in V.

Since $d_V e(0) = \mathbb{1}$ (because the differential of the exponential map $\exp_x$ at 0 is the identity), we see from (B.1.7) that $d_V F_2(0,0) = d_V F_1(0,0)$. As a consequence, by the implicit function theorem, solving $F_2(V(f), f) = 0$ in a neighborhood of $(0,0)$ amounts to proving that $d_V F_1(0,0)$ is an isomorphism. The differential of F_1 at $(0,0)$ is given by

$$D_V F_1(0,0) \cdot Z = D_g^*(\mathcal{L}_Z g) = 2 \times D_g^* D_g(Z^\sharp),$$

for $Z \in C^{k+1,\alpha}(M, TM)$, where $\sharp : TM \to T^*M$ is the musical isomorphism induced by the metric g (and this maps $C^{k+1,\alpha}(M, TM) \to C^{k-1,\alpha}(M, \otimes_S^2 T^*M)$ which is coherent). But $D_g^* D_g$ is a differential operator of order 2 which is elliptic and invertible — since D is. As a consequence $D_g^* D_g : C^{k+1,\alpha}(M, T^*M) \to C^{k-1,\alpha}(M, T^*M)$ is an isomorphism. By the implicit function theorem for Banach spaces, there exists a neighborhood $\mathcal{U} \subset \mathcal{U}_0$ and a smooth map $f \mapsto V(f)$ (from $C^{k,\alpha}(M, \otimes_S^2 T^*M) \to C^{k+1,\alpha}(M, \otimes_S^2 T^*M)$) such that $F_2(V(f), f) = 0$ for all $f \in \mathcal{U}$ (and thus $F_1(V(f), f) = 0$). Moreover, $V(f)$ is the unique solution to $F_{1,2}(Z, f) = 0$ in this neighborhood. □

Remark B.1.1. The fact that g is smooth is not essential in the proof if one does not care about the regularity of the map $g' \mapsto \psi$ (and using the map F_1 is sufficient to conclude).

Remark B.1.2. The same arguments also work in the H^s-regularity for s sufficiently large (so that the maps defined in the proof make sense). It also works in the space C_*^k, for $k \geq 2$ an integer : this is not the usual C^k-space but the Zygmund space of regularity k, defined in terms of a Littlewood-Paley decomposition (see [Tay91, Appendix A]). In particular, $C^k \subset C_*^k$. This is an artifact of the theory of pseudodifferential operators that one has to resort to Zygmund spaces rather than the usual C^k-topology.

B.2 X-ray transform and transport equations

B.2.1 The lowering and raising operators

Canonical splitting. We recall that (M, g) is a smooth closed $(n+1)$-dimensional manifold. The tangent bundle to SM can be decomposed according to :

$$T(SM) = \mathbb{V} \oplus^{\perp} \mathbb{H} \oplus^{\perp} \mathbb{R}X,$$

where $\mathbb{H}$ is the *horizontal bundle*, $\mathbb{V}$ is the *vertical bundle* and SM is endowed with the Sasaki metric g_S. If $\pi_0 : TM \to M$ denotes the projection on the base, then $d\pi_0 : \mathbb{H} \oplus^{\perp} \mathbb{R}X \to TM$ is an isomorphism, and there also exists an isomorphism $\mathcal{K} : \mathbb{V} \to TM$ called the *connection map* (see [Pat99]). We denote by ∇_S the Levi-Civita connection induced by the Sasaki metric g_S on SM. Given $u \in C^\infty(SM)$, one can decompose its gradient according to :

$$\nabla_S u = \nabla^v u + \nabla^h u + Xu \cdot X, \tag{B.2.1}$$

where $\nabla^{v,h}$ are the respective vertical and horizontal gradients (the orthogonal projection of the gradient on the vertical and horizontal bundles), i.e. $\nabla^v u \in \mathbb{V}, \nabla^h u \in \mathbb{H}$. We denote by $N_\perp$ the subbundle of $TM \to SM$ whose fiber at $(x, v) \in SM$ is given by $N_\perp(x, v) := \{v\}^\perp$. Using the maps $d\pi_0$ and $\mathcal{K}$, the vectors $\nabla^{v,h} u$ can be identified with elements of $N_\perp$, i.e. $\mathcal{K}(\nabla^v u), d\pi_0(\nabla^h u) \in N$. For the sake of simplicity, we will drop the notation of these projection maps in the following and consider $\nabla^{v,h} u$ as elements of N. The Riemannian metric on M endows $N_\perp$ with a natural L^2-scalar product and we denote by $-\operatorname{div}^{v,h}$ the formal adjoints of the maps $\nabla^{v,h} : C^\infty(SM) \to N_\perp$. In the following, $R(x, v) : N_\perp \to N_\perp$ will denote the operator $R(x, v)w = \mathcal{R}_x(w, v)v$, where $\mathbf{R}$ is the Riemannian curvature tensor.

The vertical laplacian is then defined by $\Delta^v := \operatorname{div}^v \nabla^v : C^\infty(SM) \to C^\infty(SM)$. An equivalent definition is obtained by considering the fiber-wise Laplacian induced by the Riemannian metric on each sphere $S_x M$, for $x \in M$, like in §B.1.1. We have the following commutator formulas, for which we refer to [PSU15, Proposition 2.2] for a proof :

Lemma B.2.1.

$$[X, \nabla^v] = -\nabla^h, \qquad [X, \operatorname{div}^v] = -\operatorname{div}^h,$$
$$[X, \Delta^v] = 2 \operatorname{div}^v \nabla^h + nX, \qquad [X, \nabla^h] = R\nabla^v$$

Transport equations We refer to [PSU15] for the detailed computations of this paragraph and to [GK80b] for the original arguments. From now on, we denote by $C^\infty(M, \Omega_m) := C^\infty(SM) \cap \ker(\Delta^v + m(m+n-1))$. Let us start with the important

Lemma B.2.2. $X : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m-1}) \oplus C^\infty(M, \Omega_{m+1})$

Proof. Consider normal coordinates at the point $x \in M$. Then, in these coordinates, $Xu(x, v) = \sum_i v_i \partial_{x_i} u(x, v)$ and this is in $\Omega_{m-1} \oplus \Omega_{m+1}$ if and only if $\sum_i v_i u(x, v) \in \Omega_{m-1} \oplus \Omega_{m+1}$. This boils down to the fact that the product of a degree m spherical harmonic with a degree 1 harmonic is the sum of an $(m-1)$- and an $(m+1)$ harmonic. □

This allows to decompose [2] $X|_{\Omega_m} = X_- + X_+$ with $X_- : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m-1})$ and $X_+ : C^\infty(M, \Omega_m) \to C^\infty(M, \Omega_{m+1})$. Moreover, it is easy to check that $X_+^* = -X_-$ (at least formally), where the duality is understood with respect to the L^2 scalar product on SM. Note that there is also a natural identification of the operators $X_\pm$ with the operators D and D^*. More precisely, writing $p : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(M, E_m)$ the orthogonal projection on trace-free symmetric tensors, one has the

Lemma B.2.3. *For all $f \in C^\infty(M, E_m)$, one has :*

$$X_- \pi_m^* f = \frac{m}{n+2m-1} \pi_m^* D^* f, \qquad X_+ \pi_m^* f = \pi_m^* p D f$$

The operator X_+ is elliptic (it has injective principal symbol and thus a finite dimensional kernel), whereas X_- is of divergence type (see [GK80b, Proposition 3.7]). The injectivity of X_+ is equivalent to the surjectivity of X_- on the image of X_+ that is on the L^2 orthogonal to $\ker X_-$. There exists a decomposition

$$C^\infty(M, \Omega_m) = \ker(X_-) \oplus X_+ C^\infty(M, \Omega_{m-1})$$

which is orthogonal with respect to the L^2 scalar product on SM and unique if X_+ is injective (or X_- is surjective). We call *conformal Killing tensor field* (abbreviated CKT in the following) a trace-free symmetric tensor $f \in C^\infty(M, \otimes_S^m T^*M)$ such that $X_+ \pi_m^* f = 0$ (note that $\pi_m^* f \in C^\infty(M, \Omega_m)$). We have the following

Lemma B.2.4. *Let (M, g) be an Anosov Riemannian manifold with non-positive sectional curvature. Then, there are no CKTs, except 0 for $m \geq 1$ and the constants for $m = 0$.*

This lemma actually holds in the more general case of a non-positively curved manifold with rank one (see [PSU15, Corollary 3.6]). It will be proved in the next paragraph.

Energy identities. Energy identities known as the *Pestov identity* are crucial in the study of symmetric tensors.

Lemma B.2.5 (Pestov identity)**.** *Let $u \in H^2(SM)$. Then*

$$\|\nabla^v X u\|^2 = \|\nabla_X \nabla^v u\|^2 - \int_{SM} \kappa(v, \nabla^v u) \|\nabla^v u\|^2 d\mu(x,v) + n\|Xu\|^2.$$

In particular, under the additional assumption that the sectional curvatures are non-positive :

$$\|\nabla^v X u\|^2 \geq \|\nabla_X \nabla^v u\|^2 + n\|Xu\|^2.$$

Proof. For $u \in C^\infty(SM)$, using the previous commutator formulas :

$$\begin{aligned}
\|\nabla^v X u\|^2 - \|\nabla_X \nabla^v u\|^2 &= \langle \nabla^v X u, \nabla^v X u \rangle - \langle \nabla_X \nabla^v u, \nabla_X \nabla^v u \rangle \\
&= \langle (X \operatorname{div}^v \nabla^v X - \operatorname{div}^v X^2 \nabla^v) u, u \rangle \\
&= \langle (-\operatorname{div}^h \nabla^v X + \operatorname{div}^v X \nabla^h) u, u \rangle \\
&= \langle (-\operatorname{div}^h \nabla^v X + \operatorname{div}^v \nabla^h X + \operatorname{div}^v R \nabla^v) u, u \rangle \\
&= -n \langle X^2 u, u \rangle + \langle \operatorname{div}^v R \nabla^v u, u \rangle \\
&= n\|Xu\|^2 - \langle R \nabla^v u, \nabla^v u \rangle \\
&= n\|Xu\|^2 - \int_{SM} \kappa(v, \nabla^v u) \|\nabla^v u\|^2 d\mu(x,v)
\end{aligned}$$

2. In the case of a surface, these operators can still be simplified (using the decomposition $\Omega_m = H_m \oplus H_{-m}$) and one recovers the lower and raising operators $\eta_\pm$ of Guillemin-Kazhdan [GK80a].

□

Remark B.2.1. By a density argument, the Pestov identity actually holds under the weaker assumption that $u \in H^1(SM)$ and both $\nabla^v Xu, \nabla_X \nabla^v u \in L^2(SM)$.

The previous Pestov identity specified to a function $u \in C^\infty(M, \Omega_m)$ yields the

Lemma B.2.6. *Let $u \in C^\infty(M, \Omega_m)$. Then :*

$$(2m+n-2)\|X_- u\|^2 + \|\nabla^h u\|^2 - \int_{SM} \kappa(v, \nabla^v u)\|\nabla^v u\|^2 d\mu(x,v) = (2m+n)\|X_+ u\|^2$$

The proof is similar to that of Lemma B.2.5 using the commutator identities (we refer to [PSU15, Proposition 3.4]). We can now prove Lemma B.2.4.

Proof of Lemma B.2.4. If $u \in C^\infty(M, \Omega_m)$, $X_+ u = 0$ and the sectional curvatures are non-positive, the Pestov identity B.2.6 implies that $X_- u = 0$ and $\nabla^h u = 0$. Thus $Xu = X_- + X_+ = 0$. By ergodicity, u is constant, thus $u = 0$ if $m \geq 1$. □

B.2.2 Surjectivity of π_{m*}

As explained in Section §2.5.2, the solenoidal injectivity of the X-ray transform I_m is closely related to the existence of invariant distributions with prescribed pushforward on the set of solenoidal tensors, that is of the map $\pi_{m*} : C^{-\infty}_{\text{inv}}(SM) \to C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)$, where $C^{-\infty}_{\text{inv}}(SM) = \cup_{s \leq 0} H^s(SM) \ \cap \ker X$.

Invariant distributions. Following [PZ16, Proposition 7.3] and taking advantage of the decomposition in trace-free symmetric tensors $u = \sum_{k=0}^{[m/2]} I^k(u_k)$ (see §B.1.2), the surjectivity of π_{m*} can be simplified[3] to the following

Proposition B.2.1. *Fix $m \in \mathbb{N}$. The following statements are equivalent :*

1. *For all $0 \leq k \leq m$, the map $\pi_{k*} : C^{-\infty}_{\text{inv}}(SM) \to C^\infty_{\text{sol}}(M, \otimes^k_S T^*M)$ is surjective,*
2. *For all $0 \leq k \leq m$, given $f \in C^\infty(M, \Omega_k) \cap \ker X_-$, there exists $u \in C^{-\infty}_{\text{inv}}(SM)$ such that $u_k = f$, where $u_k \in \ker(\Delta^v + k(k+n-1))$ denotes the k-th Fourier mode of u and u has only Fourier modes $\geq k$.*

Proof. First of all, observe that given $f \in \oplus_{k=0}^{[m/2]} C^\infty(M, \Omega_{m-2k})$, one has that $Xf \in C^\infty(M, \Omega_{m+1})$ if and only if $D^* \pi_{m*} f = 0$. Indeed, if $f \in \oplus_{k=0}^{[m/2]} C^\infty(M, \Omega_{m-2k})$, then by Lemma B.2.2, $Xf \in \oplus_{k=0}^{[(m+1)/2]} C^\infty(M, \Omega_{m-2k+1})$. Moreover, by Lemma B.1.4, $D^* \pi_{m*} f = -\pi_{m-1*} Xf$. Thus, if $\pi_{m-1*} Xf = 0$, then $Xf \in C^\infty(M, \Omega_{m+1})$. Conversely, if $D^* \pi_{m*} f = 0$, then $Xf \in C^\infty(M, \Omega_{m+1})$.

We first prove that (1) implies (2). Let $f \in C^\infty(M, \Omega_m) \cap \ker X_-$, then using Lemma B.1.4 :

$$D^* \pi_{m*} f = -\pi_{m-1*} Xf = -\pi_{m-1*} \underbrace{X_+ f}_{\in \Omega_{m+1}} = 0$$

Thus $\pi_{m*} f$ is a trace-free symmetric solenoidal tensor. By assumption, there exists $u \in C^{-\infty}_{\text{inv}}(SM)$ such that $\pi_{m*} f = \pi_{m*} u$. Moreover, since $\pi_{m*} : \oplus_{k=0}^{[m/2]} H^s(M, \Omega_{m-2k}) \to H^s(M, \otimes^m_S T^*M)$ is an isomorphism for all $s \in \mathbb{R}$, we have that

$$\pi_{m*}^{-1} \pi_{m*} f = f = \pi_{m*}^{-1} \pi_{m*} u = u_m + u_{m-2} + ...$$

3. Yes indeed, it is a simplification !

Thus $u_{m-2} = u_{m-4} = ... = 0$ and $u_m = f$.

Let us prove the converse, that is (2) implies (1). We proceed by induction. For $m = 0$, we consider $f \in C^\infty(M)$, then $\pi_0^* f \in C^\infty(M, \Omega_0)$ and $X_- f = 0$. Thus, there exists $u \in C^{-\infty}_{\text{inv}}(SM)$ such that $u_0 = {\pi_0}^* f$. And $\pi_{0*} u = \pi_{0*}\pi_0^* f = c_0 f$ for some constant $c_0 > 0$.

We consider $f \in C^\infty_{\text{sol}}(M, \otimes^m_S T^*M)$ and we write ${\pi_{m*}}^{-1} f = \sum_{k=0}^{[m/2]} f_{m-2k} = F$. Note that by the preliminary remark (at the beginning of this proof), $XF \in C^\infty(M, \Omega_{m+1})$ because $D^*\pi_{m*}F = D^* f = 0$ by assumption. We thus need to find $u \in C^{-\infty}_{\text{inv}}(SM)$ such that $F = \sum_{k=0}^{[m/2]} u_{m-2k}$. Consider $F' = \sum_{k=1}^{[m/2]} f_{m-2k}$, then $F = F' + f_m$ and $XF = XF' + X f_m = X_+ f_m \in \Omega_{m+1}$, thus equalizing the orders, we obtain that $XF' = X_+ f_{m-2} = -X_- f_m \in \Omega_{m-1}$. By the preliminary remark, this implies that $D^*\pi_{m-2*}F' = 0$. By induction (we use that (2) implies (1) for $m-2$), $\pi_{m-2*} : C^{-\infty}_{\text{inv}}(SM) \to C^\infty_{\text{sol}}(M, \otimes^{m-2}_S T^*M)$ is surjective, so there exists a distribution w' such that $Xw' = 0$ and $\pi_{m-2*}w' = \pi_{m-2*}F'$, that is $w'_{m-2} = f_{m-2}, w'_{m-4} = f_{m-4}, ...$. The equality $Xw' = 0$ of degree $m-1$ yields :

$$X_- w'_m + X_+ w'_{m-2} = X_- w'_m + X_+ f_{m-2} = X_-(w'_m - f_m) = 0$$

and by assumption, there exists a distribution $w = \sum_{k \geq 0} w_{m+2k}$ such that $Xw = 0$ and $-w_m = w'_m - f_m$. Then, setting $W = w + w'$, one has $XW = 0$ and $W_m = f_m, W_{m-2} = f_{m-2}, ...$. This proves the surjectivity. □

Beurling transform in non-positive curvature. We now assume that (M, g) is an Anosov Riemannian manifold. By Lemmas 2.5.4, 2.5.8 and Proposition B.2.1, proving the solenoidal injectivity of the X-ray transform I_m amounts to proving the second item of Proposition B.2.1 : given $f \in C^\infty(M, \Omega_m) \cap \ker X_-$, there exists $u \in C^{-\infty}_{\text{inv}}(SM)$ such that $u_m = f$, where $u_m \in \ker(\Delta^v + m(m+n-1))$ denotes the m-th Fourier mode of u.

Proposition B.2.2. *Assume (M, g) is Anosov with non-positive curvature. Then, the second item of Proposition B.2.1 holds.*

An immediate consequence is that we recover the celebrated result of [CS98] :

Theorem B.2.1. *Assume (M, g) is Anosov with non-positive curvature. Then I_m is solenoidal injective for all $m \geq 0$.*

Let us explain the heuristic behind Proposition B.2.2. Let $f \in C^\infty(M, \Omega_m) \cap \ker X_-$. We are looking for a distribution $u \in C^{-\infty}(SM)$ such that $Xu = 0$ and $u_m = f$. Assume $u = \sum_{k \geq 0} u_k$, then equalizing $Xu = 0$ gives

$$X_- u_{k+1} + X_+ u_{k-1} = 0, \tag{B.2.2}$$

for all $k \geq 1$. Note that we also have the "initial conditions" :

$$\begin{aligned} X_- u_m + X_+ u_{m-2} = 0 &= \underbrace{X_- f}_{=0} + X_+ u_{m-2}, \\ X_- u_{m+2} + X_+ u_m = 0 &= X_- u_{m+2} + X_+ f \end{aligned}$$

We can immediately take $u_k = 0$ for $k \neq m \pmod 2$ and by the first initial condition, we can take $u_{m-2} = u_{m-4} = ... = 0$. Under the assumption that (M, g) is Anosov and non-positively curved, we know by Lemma B.2.4 that there are no CKTs. This implies that X_- is surjective on the image of X_+ (the orthogonal of the kernel of X_-)

and thus there exists a (unique) u_{m+2} which is orthogonal to $\ker X_-$ and such that $X_- u_{m+2} = -X_+ f$. Then, we solve by induction (B.2.2) to obtain the modes u_{m+2k} for $k \geq 0$. This allows to construct $u = f + u_{m+2} + u_{m+4} + ...$ such that $Xu = 0$ and $u_m = f$. Of course, this is only a formal argument and one needs to check that the formal series $\sum_{k\geq 0} u_{m+2k}$ converges in some suitable norm. Let us give more formal definitions.

Definition B.2.1. Assume (M, g) is Anosov with non-positive curvature. Let $k \geq 0$. Given $f \in C^\infty(M, \Omega_k)$, there exists a unique $B_k f \in C^\infty(M, \Omega_{k+2})$ which is orthogonal to $\ker X_-$ and solves (B.2.2), that is $X_- B_k f + X_+ f = 0$. The map $B_k : \Omega_k \to \Omega_{k+2}$ is called the *Beurling transform*.

The formal solution we are looking for can then be written

$$u = \sum_{k\geq 0} B^k f, \tag{B.2.3}$$

where $B^k := B_{m+2k} B_{m+2(k-2)} ... B_m$. Following [PSU15, Theorem 1.1], we have the following bounds :

Lemma B.2.7. *Let (M, g) be an Anosov Riemannian $(n+1)$-dimensional manifold with non-positive curvature. Then for all $k \geq 0$:*

$$\forall f \in C^\infty(M, \Omega_k), \qquad \|B_k f\|_{L^2} \leq b_{n,k} \|f\|_{L^2},$$

where :

$$\begin{aligned} & b_{1,0} = \sqrt{2}, b_{1,k} = 1, \quad \forall k \geq 1, \\ & b_{2,k} = \left(1 + \frac{1}{(k+2)^2(2k+1)}\right)^{1/2}, \quad \forall k \geq 0 \\ & b_{n,k} \leq 1, \quad \forall n \geq 3, k \geq 0 \end{aligned}$$

Proof. This bound is actually implied by the bound

$$\|X_- u\|_{L^2} \leq b_{n,k} \|X_+ u\|_{L^2}, \qquad \forall u \in C^\infty(M, \Omega_{k+1}) \tag{B.2.4}$$

Indeed, if $f \in C^\infty(M, \Omega_k)$ and $u = B_k f \in C^\infty(M, \Omega_{k+2})$, then $X_- u = -X_+ f$ and u is orthogonal to $\ker X_-$ and in the image of X_+. Thus, there exists $v \in C^\infty(M, \Omega_{k+1})$ such that $X_+ v = u$. And :

$$\|u\|^2 = \langle u, X_+ v\rangle = -\langle X_- u, v\rangle = \langle X_+ f, v\rangle = -\langle f, X_- v\rangle$$

Applying the Cauchy-Schwarz estimate together with (B.2.4), we obtain

$$\|u\|^2 \leq b_{n,k} \|f\| \|X_+ v\| = b_{n,k} \|f\| \|u\|,$$

which gives the sought result.

We are thus left to prove (B.2.4). We consider $u \in C^\infty(M, \Omega_{k+1})$. Applying the Pestov identity (see Lemma B.2.5) in non-positive curvature, we have :

$$\|\nabla^v X u\|^2 \geq \|\nabla_X \nabla^v u\|^2 + n\|Xu\|^2$$

But $Xu = X_- u + X_+ u \in \Omega_k \oplus \Omega_{k+2}$ so $\|Xu\|_{L^2}^2 = \|X_- u\|_{L^2}^2 + \|X_+ u\|_{L^2}^2$ and

$$\|\nabla^v X u\|_{L^2}^2 = k(k+n-1)\|X_- u\|^2 + (k+2)(k+n+1)\|X_+ u\|^2$$

Moreover, following [PSU15, Lemma 4.3], one can prove that :

$$\|\nabla_X \nabla^v u\|^2 \geq \frac{k(k+n)}{k+n-1} \|X_- u\|^2 + \frac{(k+1)^2(k+n+1)}{k+2} \|X_+ u\|^2$$

Combining these three bounds and after some tedious computations, we obtain the sought result. □

In the following, we define for $s \in \mathbb{R}$, the Hilbert spaces $L^2_x H^s_v(SM)$ as the completion of $C^\infty(SM)$ with respect to the norm

$$\|f\|_{L^2_x H^s_v} := \left(\sum_{k \geq 0} \langle k \rangle^{2s} \|f_k\|^2_{L^2} \right)^{1/2},$$

where $\langle k \rangle = \sqrt{1+k^2}$. In turn, Lemma B.2.7 implies the

Lemma B.2.8. *Let $m \geq 0$ and $f \in C^\infty(M, \Omega_m)$ such that $X_- f = 0$. Then for all $k \geq 0$, $\|B^k f\|_{L^2} \leq 2\|f\|_{L^2}$.*

Proof. The proof is an immediate consequence of the following computation :

$$\prod_{k=0}^{+\infty} b_{n,m+2k} \leq 2,$$

for all $n \geq 1, m \geq 0$. Then $\sum_{k \geq 0} B^k f \in L^2_x H_v^{-1/2-\varepsilon}(SM)$, for all $\varepsilon > 0$. □

Proof of Proposition B.2.2. In particular, using the previous lemma, in (B.2.3), one has the convergence of $u = \sum_{k \geq 0} B^k f \in L^2_x H_v^{-1/2-}(SM)$. This concludes the proof. □

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université PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

Titre : Sur la rigidité des variétés riemanniennes

Mots Clefs : Problèmes inverses, rigidité, analyse microlocale, dynamique hyperbolique

Résumé : Une variété riemannienne est dite rigide lorsque la longueur des géodésiques périodiques (cas des variétés fermées) ou des géodésiques diffusées (cas des variétés ouvertes) permet de reconstruire globalement la géométrie de la variété. Cette notion trouve naturellement son origine dans des dispositifs d'imagerie numérique tels que la tomographie par rayons X. Grâce à une approche résolument analytique initiée par Guillarmou et fondée sur de l'analyse microlocale (plus particulièrement sur certaines techniques récentes dues à Faure-Sjostrand et Dyatlov-Zworski permettant une étude analytique fine des flots Anosov), nous montrons que le spectre marqué des longueurs, c'est-à-dire la donnée des longueurs des géodésiques périodiques marquées par l'homotopie, d'une variété fermée Anosov ou Anosov à pointes hyperboliques détermine localement la métrique de la variété. Dans le cas d'une variété ouverte avec ensemble capté hyperbolique, nous montrons que la distance marquée au bord, c'est-à-dire la donnée de la longueur des géodésiques diffusées marquées par l'homotopie, détermine localement la métrique. Enfin, dans le cas d'une surface asymptotiquement hyperbolique, nous montrons qu'une notion de distance renormalisée entre paire de points au bord à l'infini permet de reconstruire globalement la géométrie de la surface.

Title : On the rigidity of Riemannian manifolds

Keys words : Inverse problems, rigidity, microlocal analysis, hyperbolic dynamics

Abstract : A Riemannian manifold is said to be rigid if the length of periodic geodesics (in the case of a closed manifold) or scattered geodesics (in the case of an open manifold) allows to recover the full geometry of the manifold. This notion naturally arises in imaging devices such as X-ray tomography. Thanks to an analytic framework introduced by Guillarmou and based on microlocal analysis (and more precisely on the analytic study of hyperbolic flows of Faure-Sjostrand and Dyatlov-Zworski), we show that the marked length spectrum, that is the lengths of the periodic geodesics marked by homotopy, of a closed Anosov manifold or of an Anosov manifold with hyperbolic cusps locally determines its metric. In the case of an open manifold with hyperbolic trapped set, we show that the lengths of the scattered geodesics marked by homotopy locally determines the metric. Eventually, in the case of an asymptotically hyperbolic surface, we show that a suitable notion of renormalized distance between pair of points on the boundary at infinity allows to globally reconstruct the geometry of the surface.